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ON THE STABILITY OF RICCATI DIFFERENTIAL EQUATION $\dot{X} = TX + Q(X)$ IN $\mathbb{R}^n$

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Abstract The system $\dot{X} = TX + Q(X)$ (in $\mathbb{R}^n$), where T is linear and Q is quadratic, is considered via commutative algebras. The case of the linearized system having a centre manifold spanned on vectors E_1, E_2 (and $TE_1 = \omega E_2$, $TE_2 = -\omega E_1$) is studied. It is shown that for $\mathrm{span}(E_1, E_2)$ being a subalgebra (of the algebra corresponding to the form $Q(X)$), the system is stable. Necessary and sufficient conditions are given for stability of the system in the case where $\mathrm{span}(E_1, E_2)$ is not a subalgebra.

Keywords: Riccati differential equation; autonomous differential equation; stability; centre manifold

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1. Introduction

It is well known (see, for example, [**2**–**4**]) that one can completely determine the stability of a nonlinear flow $\dot{x} = f(x)$ near a hyperbolic stationary point $x = 0$ by considering the Jordan form of the Jacobian matrix $Df(0)$ of the nonlinear flow. This is the statement of the stable manifold theorem and Hartman's theorem. The first theorem shows that the local structure of hyperbolic stationary points of nonlinear flows, in terms of the existence and transversality of local stable and unstable manifolds, is the same as the linearized flow, and the second theorem asserts that there is a continuous invertible map in some neighbourhood of the stationary point which takes the nonlinear flow to the linear flow preserving the sense of time.

For the non-hyperbolic stationary points of the nonlinear flow, the centre manifold theorem implies that the system can be written locally in coordinates $(x, y, z) \in W^{\mathrm{c}} \times W^{\mathrm{s}} \times W^{\mathrm{u}}$ on the invariant manifolds as

$$\left.\begin{aligned} \dot{x} &= g(x), \\ \dot{y} &= -By, \\ \dot{z} &= Cz, \end{aligned}\right\} \tag{1.1}$$

where B and C are positive definite matrices. The motion on W^{s} (W^{u}) is unequivocally towards (away from) the stationary point, so the local behaviour can be understood by solving or analysing the system $\dot{x} = g(x)$.

In general, the stability of non-hyperbolic stationary points of a nonlinear non-autonomous system is not completely understood; especially if $Df(x_0) = 0$. There are results on normal forms for non-hyperbolic stationary points in $\mathbb{R}^2$ (see, for example, [**2**, pp. 79–83]), the averaging theory for small perturbations of oscillations, i.e. for systems of the form $\dot{x} = f(x) + \varepsilon g(x,t)$, where $g(x,t)$ is periodic (see, for example, [**4**, pp. 167–226]). Some partial and special results from the algebraic point of view (see, for example, Theorem 1 in [**16**], and § 3 in [**7**], and Theorems 4.1 and 4.2 in [**6**], and Theorem 1 in [**10**]) on (non-)stability of non-hyperbolic stationary points are given for equations of the form $\dot{X} = Q(X)$ and $\dot{X} = TX + Q(X)$, where $X \in \mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, and $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a quadratic form. Many theorems on stability of the quadratic equation $\dot{X} = Q(X)$ are also valid if $Q(X)$ becomes homogeneous of any degree (see, for example, [**6**, Theorem 4.1]). Koditschek and Narendra (see [**8**]) suggested a useful approach to the investigation of the stability characteristics of a class of second-order differential equations of the type $\dot{X} = Q(X)$ and $\dot{X} = TX + Q(X)$ in the plane. They gave necessary and sufficient conditions for stability in the large (i.e. all solutions are stable and bounded) for the system $\dot{X} = Q(X)$ and necessary and sufficient conditions for asymptotical stability in the large (i.e. all solutions are asymptotically stable and the domain of the attraction is the entire space) for the system $\dot{X} = TX + Q(X)$.

A direct motivation for writing this article is [**16**]. The method of approach to the polynomial autonomous dynamical system in [**16**], as well as in the present article, is via commutative (non-associative, in general) finite-dimensional algebras. It seems that this idea originated in 1960 with Marcus [**9**], where all the commutative algebras in $\mathbb{R}^2$ were classified. There is a one-to-one correspondence between quadratic systems and the corresponding algebra, and between homogeneous systems of degree n and the corresponding n-ary algebra (see [**14, 15, 17**]). The 3-ary algebras in $\mathbb{R}^2$ are classified in [**12**] and [**13**] as well. For a full survey of this theory the reader can consult, for example, [**17**], [**7**], [**6**] and [**11**]. Walcher's monograph [**17**] is also a standard reference for the state of the art in 1990, with many references to older papers.

The quadratic form $Q(X)$ ($X \in \mathbb{R}^n$) in the system $\dot{X} = Q(X)$ (or $\dot{X} = TX + Q(X)$) can be interpreted as diagonal of the following bilinear form

$$B(X,Y) := \tfrac{1}{2}[Q(X+Y) - Q(X) - Q(Y)], \qquad B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n.$$

Defining $X \cdot Y = B(X,Y)$, we can interpret the resulting system of ordinary differential equations (ODEs) as a Riccati equation $\dot{X} = X^2$ (or $\dot{X} = TX + X^2$) in the commutative algebra $\mathcal{A} = (\mathbb{R}^n, \cdot)$.

In [**16**] the following theorem is proved.

Theorem 1.1. *Let $F_t(X)$ denote the solution to $\dot{X} = TX + X^2$ in $\mathcal{A}$ such that $F_0(X) = X$. Let $E \in \mathcal{A}$ be an idempotent satisfying $TE = 0$. Then*

(1) *if $a \neq 0$, $F_t(aX)$ blows up in finite time; and*

(2) *the origin is an unstable equilibrium.*

For the proof the reader should refer to [**16**].

