DISCRETE GROUPS OF MOTIONS

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1. Introduction. This paper deals with the discrete groups of rigid motions of the hyperbolic plane. It is known **(12)** that the finitely generated, orientation-preserving groups have the following presentations:

Generators: $a_1, b_1, \ldots, a_p, b_p, S_1, \ldots, S_d, c_1, \ldots, c_r.$

Defining relations: $k_1 \ldots k_p S_1 \ldots S_d c_1 \ldots c_r = 1,$

$$S_1^{n1} = S_2^{n2} = \ldots = S_d^{nd} = 1,$$

where $k_m = a_m b_m a_m^{-1} b_m^{-1}$. We shall denote this group by $F(p; n_1, \ldots, n_d; r)$.

In particular, the finitely generated free groups are contained among these. Indeed, one purpose of this paper is to indicate some geometrical methods for investigating free groups.

The above groups also include the orientation-preserving discrete groups of motions of the sphere and Euclidean plane **(3)**. But the results we shall obtain are mainly concerned with the hyperbolic groups and are either easy or false for the Euclidean and spherical groups. For instance, we shall extend the following theorem of Howson **(7)** to discrete groups of motions: if S and T are finitely generated subgroups of a free group, then $S \cap T$ is also finitely generated. This theorem is trivial for the Euclidean and spherical groups, which contain no infinitely generated subgroups. We shall generalize the theorem of Karrass and Solitar **(8)** that if F is a free group and H is a finitely generated subgroup which contains a normal subgroup of F, then H is of finite index. This theorem is trivial for the spherical groups and is false for most of the Euclidean groups. For the above reasons we shall consider only the case of discrete hyperbolic groups. We shall usually omit "discrete hyperbolic." We mention the interesting result of Nielsen **(11)**, Bundgaard **(1)**, and Fox **(5)** that the above groups all contain subgroups of finite index with no elements of finite order.

2. Hyperbolic groups. Let D be the disk $\{z | |z| < 1\}$ in the complex plane, $\bar{D}$ its closure and E its boundary. D can be given a Riemannian metric so that it becomes the Poincaré model of the hyperbolic plane. The geodesics, which we shall call h-lines, are arcs of circles orthogonal to E. The isometries are the linear fractional transformations which preserve D. They are of the forms

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$$S(z) = \frac{az + \bar{b}}{bz + \bar{a}} \quad \text{and} \quad T(z) = \frac{c\bar{z} + \bar{d}}{d\bar{z} + \bar{c}},$$

where $a\bar{a} - b\bar{b} = c\bar{c} - d\bar{d} = 1$.

The transformation S is called a translation if it has two fixed points which are on E. This is equivalent to the condition $|a + \bar{a}| > 2$. A translation maps each circle through the fixed points onto itself. In particular, the h-line through the fixed points is invariant and is called the axis of the transformation. S is called a rotation if it has a fixed point in D, and a limit-rotation if it has a single fixed point on E. These conditions are respectively equivalent to $|a + \bar{a}| < 2$ and $|a + \bar{a}| = 2$. T is a reflection in an h-line if $d + \bar{d} = 0$. Otherwise T is a glide-reflection, that is, the product of a translation along an axis λ with a reflection in λ.

For each transformation S or T, there are a pair of h-lines λ and λ', called the isometric circles of the transformation (see **(4)**). S is the product of a reflection in λ and a reflection in the perpendicular bisector of the Euclidean line through the centres of the circles λ and λ'. If T is a glide-reflection, this product must be combined with a reflection in the h-line through the fixed points of T; if T is a reflection, $\lambda = \lambda'$ is the h-line of reflection. S is a translation, rotation or limit-rotation, according as λ and λ' do not intersect, do intersect, or are tangent. A discrete, hyperbolic group G has a canonical fundamental region, denoted R_G, which consists of the region in D outside of the isometric circles of all elements of G.