Remark 1.2. For $f(X) = TX + X^2$ we have $f(0) = 0$ and the Jacobian at the origin is $Df(0) = T$, so the condition $TE = 0$ implies non-hyperbolicity of the origin. Thus the origin is an unstable non-hyperbolic stationary point of the equation

$$\dot{X} = TX + X^2. \tag{1.2}$$

We can interpret the existence of the idempotent E in Theorem 1.1 as $\mathbb{R}E = \{xE;\ x \in \mathbb{R}\}$ being a one-dimensional subalgebra of $\mathcal{A}$.

In this article, however, we would like to consider the case of purely imaginary eigenvalues of $Df(0) = T$ with corresponding eigenvectors forming a two-dimensional subalgebra of $\mathcal{A}$. So there exist vectors $E_1 \neq E_2 \neq 0$ such that

$$\left.\begin{aligned} TE_1 &= \omega E_2, \\ TE_2 &= -\omega E_1 \end{aligned}\right\} \tag{1.3}$$

and $\operatorname{span}(E_1, E_2)$ is a subalgebra of $\mathcal{A}$. This implies that in (real) normal form the matrix T contains a block of the form

$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

for some positive real ω.

2. Main theorems

Before proving the main theorem let us consider some additional special algebraic properties of equation (1.2).

Let E be an idempotent of the algebra $\mathcal{A}$ corresponding to equation (1.2) and let E be an eigenvector of the matrix T for some non-zero real eigenvalue λ. Let us consider the initial-value problem

$$F_0(X) = aE, \qquad \dot{X} = TX + X^2.$$

We seek a solution of the form $F_t(X) = f(t)E$:

$$\begin{aligned} f'(t)E &= Tf(t)E + (f(t)E)^2 \\ &= f(t)TE + f^2(t)E^2 \\ &= \lambda f(t)E + f^2(t)E, \end{aligned}$$

and obtain the differential equation

$$f'(t) = \lambda f(t) + f^2(t) \quad \text{in } \mathbb{R}.$$

Thus, a special solution on the subspace $\mathbb{R}E$ is

$$F_t(X) = \frac{(\lambda a/(a+\lambda))\mathrm{e}^{t\lambda}}{1 - (a/(a+\lambda))\mathrm{e}^{t\lambda}} E. \tag{2.1}$$

From (1.2) we see that the point $X = -\lambda E$ is stationary:

$$TX + X^2 = T(-\lambda E) + (-\lambda E)^2 = -\lambda TE + \lambda^2 E^2 = -\lambda^2 E + \lambda^2 E = 0.$$

If $\lambda > 0$, then the origin is of course unstable. And from (2.1) we can deduce that for $a > 0$ we have blow up in finite time

$$t_0 = \frac{1}{\lambda} \ln\left(\frac{a+\lambda}{a}\right).$$

If we linearize (1.2) around the stationary point $X = -\lambda E$, we get the system

$$\dot{Y} = \underbrace{(T - 2\lambda L_E)}_{\mathbb{T}} Y,$$

where L_E is the left multiplication by E. We define $\mathbb{T} := T - 2\lambda L_E$. Using $E \cdot E = E$ and $TE = \lambda E$ we get

$$\mathbb{T}E = TE - 2\lambda E \cdot E = \lambda E - 2\lambda E = -\lambda E.$$

Thus E is an eigenvector of $\mathbb{T}$ and $-\lambda$ is the corresponding eigenvalue. We have shown the following proposition.

Proposition 2.1. *Let E be the eigenvector of T corresponding to eigenvalue λ and let E be an idempotent of algebra $\mathcal{A}$ corresponding to the equation (1.2). If $\lambda \neq 0$, the system (1.2) has at least two critical points $X = 0$ and $X = -\lambda E$ and they cannot both be stable at the same time.*

It is well known that the system $\dot{X} = X^2$ may have infinitely many unstable critical points. In $\mathbb{R}^2$ every commutative algebra which contains a nilpotent of index 2 (except for the trivial case of the nil algebra) has infinitely many unstable critical points. In the next proposition we will see that the system (1.2) may also have infinitely many unstable critical points.

Proposition 2.2. *Let $N \neq 0$ be a nilpotent of order 2 of algebra $\mathcal{A}$, corresponding to equation (1.2). Let E and N be eigenvectors of matrix T corresponding to eigenvalues λ and 0, respectively. Assume that $NE = \mu E$, where μ is non-zero. Then there exist infinitely many unstable critical points of system (1.2). (If $\lambda > 0$, the origin is one of them.)*

Proof. Obviously, the line $\mathbb{R}N$ is the line of critical points for

$$T(yN) + (yN)^2 = yTN + y^2 N^2 = 0 + 0 = 0 \quad \text{for all } y \in \mathbb{R}.$$

The linearized equation around $X = yN$ is

$$\dot{Y} = \underbrace{(T + 2yL_N)}_{\Upsilon} Y,$$

where L_N is the left multiplication by N. We define $\Upsilon := T + 2yL_N$. Using $E \cdot E = E$, $TE = \lambda E$, $TN = 0$ and $N \cdot E = \mu E$, $\mu \neq 0$, we get

$$\Upsilon E = TE + 2yN \cdot E = \lambda E + 2y\mu E = (\lambda + 2y\mu)E.$$

Thus for $y > -\lambda/2\mu$ the matrix Υ has at least one positive eigenvalue, and this finishes the proof. $\square$

Corollary 2.3. *If $\lambda > 0$ and $N \cdot E = 0$, then $X = yN$ is unstable for every real y.*

For the very simple case of an algebra (and the corresponding system) discussed in the above proposition see the following example.

Example 2.4. Let us consider the system

$$\begin{aligned}\dot{x} &= 0,\\ \dot{y} &= \lambda y + 2\mu xy + y^2,\end{aligned}$$

of the form $\dot{X} = TX + X^2$. Its linear part is defined by the matrix

$$T = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$$

with eigenvectors $N = (1, 0)$ and $E = (0, 1)$ corresponding to eigenvalues 0 and λ, respectively. Its nonlinear part $X^2 = X \circ X$ is defined by the following multiplication table

$\circ$	N	E
N	0	μE
E	μE	E

$\mu \neq 0.$

By Proposition 2.1 for every $y > -\lambda/2\mu$ the critical point $X = yN$ is unstable.

In the next theorem let us consider the main result; the stability of the origin in system (1.2) with a two-dimensional centre. *For the unstable, stable and centre manifold of the linearized equation (as well as for the nonlinearized equation), we will use the commonly used notation E^{u}, E^{s} and E^{c} (and W^{u}, W^{s} and W^{c}), respectively.*

Theorem 2.5. *Let $E_1 \neq E_2 \neq 0$ satisfy condition (1.3) and suppose $\mathrm{span}(E_1, E_2)$ is a subalgebra of $\mathcal{A}$. Assume $E^{\mathrm{u}} = \emptyset$ and $E^{\mathrm{c}} = \mathrm{span}(E_1, E_2)$. Then the origin is a stable critical point of the ODE (1.2).*

Proof. The unstable manifold is empty by the centre manifold theorem. Equation (1.2) can be written (in coordinates $\boldsymbol{x}$ in the direction of E^{c} and $\boldsymbol{y}$ in the direction of E^{s}) as

$$\begin{aligned}\boldsymbol{x}' &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \boldsymbol{x} + f_1(\boldsymbol{x}, \boldsymbol{y}),\\ \boldsymbol{y}' &= -B\boldsymbol{y} + f_1(\boldsymbol{x}, \boldsymbol{y}),\end{aligned}$$

where the eigenvalues of B all have strictly positive real parts and the functions f_i, $i = 1, 2$, represent nonlinear terms. Since $\mathrm{span}(E_1, E_2)$ is a subalgebra of $\mathcal{A}$, we have

$$\begin{aligned} \boldsymbol{x}' &= T_{/E^{\mathrm{c}}}\boldsymbol{x} + Q_{/E^{\mathrm{c}}}(\boldsymbol{x}), \\ \boldsymbol{y}' &= T_{/E^{\mathrm{s}}}\boldsymbol{y} + Q_{/E^{\mathrm{s}}}(\boldsymbol{x}, \boldsymbol{y}). \end{aligned}$$

By the centre manifold theorem W^{c} always exists and, since $\mathrm{span}(E_1, E_2)$ is a subalgebra of $\mathcal{A}$ (i.e. $f_1(\boldsymbol{x}, \boldsymbol{y}) = f_1(\boldsymbol{x})$) for any choice of W^{c} (i.e. for any centre manifold with equation $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$, where $\boldsymbol{h}$ must be at least a quadratic homogeneous function in every component), equation (1.1), which defines the stability on the centre manifold, is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a_1x^2 + 2b_1xy + c_1y^2 \\ a_2x^2 + 2b_2xy + c_2y^2 \end{bmatrix}. \tag{2.2}$$

To analyse the stability of such a centre (i.e. the stability of a perturbed Hamiltonian system) let us refer to Arnold's book [**1**, pp. 149–151], where the stability of the system

$$\left.\begin{aligned} \dot{x} &= -\omega y + \varepsilon f_1(x, y), \\ \dot{y} &= \omega x + \varepsilon f_2(x, y), \end{aligned}\right\} \quad \varepsilon \ll 1, \quad x^2 + y^2 \leqslant R^2, \tag{2.3}$$

is studied via the increment of energy (the Hamiltonian) $E = -\frac{1}{2}\omega(x^2 + y^2)$ over one revolution about the origin. We can study the increment of energy over the trajectories of the linear system (i.e. over circles $C_R = \{(x, y);\ x = R\cos\phi,\ y = R\sin\phi\}$). Thus

$$\Delta E = \varepsilon \oint [f_2(x, y)\,\mathrm{d}y - f_1(x, y)\,\mathrm{d}x] + O(\varepsilon^2),$$

where the integral $I = \oint [f_2(x, y)\,\mathrm{d}y - f_1(x, y)\,\mathrm{d}x]$ is taken counterclockwise over C_R. By Green's Theorem we have

$$I = -\iint_{\mathrm{int}(C_R)} \left(\frac{\partial f_1(x, y)}{\partial x} + \frac{\partial f_2(x, y)}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y. \tag{2.4}$$

If the increment of energy (i.e. the integral (2.4)) is positive (negative), the trajectories near the origin of the perturbed equation (2.3) are expanding (contracting) spirals, i.e. the origin is unstable (stable). If the integral (2.4) equals zero, the trajectories near the origin of the perturbed equation (2.3) are cycles and the origin is stable.