A subset M of $\bar{D}$ is called *h-convex* if with every two of its points it contains the h-line segment between them. For any subset M of $\bar{D}$, we denote by $[M]$, the h-convex closure of M, that is, the intersection of all h-convex subsets of $\bar{D}$ which contain M.

The set of limit points L_G of a group G is the intersection with E of the set of limit points of $\{g(z) | g \in G\}$, where z is any point in D. This set is independent of $z \in D$, because the transformations in G preserve hyperbolic distances, and these become arbitrarily small relative to Euclidean distance, as E is approached. L_G is a closed set, invariant under G. The *convex figure* of G is the set $K_G = [L_G] \cap D$. This is an h-convex set which is invariant under G.

For each limit-rotation $g \in G$, it is possible to find a limit-circle C_g so that:

(a) C_g is tangent to E at the fixed point of g,

(b) $C_g \subset K_G$,

(c) If g_1 and g_2 are limit-rotations such that $g_2 = fg_1f^{-1}$, where $f \in G$, then

$$C_{g_2} = fC_{g_1},$$

(d) If z_1 and z_2 are two points interior to C_g, u is the fixed point of g, and $z_2 = f(z_1)$, where $f \in G$, then f is either a limit-rotation with fixed point u, or f is a reflection in an h-line with one endpoint at u.

We shall denote by K^*_G the region obtained from K_G by deleting the interior of each C_g. K^*_G is neither unique nor h-convex, but it is invariant under G. We shall say that K^*_G *is compact mod* G, if there exists a disk $\Gamma = \{z \mid |z| < r < 1\}$ such that

$$K^*_G \subset G\Gamma = \bigcup_{g \in G} g\Gamma.$$

This is equivalent to the compactness of the surface obtained from K^*_G by identifying points congruent under G. Nielsen **(10; 12)** has proved that *G is finitely generated, if and only if K^*_G is compact* mod G.

It is not hard to see (by constructing the fundamental region) that every hyperbolic group $F(p; n_1, \ldots, n_d; r)$ is realized as a group without limit-rotations. In fact, according to Nielsen **(12)**, for any finitely generated, hyperbolic group G, there is a homeomorphism s of D, such that sGs^{-1} is a group of motions without limit-rotations. When G contains no limit-rotations, $K^*_G = K_G$.

3. The results. Coxeter **(2)** and Goldberg **(6)** have shown that every abelian subgroup of the modular group $F(0; 2, 3; 1)$ is cyclic. We shall prove the following stronger version of this for the discrete, orientation-preserving, hyperbolic groups.

THEOREM 1. *If $F(p; n_1, n_2, \ldots, n_d; r)$ is hyperbolic, then the centralizer of any element is cyclic. The possible finite orders are the divisors of $n_1, n_2, \ldots, n_d$. Any finite subgroup is a cyclic group, conjugate to a subgroup of $\langle S_1 \rangle, \langle S_2 \rangle, \ldots,$* or *$\langle S_d \rangle$.*

Proof. It is well-known that two orientation-preserving linear fractional transformations commute if and only if they have the same fixed points. Therefore the centralizer of a rotation or limit-rotation is a group of rotations or limit-rotations with the same fixed point, and the centralizer of a translation is a group of translations with the same invariant axis. Each of these groups leaves a curve (or curves) invariant—a circle in D, for a group of rotations, a limit-circle for a group of limit-rotations, an h-line for a group of translations. Because the group is discrete, there must be an element which transforms a given point (on the invariant curve) the least distance in a fixed direction. This element generates the group, which is therefore cyclic.

The group $F(p; n_1, n_2, \ldots, n_d; r)$ has a fundamental region, which has among its vertices, the points $z_1, z_2, \ldots, z_d$ which are fixed points for $S_1, S_2, \ldots, S_d$ respectively. If z is a fixed point of a rotation S, there is an element f which maps z into one of the points z_k. Then fSf^{-1} is in the subgroup generated by S_k. Therefore the order of fSf^{-1}, which is the same as the order of S, divides n_k.