Introducing the coordinates $\varepsilon X = x$, $\varepsilon Y = y$, $\varepsilon \ll 1$, into (2.2) yields a (2.3)-like system

$$\left.\begin{aligned} \dot{X} &= -\omega Y + \varepsilon(a_1X^2 + 2b_1XY + c_1Y^2), \\ \dot{Y} &= \omega X + \varepsilon(a_2X^2 + 2b_2XY + c_2Y^2), \end{aligned}\right\} \tag{2.5}$$

which can be analysed by the sign of (2.4). The straightforward computation

$$\begin{aligned} I &= -\iint_{\mathrm{int}(C_R)} \left(\frac{\partial f_1}{\partial X} + \frac{\partial f_2}{\partial Y}\right) \mathrm{d}X\,\mathrm{d}Y \\ &= -\iint_{\mathrm{int}(C_R)} (2a_1X + 2b_1Y + 2b_2X + 2c_2Y)\,\mathrm{d}X\,\mathrm{d}Y \\ &= -2\int_0^{2\pi} ((a_1+b_2)\cos\phi + (b_1+c_2)\sin\phi)\,\mathrm{d}\phi \cdot \int_0^R r^2\,\mathrm{d}r \\ &= 0 \end{aligned}$$

shows that system (2.5) has cycle orbits near the origin for every $R > 0$, and the equations $\varepsilon X = x$, $\varepsilon Y = y$ imply cycle orbits near the origin for every $0 < R < \varepsilon$, $\varepsilon \ll 1$. Thus the origin of (2.2) is stable. □

The similar change of coordinates

$$\sqrt[m-1]{\varepsilon}X = x, \qquad \sqrt[m-1]{\varepsilon}Y = y$$

into the equation

$$\boldsymbol{x}' = T\boldsymbol{x} + Q(\boldsymbol{x}), \tag{2.6}$$

where Q is homogeneous of degree $m > 2$, yields a (2.3)-like system. A straightforward computation shows that the integral (2.4) equals zero for every even m. Thus we can state the following theorem.

Theorem 2.6. *Let E_1, E_2 satisfy the condition (1.3) and let* $\mathrm{span}(E_1, E_2)$ *be a subalgebra of a m-ary algebra $\mathcal{A}$ (where m is even). Suppose $E^{\mathrm{u}} = \emptyset$ and $E^{\mathrm{c}} = \mathrm{span}(E_1, E_2)$. Then the origin is a stable critical point for (2.6).*

For $m = 3$, however, we have the following theorem.

Theorem 2.7. *Let E_1, E_2 satisfy the condition (1.3) and let* $\mathrm{span}(E_1, E_2)$ *be a subalgebra of a 3-ary algebra $\mathcal{A}$. Suppose $E^{\mathrm{u}} = \emptyset$ and $E^{\mathrm{c}} = \mathrm{span}(E_1, E_2)$. Then the origin is stable for (2.6) if*

$$3(a_1 + a_2) + 2(c_1 + c_2) \geqslant 0,$$

where $a_1, a_2, c_1, c_2 \in \mathbb{R}$ are coefficients from (2.7).

Proof. Let us consider the system

$$\left.\begin{aligned} \dot{X} &= -\omega Y + \varepsilon(a_1X^3 + 2b_1X^2Y + 2c_1XY^2 + d_1Y^3), \\ \dot{Y} &= \omega X + \varepsilon(d_2X^3 + 2c_2X^2Y + 2b_2XY^2 + a_2Y^3), \end{aligned}\right\} \tag{2.7}$$

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in $\operatorname{span}(E_1, E_2)$. From (2.4) we see that

$$\begin{aligned}
I &= -\iint_{\operatorname{int}(C_R)} \left(\frac{\partial f_1}{\partial X} + \frac{\partial f_2}{\partial Y}\right) \mathrm{d}X\,\mathrm{d}Y \\
&= -\iint_{\operatorname{int}(C_R)} (3a_1X^2 + 4b_1XY + 2c_1Y^2 + 3a_2Y^2 + 4b_2XY + 2c_2X^2)\,\mathrm{d}X\,\mathrm{d}Y \\
&= -\int_0^R r^3\,\mathrm{d}r \int_0^{2\pi} ((3a_1 + 2c_2)\cos^2\phi + (3a_2 + 2c_1)\sin^2\phi + 2(b_1 + b_2)\sin 2\phi)\,\mathrm{d}\phi \\
&= -\tfrac{1}{4}R^4(\pi(3a_1 + 2c_2) + \pi(3a_2 + 2c_1)) \\
&= -\tfrac{1}{4}R^4\pi(3a_1 + 3a_2 + 2c_1 + 2c_2).
\end{aligned}$$

This finishes the proof. □

In the remainder of this section we want to improve Theorem 2.5. Let us now consider the system

$$\left.\begin{aligned}
\dot{x} &= -\omega y + q_1(x, y, z), \\
\dot{y} &= \omega x + q_2(x, y, z), \\
\dot{z} &= -Az + Q(x, y, z),
\end{aligned}\right\} \tag{2.8}$$

where $x, y \in \mathbb{R}$, $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, A has strictly positive (different) eigenvalues, and $Q(x, y, z) = (Q_1, \dots, Q_n)$, $q_1(x, y, z)$, $q_2(x, y, z)$ are quadratic. Let us assume that the upper system is written in normal form (i.e. A is diagonal):

$$\left.\begin{aligned}
\dot{x} &= -\omega y + a_1x^2 + 2b_1xy + c_1y^2 + \mu_{1i}\sum_i xz_i + \nu_{1j}\sum_j yz_j + \xi_{1kl}\sum_{k,l} z_kz_l, \\
\dot{y} &= \omega x + a_2x^2 + 2b_2xy + c_2y^2 + \mu_{2i}\sum_i xz_i + \nu_{2j}\sum_j yz_j + \xi_{2kl}\sum_{k,l} z_kz_l, \\
\dot{z}_1 &= -\lambda_1z_1 + \alpha_1x^2 + 2\beta_1xy + \gamma_1y^2 + M_{1i}\sum_i xz_i + N_{1j}\sum_j yz_j + \Xi_{1,kl}\sum_{k,l} z_kz_l, \\
\dot{z}_2 &= -\lambda_2z_2 + \alpha_2x^2 + 2\beta_2xy + \gamma_2y^2 + M_{2i}\sum_i xz_i + N_{2j}\sum_j yz_j + \Xi_{2,kl}\sum_{k,l} z_kz_l, \\
&\vdots \\
\dot{z}_n &= -\lambda_nz_n + \alpha_nx^2 + 2\beta_nxy + \gamma_ny^2 + M_{ni}\sum_i xz_i + N_{nj}\sum_j yz_j + \Xi_{n,kl}\sum_{k,l} z_kz_l.
\end{aligned}\right\} \tag{2.9}$$

By the centre manifold theorem (see [**3**, p. 204]) we seek the centre manifold

$$z(x, y) = h(x, y) = (h_1(x, y), h_2(x, y), \dots, h_n(x, y))$$

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with equation

$$\begin{bmatrix} \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_1}{\partial y} \\ \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_2}{\partial y} \\ \vdots & \vdots \\ \dfrac{\partial h_n}{\partial x} & \dfrac{\partial h_n}{\partial y} \end{bmatrix} \left(\begin{bmatrix} -\omega y \\ \omega x \end{bmatrix} + \begin{bmatrix} q_1(x,y,h(x,y)) \\ q_2(x,y,h(x,y)) \end{bmatrix} \right) = \begin{bmatrix} -\lambda_1 z_1 \\ -\lambda_2 z_2 \\ \vdots \\ -\lambda_n z_n \end{bmatrix} + \begin{bmatrix} Q_1(x,y,h(x,y)) \\ Q_2(x,y,h(x,y)) \\ \vdots \\ Q_n(x,y,h(x,y)) \end{bmatrix}. \tag{2.10}$$

The functions $h_i(x,y)$ must be at least quadratic, for $h(0,0)=0$ and $Dh(0,0)=0$ must hold on the centre manifold. Thus, for $i=1,2,\dots,n$,

$$h_i(x,y) = A_i x^2 + 2B_i xy + C_i y^2 + \text{higher terms}.$$

From (2.10) we get the following system (restricted to quadratic terms x^2, xy and y^2) for $i=1,2,\dots,n$:

$$\left.\begin{aligned} xy:&\quad -2A_i\omega + 2B_i\lambda_i + 2C_i\omega = 2\beta_i, \\ x^2:&\quad A_i\lambda_i + 2B_i\omega = \alpha_i, \\ y^2:&\quad -2\omega B_i + C_i\lambda_i = \gamma_i, \end{aligned}\right\} \tag{2.11}$$

whose solutions are

$$\left.\begin{aligned} A_i &= \frac{2\omega^2\gamma_i + 2\omega^2\alpha_i - 2\omega\beta_i\lambda_i + \alpha_i\lambda_i^2}{(4\omega^2+\lambda_i^2)\lambda_i}, \\ B_i &= \frac{\omega\alpha_i + \beta_i\lambda_i - \omega\gamma_i}{4\omega^2+\lambda_i^2} \\ C_i &= \frac{2\omega^2\gamma_i + 2\omega^2\alpha_i + 2\omega\beta_i\lambda_i + \gamma_i\lambda_i^2}{(4\omega^2+\lambda_i^2)\lambda_i}. \end{aligned}\right\} \tag{2.12}$$

Due to the fact that $\lambda_i(4\omega^2+\lambda_i^2) \neq 0$, the coefficients A_i, B_i and C_i are unique.

Putting $z_i = h_i(x,y) = A_i x^2 + 2B_i xy + C_i y^2$ into

$$\left.\begin{aligned} \dot{x} &= -\omega y + a_1x^2 + 2b_1xy + c_1y^2 + \mu_{1i}\sum_i xz_i + \nu_{1j}\sum_j yz_j + \xi_{1kl}\sum_{k,l} z_kz_l, \\ \dot{y} &= \omega x + a_2x^2 + 2b_2xy + c_2y^2 + \mu_{2i}\sum_i xz_i + \nu_{2j}\sum_j yz_j + \xi_{2kl}\sum_{k,l} z_kz_l, \end{aligned}\right\} \tag{2.13}$$

yields the centre manifold system

$$\left.\begin{aligned} \dot{x} &= -\omega y + a_1x^2 + 2b_1xy + c_1y^2 + \mu_{1i}\sum_i xh_i + \nu_{1j}\sum_j yh_j + \xi_{1kl}\sum_{k,l} h_kh_l, \\ \dot{y} &= \omega x + a_2x^2 + 2b_2xy + c_2y^2 + \mu_{2i}\sum_i xh_i + \nu_{2j}\sum_j yh_j + \xi_{2kl}\sum_{k,l} h_kh_l. \end{aligned}\right\} \tag{2.14}$$

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If at least one of the coefficients a_1, b_1, c_1, a_2, b_2, c_2 does not vanish, then by change of coordinates $\varepsilon X = x$, $\varepsilon Y = y$, the above system yields (2.5). On the other hand, if $a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0$, then by change of coordinates $\sqrt{\varepsilon}X = x$, $\sqrt{\varepsilon}Y = y$, the above system yields a (2.7)-like system

$$\begin{aligned}\dot{x} &= -\omega y + \varepsilon\biggl[\sum_i \mu_{1i}x(A_i x^2 + 2B_i xy + C_i y^2) + \nu_{1j}\sum_j \nu_{1j}y(A_j x^2 + 2B_j xy + C_j y^2)\biggr],\\ \dot{y} &= \omega x + \varepsilon\biggl[\sum_i \mu_{2i}x(A_i x^2 + 2B_i xy + C_i y^2) + \sum_j \nu_{2j}y(A_j x^2 + 2B_j xy + C_j y^2)\biggr],\end{aligned}$$

which can be treated by Theorem 2.7 using the following coefficients

$$\begin{aligned}\tilde{a}_1 &= \sum_{i=1}^n \mu_{1i}A_i, &\quad \tilde{c}_1 &= \sum_{i=1}^n \mu_{1i}C_i + 2\sum_{i=1}^n \nu_{1i}B_i,\\ \tilde{a}_2 &= \sum_{i=1}^n \nu_{2i}C_i, &\quad \tilde{c}_2 &= 2\sum_{i=1}^n \mu_{2i}B_i + \sum_{i=1}^n \nu_{2i}A_i.\end{aligned}$$