Let G be any finite subgroup of $F(p; n_1, \ldots, n_d; r)$, and let T_1 and T_2 be two elements (necessarily rotations) with fixed points t_1 and t_2 in D. We shall show that $t_1 = t_2$. Assuming otherwise, let λ_3 be the h-line through t_1 and t_2, and r_3 the reflection in λ_3. There are h-lines λ_2 and λ_1 through the points t_1

and t_2 respectively, such that if r_i is the reflection in λ_i, then $T_1 = r_2r_3$ and $T_2 = r_3r_1$. Therefore $T_1T_2 = r_2r_1$. If λ_1 and λ_2 diverge, r_2r_1 is a translation; if λ_1 and λ_2 are asymptotic (meet at a point on E), then r_2r_1 is a limit-rotation. Since these are transformations of infinite order, it follows that λ_1 and λ_2 must meet at a point t_3 in D, and $T_3 = r_1r_2$ is a rotation whose fixed point is t_3. The group $\langle r_1, r_2, r_3\rangle$ has the triangle $t_1t_2t_3$ as fundamental region. This group is infinite, since the images of $t_1t_2t_3$ under $\langle r_1, r_2, r_3\rangle$ cover D; since $\langle T_1, T_2, T_3\rangle$ is of index 2 in $\langle r_1, r_2, r_3\rangle$, the former subgroup is also infinite. We conclude that $t_1 = t_2$, and G is a cyclic group conjugate to a subgroup of $\langle S_1\rangle, \langle S_2\rangle, \ldots,$ or $\langle S_d\rangle$.

For hyperbolic groups which contain orientation-reversing transformations, the only exceptions are the following. The centralizer of a translation or glide-reflection can be a product of cyclic groups $C_\infty \times C_2$. The centralizer of a reflection can be the group $C_\infty \times C_2$ or $F(0; 2, 2; 1) \times C_2$. A finite subgroup can be a dihedral group.

THEOREM 2. *If S and T are finitely generated subgroups of a discrete group, then $S \cap T$ is also finitely generated.*

Proof. Let $H = S \cap T$ and let G be the finitely generated discrete group generated by S and T. As we remarked in § 2, we can suppose that G contains no limit-rotations. By Nielsen's theorem, there exist disks

$$\Gamma_S = \{z \mid |z| < r_s < 1\}, \qquad \Gamma_T = \{z \mid |z| < r_t < 1\}$$

such that $K_S \subset S\Gamma_S$ and $K_T \subset T\Gamma_T$. Let $r = \max(r_s, r_t)$ and $\Gamma = \{z \mid |z| < r\}$. Then

$$K_S \subset S\Gamma \quad \text{and} \quad K_T \subset T\Gamma.$$

Choose coset representatives $\{s_i\}$, $\{t_j\}$ so that

$$S = \bigcup_i Hs_i \qquad \text{and} \qquad T = \bigcup_j Ht_j.$$

Then

$$K_S \subset S\Gamma = H \bigcup_i s_i\Gamma,$$

$$K_T \subset T\Gamma = H \bigcup_j t_j\Gamma.$$

Also

$$K_H \subset K_S \cap K_T,$$

since

$$L_H \subset L_S \cap L_T.$$

We now show that $s_i\Gamma \cap K_T \neq \phi$ for only a finite number of representatives s_i. For any $h \in H$, $s_i\Gamma \cap ht_j\Gamma \neq \phi$ if and only if $\Gamma \cap s_i^{-1}ht_j\Gamma \neq \phi$. Now if $d(z_1, z_2)$ is the hyperbolic distance between the points z_1 and z_2 in D, and the