Thus in the case of $a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0$ the system (2.9) is stable if

$$\begin{aligned}3(\tilde{a}_1 + \tilde{a}_2) + 2(\tilde{c}_1 + \tilde{c}_2) &\geqslant 0,\\ 3\sum_{i=1}^n(\mu_{1i}A_i + \nu_{2i}C_i) + 2\sum_{i=1}^n(\mu_{1i}C_i + 2\nu_{1i}B_i + 2\mu_{2i}B_i + \nu_{2i}A_i) &\geqslant 0,\\ \sum_{i=1}^n(3\mu_{1i}A_i + 3\nu_{2i}C_i + 2\mu_{1i}C_i + 4\nu_{1i}B_i + 4\mu_{2i}B_i + 2\nu_{2i}A_i) &\geqslant 0,\\ \sum_{i=1}^n[(3\mu_{1i} + 2\nu_{2i})A_i + 4(\nu_{1i} + \mu_{2i})B_i + (3\nu_{2i} + \mu_{1i})C_i] &\geqslant 0.\end{aligned}$$

This yields the following theorems for systems of the form (2.9).

Theorem 2.8. *The origin is the stable critical point of the equation (2.9) for all algebras $\mathcal{A}$ in which at least one of the products E_1E_1, E_1E_2 or E_2E_2 contain some non-zero vector ξE_i, $i \in \{1, 2\}$.*

Theorem 2.9. *The stability of system (2.9) in the case when neither E_1E_1, E_1E_2 nor E_2E_2 contain any vectors ξE_i, $i \in \{1, 2\}$, is completely determined by the following inequality:*

$$\sum_{i=1}^n[(3\mu_{1i} + 2\nu_{2i})A_i + 4(\nu_{1i} + \mu_{2i})B_i + (3\nu_{2i} + \mu_{1i})C_i] \geqslant 0, \tag{2.15}$$

i.e. for all algebras $\mathcal{A}$ for which condition (2.15) holds, the origin is a stable critical point.

Let us now consider the case when the matrix A in (2.8) has a double (real) eigenvalue. Let us assume that A contains a block

$$\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$

for some z_i, z_{i+1}. Without loss of generality we can take $i = 1$ (i.e. $\lambda_1 = \lambda_2 = \lambda$). The centre manifold equation is the following:

$$\begin{bmatrix} \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_1}{\partial y} \\ \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_2}{\partial y} \\ \vdots & \vdots \\ \dfrac{\partial h_n}{\partial x} & \dfrac{\partial h_n}{\partial y} \end{bmatrix} \left(\begin{bmatrix} -\omega y \\ \omega x \end{bmatrix} + \begin{bmatrix} q_1(x, y, h(x, y)) \\ q_2(x, y, h(x, y)) \end{bmatrix} \right) = \begin{bmatrix} -\lambda z_1 + z_2 \\ -\lambda z_2 \\ \vdots \\ -\lambda_n z_n \end{bmatrix} + \begin{bmatrix} Q_1(x, y, h(x, y)) \\ Q_2(x, y, h(x, y)) \\ \vdots \\ Q_n(x, y, h(x, y)) \end{bmatrix}.$$

Hence for $i = 1$ the coefficients A_1, B_1 and C_1 are given by

$$\begin{aligned} xy: &\quad -2A_1\omega + 2B_1\lambda + 2C_1\omega = 2\beta_1 + 2B_2, \\ x^2: &\quad A_1\lambda + 2B_1\omega = \alpha_1 + A_2, \\ y^2: &\quad -2\omega B_1 + C_1\lambda = \gamma_1 + C_2, \end{aligned}$$

and for $i = 2$ we have

$$\begin{aligned} xy: &\quad -2A_2\omega + 2B_2\lambda + 2C_2\omega = 2\beta_2, \\ x^2: &\quad A_2\lambda + 2B_2\omega = \alpha_2, \\ y^2: &\quad -2\omega B_2 + C_2\lambda = \gamma_2. \end{aligned}$$

The solutions for A_1, B_1, C_1, A_2, B_2 and C_2 still remain unique:

$$\left.\begin{aligned} A_2 &= \frac{2\omega^2\gamma_2 + 2\omega^2\alpha_2 - 2\omega\beta_2\lambda + \alpha_2\lambda^2}{(4\omega^2 + \lambda^2)\lambda}, \\ B_2 &= \frac{\omega\alpha_2 + \beta_2\lambda - \omega\gamma_2}{4\omega^2 + \lambda^2}, \\ C_2 &= \frac{2\omega^2\gamma_2 + 2\omega^2\alpha_2 + 2\omega\beta_2\lambda + \gamma_2\lambda^2}{(4\omega^2 + \lambda^2)\lambda}, \\ A_1 &= \frac{2\gamma_1\omega^2 + 2\omega^2\alpha_1 - 2\beta_1\lambda\omega + \alpha_1\lambda^2}{(4\omega^2 + \lambda^2)\lambda} + \frac{A_2(\lambda^2 + 2\omega^2) - 2B_2\lambda\omega + 2C_2\omega^2}{(4\omega^2 + \lambda^2)\lambda}, \\ B_1 &= \frac{-\gamma_1\omega + \omega\alpha_1 + \beta_1\lambda}{4\omega^2 + \lambda^2} + \frac{A_2\omega + B_2\lambda - C_2\omega}{4\omega^2 + \lambda^2}, \\ C_2 &= \frac{2\gamma_1\omega^2 + 2\omega^2\alpha_1 + 2\beta_1\lambda\omega + \lambda^2\gamma_1}{(4\omega^2 + \lambda^2)\lambda} + \frac{2A_2\omega^2 + 2B_2\lambda\omega + C_2(\lambda^2 + 2\omega^2)}{(4\omega^2 + \lambda^2)\lambda}. \end{aligned}\right\} \quad (2.16)$$