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hyperbolic radius of Γ is ρ, then $\Gamma \cap g\Gamma \neq \phi$ if and only if $d(0, g(0)) < 2\rho$. But the discreteness of G implies that there are only a finite number of elements $g \in G$ with this last property. Therefore there are only a finite number of elements $g = s_i^{-1}ht_j$ with $\Gamma \cap g\Gamma \neq \phi$. Note that if

$$s_{i1}^{-1}h_1t_{j1} = s_{i2}^{-1}h_2t_{j2},$$

then

$$s_{i2}s_{i1}^{-1}h_1 = h_2t_{j2}t_{j1}^{-1} \in S \cap T = H.$$

Therefore $s_{i_2}s_{i_1}^{-1}$ and $t_{j_2}t_{j_1}^{-1} \in H$, so

$$s_{i1} = s_{i2}, \qquad t_{j1} = t_{j2}$$

and $h_1 = h_2$. It follows that there are only a finite number of the s_i, t_j, h for which $s_i\Gamma \cap ht_j\Gamma \neq \phi$, and therefore only a finite number of the s_i for which $s_i\Gamma \cap K_T \neq \phi$.

Since $K_H \subset K_T$, there are only a finite number of the s_i, say $s_{i1}, s_{i2}, \ldots, s_{i_n}$, so that $s_i\Gamma \cap K_H \neq \phi$. Furthermore, the elements of H map K_H and K_S onto themselves and consequently $K_S - K_H$ onto itself. It follows that $s_i\Gamma \cap K_H \neq \phi$ if and only if $Hs_i\Gamma \cap K_H \neq \phi$. Recalling that

$$K_H \subset H \bigcup_i s_i\Gamma,$$

we now obtain

$$K_H \subset H \bigcup_{k=1}^{n} s_{ik}\Gamma.$$

Let Γ' be a disk with centre 0 and radius $r' < 1$, which is large enough to contain

$$\bigcup_{k=1}^{n} s_{ik}\Gamma.$$

Then $K_H \subset H\Gamma'$, or K_H is compact mod H. Nielsen's theorem now implies that H is finitely generated.

THEOREM 3. *If H is a finitely generated subgroup of G and if $L_H = L_G$, then $[G\colon H]$ is finite.*

Proof. If $L_G = \phi$, then G must be finite. If L_G consists of a single point z, then the elements of G and H are limit-rotations whose fixed point is z, and possibly reflections in h-lines with one endpoint at z. It is easy to see that the index $[G\colon H]$ is finite in this case. If L_G contains more than one point, then K^*_G and K^*_H are non-empty sets. By Nielsen's theorem there is a disk $\Gamma = \{z|\ |z| < r < 1\}$ so that $K^*_H \subset H\Gamma$. Since G is discrete, there can be only a finite number of elements $g \in G$ so that $\Gamma \cap g\Gamma \neq \phi$. We shall show that every $g \in G$ is congruent mod H to one of these elements, which we denote by $g_1, g_2, \ldots, g_n$. Let $z \in \Gamma \cap K^*_H$ (we suppose that Γ is large enough

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so that this intersection is not empty) and let $g \in G$. Since $K_G = K_H$, we have $K^*_G \subset K^*_H$. K^*_G is invariant under G, so $g(z) \in K^*_H$. Therefore there exists $h \in H$ so that $hg(z) \in \Gamma$. Thus $\Gamma \cap hg\Gamma \neq \phi$ (since $hg(z) \in \Gamma \cap hg\Gamma$) and $hg = g_k$ for some k. It follows that $[G:H]$ is finite.

The following is proved in (**4**, p. 43).

LEMMA 1. *If S is a closed subset of E which contains more than one point, and S is invariant under a group G, then $S \supset L_G$.*

Definition. An N-chain of a group G is a sequence of subgroups $G_1, G_2, \ldots, G_n$ such that:

(a) $G_k \neq \{1\}$ $(k = 1, 2, \ldots, n)$,

(b) either G_k is a normal subgroup of G_{k+1}, or G_{k+1} is a normal subgroup of G_k.

We shall say that two subgroups H and K are N-equivalent if there is an N-chain $H = G_1, G_2, \ldots, G_n = K$. A subgroup which is N-equivalent to G will be called an N-subgroup.

We shall call a group *quasi-abelian* if it leaves invariant an h-line or a point in D. Such a group is either abelian or has an abelian subgroup of index 2. G is quasi-abelian if and only if L_G consists of 0, 1, or 2 points. The following Lemma shows than an N-equivalence class consists entirely of quasi-abelian groups if it contains one such group.

LEMMA 2. *If G and H are N-equivalent subgroups of a discrete group and G is not quasi-abelian, then $L_G = L_H$.*

Proof. Let the N-chain be $G = G_1, G_2, \ldots, G_n = H$. We proceed to prove by induction that

$$L_{G_k} = L_G \qquad (k = 1, 2, \ldots, n).$$

Clearly $L_{G_1} = L_G$. Assume $L_{G_k} = L_G$. If $G_k \subset G_{k+1}$, then $L_{G_k} \subset L_{G_{k+1}}$. On the other hand, L_{G_k} is invariant under G_{k+1}. For let $g \in G_{k+1}$ and $z_0 \in L_{G_k}$. There is a sequence $\{h_j\} \subset G_k$ so that for any $z \in D$,

$$\lim_{j\to\infty} h_j(z) = z_0.$$

Now $gh_jg^{-1} \in G_k$ and

$$\lim_{j\to\infty} h_jg^{-1}(z) = z_0,$$

so that

$$\lim_{j\to\infty} gh_jg^{-1}(z) = g(z_0).$$

Therefore $g(z_0) \in L_{G_k}$, and L_{G_k} is invariant under G_{k+1}. Since $L_{G_k} = L_G$ and G is not quasi-abelian, L_{G_k} contains more than 2 points. By Lemma 1,

$$L_{G_k} \supset L_{G_{k+1}}.$$

Thus

$$L_G = L_{G_k} = L_{G_{k+1}}.$$

It remains to consider the case where G_{k+1} is a normal subgroup of G_k. In this case $L_{G_{k+1}} \subset L_{G_k}$. Moreover, in the same manner as above we can show that $L_{G_{k+1}}$ is invariant under G_k.

We assert that $L_{G_{k+1}}$ contains more than one point. If $L_{G_{k+1}} = \phi$, then G_{k+1} is a finite group. G_{k+1} is either a group of rotations (and possibly reflections) with a common fixed point $z \in D$ or a reflection group of order 2. In the first case the point z must be invariant under all transformations in G_k. Then G_k is also a finite group, so that

$$L_G = L_{G_k} = \phi.$$

But this implies that G is quasi-abelian. In the second case, G_{k+1} consists of the identity and a reflection r in some h-line λ. The elements of G_k leave λ invariant. L_{G_k} is either empty or consists of the endpoints of λ. This is true also of L_G, so that G must be quasi-abelian. If $L_{G_{k+1}}$ contains only a single point z, then this point is invariant under G_k. G_k is a group of limit-rotations with limit-centre z (and possibly reflections in h-lines with one endpoint at z). Then

$$L_G = L_{G_k} = \{z\}$$

and it follows that G is quasi-abelian.

Lemma 1 now implies that $L_{G_{k+1}} \supset L_{G_k}$, so that

$$L_G = L_{G_k} = L_{G_{k+1}}.$$

It now follows that $L_H = L_{G_k} = L_G$.

The previous lemma and Theorem 3 imply the following.

THEOREM 4. *Let H be a finitely generated N-subgroup of a non-quasi-abelian group G. Then $[G{:}\,H]$ is finite.*

LEMMA 3. *If U and V are subnormal subgroups of a non-quasi-abelian group, then $U \cap V \neq \{1\}$.*