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On the other hand, let us assume that A contains a block

$$\begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}$$

for some z_i, z_{i+1}. Without loss of generality we can take $i = 1$ (i.e. $\lambda_1 = \lambda_2 = \lambda$ with two different corresponding eigenvectors). The centre manifold equation is as follows:

$$\begin{bmatrix} \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_1}{\partial y} \\ \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_2}{\partial y} \\ \vdots & \vdots \\ \dfrac{\partial h_n}{\partial x} & \dfrac{\partial h_n}{\partial y} \end{bmatrix} \left(\begin{bmatrix} -\omega y \\ \omega x \end{bmatrix} + \begin{bmatrix} q_1(x, y, h(x, y)) \\ q_2(x, y, h(x, y)) \end{bmatrix} \right) = \begin{bmatrix} -\lambda z_1 \\ -\lambda z_2 \\ \vdots \\ -\lambda_n z_n \end{bmatrix} + \begin{bmatrix} Q_1(x, y, h(x, y)) \\ Q_2(x, y, h(x, y)) \\ \vdots \\ Q_n(x, y, h(x, y)) \end{bmatrix}.$$

Hence for $i = 1, 2$ we have

$$\left.\begin{aligned} A_i &= \frac{2\omega^2\gamma_i + 2\omega^2\alpha_i - 2\omega\beta_i\lambda + \alpha_i\lambda^2}{(4\omega^2 + \lambda^2)\lambda}, \\ B_i &= \frac{\omega\alpha_i + \beta_i\lambda - \omega\gamma_i}{4\omega^2 + \lambda^2}, \\ C_i &= \frac{2\omega^2\gamma_i + 2\omega^2\alpha_i + 2\omega\beta_i\lambda + \gamma_i\lambda^2}{(4\omega^2 + \lambda^2)\lambda}. \end{aligned}\right\} \tag{2.17}$$

In a similar way we can show that the coefficients A_i, B_i, C_i remain unique if A has triple eigenvalues. For the block

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix},$$

the coefficients A_i, B_i and C_i for $i = 1, 2, 3$ are determined by the following system:

$$\begin{aligned} xy: &\quad -2A_1\omega + 2C_1\omega = -2B_1\lambda + 2B_2 + 2\beta_1 + 2\beta_2, \\ x^2: &\quad 2B_1\omega = -A_1\lambda + A_2 + \alpha_1 + \alpha_2, \\ y^2: &\quad -2\omega B_1 = -C_1\lambda + C_2 + \gamma_1 + \gamma_2. \end{aligned}$$

$$\begin{aligned} xy: &\quad -2A_2\omega + 2C_2\omega = -2B_2\lambda + 2B_3 + 2\beta_2 + 2\beta_3, \\ x^2: &\quad 2B_2\omega = -A_2\lambda + A_3 + \alpha_2 + \alpha_3, \\ y^2: &\quad -2\omega B_2 = -C_2\lambda + C_3 + \gamma_2 + \gamma_3, \end{aligned}$$

$$\begin{aligned} xy: &\quad -2A_3\omega + 2C_3\omega = -2B_3\lambda + 2\beta_3, \\ x^2: &\quad 2B_3\omega = -A_3\lambda + \alpha_3, \\ y^2: &\quad -2\omega B_3 = -C_3\lambda + \gamma_3, \end{aligned}$$

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whose solution is

$$\left.\begin{aligned}
A_3 &= \frac{2\omega^2\gamma_3 + 2\omega^2\alpha_3 - 2\omega\beta_3\lambda + \alpha_3\lambda^2}{(4\omega^2+\lambda^2)\lambda},\\
B_3 &= \frac{-\omega\gamma_3 + \omega\alpha_3 + \beta_3\lambda}{4\omega^2+\lambda^2},\\
C_3 &= \frac{2\omega^2\gamma_3 + 2\omega^2\alpha_3 + 2\omega\beta_3\lambda + \gamma_3\lambda^2}{(4\omega^2+\lambda^2)\lambda},\\
A_2 &= \frac{2\omega^2\gamma_3 + 2\omega^2\alpha_3 + 2\omega^2\gamma_2 + 2\alpha_2\omega^2 - 2\beta_2\lambda\omega - 2\omega\beta_3\lambda + \alpha_3\lambda^2 + \lambda^2\alpha_2}{(4\omega^2+\lambda^2)\lambda}\\
&\quad + \frac{2\omega^2 C_3 + 2A_3\omega^2 - 2\omega B_3\lambda + \lambda^2 A_3}{(4\omega^2+\lambda^2)\lambda},\\
B_2 &= \frac{\omega\alpha_2 + A_3\omega - \omega\gamma_3 + \omega\alpha_3 - C_3\omega - \omega\gamma_2 + \beta_3\lambda + \beta_2\lambda + B_3\lambda}{4\omega^2+\lambda^2},\\
C_2 &= \frac{2\omega^2\gamma_3 + 2\omega^2\alpha_3 + 2\omega^2\gamma_2 + 2\alpha_2\omega^2 + 2\beta_2\lambda\omega + 2\omega\beta_3\lambda + \lambda^2\gamma_2 + \gamma_3\lambda^2}{(4\omega^2+\lambda^2)\lambda}\\
&\quad + \frac{2\omega^2 C_3 + 2A_3\omega^2 + 2\omega B_3\lambda + \lambda^2 C_3}{(4\omega^2+\lambda^2)\lambda},\\
A_1 &= \frac{2\lambda^2\alpha_2\omega^2 - 8\lambda^2\omega^3\beta_2 + 2\lambda^3\omega^2\gamma_2 - 2\lambda^4\omega\beta_2 + 6\lambda^3\omega^2\alpha_2 + 8\lambda\omega^4\gamma_1}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)}\\
&\quad + \frac{8\omega^4\lambda\alpha_2 + 8\lambda\gamma_2\omega^4 + 8\gamma_3\omega^4 + 8\omega^4\alpha_3 + 2\lambda^3\omega^2\gamma_1}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)}\\
&\quad + \frac{-2\lambda^4\omega\beta_1 + 6\lambda^3\omega^2\alpha_1 + 6\lambda^2\omega^2\gamma_3 + 2\lambda^2\omega^2\alpha_3 - 8\lambda^2\omega^3\beta_1}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)}\\
&\quad + \frac{-4\lambda^3\omega\beta_3 + 8\lambda\omega^4\alpha_1 + \lambda^4\alpha_2 + \lambda^5\alpha_2 + \lambda^4\alpha_3}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)}\\
&\quad + \frac{\lambda^5\alpha_1 + 6\omega^2\lambda^2\gamma_2 - 4\lambda^3\omega\beta_2 + 8\omega^4\gamma_2 + 8\omega^4\alpha_2}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)}\\
&\quad + \frac{8C_3\omega^4 + 8\omega^4 A_3 + 2\lambda^2\omega^2 A_3 - 4\lambda^3\omega B_3 + 6\lambda^2\omega^2 C_3 + \lambda^4 A_3}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)},\\
B_1 &= -\frac{-\lambda^3\beta_2 + \omega\lambda^2\gamma_1 + \omega\lambda^2\gamma_2 - \lambda^3\beta_1 - \lambda^2\alpha_2\omega - \lambda^2\alpha_1\omega - 2A_3\omega\lambda}{\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4}\\
&\quad - \frac{2\omega\gamma_3\lambda - 2\omega\alpha_3\lambda - 4\omega^2\beta_1\lambda - 4\omega^2\beta_2\lambda - \lambda^2 B_3}{\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4}\\
&\quad - \frac{-\lambda^2\beta_3 - 2\lambda\alpha_2\omega - \lambda^2\beta_2 + 2\omega\lambda\gamma_2 + 4\omega^2\beta_3 + 4\omega^2 B_3}{\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4}\\
&\quad - \frac{2\omega C_3\lambda + 4\omega^2\beta_2 - 4\omega^3\alpha_1 + 4\omega^3\gamma_2 - 4\omega^3\alpha_2 + 4\omega^3\gamma_1}{\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4},
\end{aligned}\right\} \tag{2.18}$$

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$$
\left.
\begin{aligned}
C_1 = {} & \frac{6\lambda^2\alpha_2\omega^2 + 8\lambda^2\omega^3\beta_2 + 6\lambda^3\omega^2\gamma_2 + 2\lambda^4\omega\beta_2 + 2\lambda^3\omega^2\alpha_2}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)} \\
& + \frac{8\lambda\omega^4\gamma_1 + 8\omega^4\lambda\alpha_2 + 8\lambda\gamma_2\omega^4 + 8C_3\omega^4}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)} \\
& + \frac{8\gamma_3\omega^4 + 8\omega^4 A_3 + 8\omega^4\alpha_3 + 6\lambda^3\omega^2\gamma_1 + 2\lambda^4\omega\beta_1 + 2\lambda^3\omega^2\alpha_1}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)} \\
& + \frac{6\lambda^2\omega^2 A_3 + 2\lambda^2\omega^2\gamma_3 + 6\lambda^2\omega^2\alpha_3 + 8\lambda^2\omega^3\beta_1}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)} \\
& + \frac{4\lambda^3\omega B_3 + 4\lambda^3\omega\beta_3 + 2\lambda^2\omega^2 C_3 + 8\lambda\omega^4\alpha_1^3}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)} \\
& + \frac{2\omega^2\lambda^2\gamma_2 + 4\lambda^3\omega\beta_2 + \lambda^4\gamma_3 + \lambda^5\gamma_1 + \lambda^4 C_3}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)^4} \\
& + \frac{\lambda^4\gamma_2 + \lambda^5\gamma_2 + 8\omega^4\gamma_2 + 8\omega^4\alpha_2}{\lambda^2(\lambda^4 + 8\lambda^2\omega^2 + 16\omega^4)}.
\end{aligned}
\right\} \quad (2.18\ \textit{cont.})
$$

For A having multiple (we have just shown for double and triple) eigenvalues, Theorems 2.8 and 2.9 still remain valid for equations of type (2.8).

Theorem 2.10. *The origin is a stable critical point of the equation (2.8) for all algebras $\mathcal{A}$ in which at least one of the products E_1E_1, E_1E_2 or E_2E_2 contain some non-zero vector ξE_i, $i \in \{1, 2\}$.*

Theorem 2.11. *The stability of system (2.8) in the case when neither E_1E_1, E_1E_2 nor E_2E_2 contain any vectors ξE_i, $i \in \{1, 2\}$, is completely determined by inequality (2.15). The coefficients A_i, B_i and C_i are determined by (2.16), (2.17) or (2.18) for double and triple eigenvalue λ. For other multiple eigenvalues, similar formulae for the coefficients A_i, B_i and C_i could be derived.*

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