Proof. Let F_1, U_1, and V_1 be the orientation-preserving subgroups of index 2 in F, U, and V respectively. In the proof of Lemma 2, we saw that if an N-subgroup of F is a reflection group or order 2, then F is quasi-abelian. Therefore neither U nor V are reflection groups, so U_1 and V_1 are non-trivial, subnormal subgroups of F_1. There exist normal series

$$F_1 \supset F_2 \supset \ldots \supset F_n = U_1,$$

$$F_1 \supset F_2' \supset \ldots \supset F_n' = V_1,$$

where some of the F_k or some of the F_k' might coincide. We shall prove inductively that $F_k \cap F_k'$ is a non-trivial non-abelian group. If $F_2 \cap F_2' = \{1\}$,

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then each element of F_2 commutes with each element of F_2' But two orientation-preserving transformations commute, if and only if they have the same fixed points. This implies that F_2 (and F_2') is a commutative group. The elements of F_2 must be rotations with a common fixed point $z_1 \in D$, limit-rotations with a common fixed point $z_2 \in E$, or translations with a common axis λ. Since F_2 is normal in F_1, the elements of F_1 must have the same invariant point or h-line. Therefore F_1 is abelian, and F is quasi-abelian. From this contradiction we conclude that $F_2 \cap F_2' \neq \{1\}$. Furthermore $F_2 \cap F_2'$ is not abelian, since this together with its normality in F_1 would imply that F_1 is abelian. Now suppose that $F_k \cap F_k' \neq \{1\}$ and is non-abelian. $F_{k+1} \cap F_k'$ and $F_k \cap F_{k+1}'$ are normal subgroups of $F_k \cap F_k'$. By the same argument as before, we conclude that $F_{k+1} \cap F_{k+1}' = (F_{k+1} \cap F_k') \cap (F_k \cap F_{k+1}') \neq \{1\}$ and is not abelian. It now follows that $U \cap V \neq \{1\}$.

THEOREM 5. *Let H and K be two non-quasi-abelian subgroups of a discrete group. Then H and K are N-equivalent, if and only if there is a non-trivial subgroup J which is simultaneously subnormal in H and K.*

Proof. The "if" part is obvious; we shall prove the "only if" part. There is an N-chain $H = G_1, G_2, \ldots, G_n = K$. The series

$$G_1 \supset G_1 \cap G_2 \supset G_1 \cap G_2 \cap G_3 \supset \ldots \bigcap_{k=1}^{n} G_k$$

is a normal series. We shall prove inductively that

$$\bigcap_{k=1}^{m} G_k$$

is a non-trivial, subnormal subgroup of G_m. This is certainly true for $m = 1$; assume that this is true for $m = p$. If G_p is a normal subgroup of G_{p+1}, then

$$\bigcap_{k=1}^{p+1} G_k = \bigcap_{k=1}^{p} G_k \neq \{1\}.$$

Since

$$\bigcap_{k=1}^{p+1} G_k$$

is subnormal in G_p, which is normal in G_{p+1}, it follows that

$$\bigcap_{k=1}^{p+1} G_k$$

is subnormal in G_{p+1}. Now suppose that G_{p+1} is a normal subgroup of G_p. G_p cannot be quasi-abelian. The conditions of Lemma 3 are fulfilled, with

$$F = G_p, \qquad U = \bigcap_{k=1}^{p} G_k, \qquad V = G_{p+1}..$$

Therefore

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$$\bigcap_{k=1}^{p+1} G_k \neq \{1\}.$$

Since

$$\bigcap_{k=1}^{p} G_k$$

is subnormal in G_p,

$$\bigcap_{k=1}^{p} G_k \cap G_{p+1}$$

is subnormal in $G_p \cap G_{p+1} = G_{p+1}$. It now follows that the group

$$J = \bigcap_{k=1}^{n} G_k$$

is a non-trivial, subnormal subgroup of H and K.

This theorem implies that if G is not quasi-abelian, then a subgroup H is an N-subgroup if and only if it contains a subnormal subgroup of G.

THEOREM 6. *Let H be a finitely generated non-quasi-abelian subgroup of G. Then there is a subgroup G_H of G such that*

(a) *G_H is N-equivalent to H,*

(b) *if $K \subset G$ and K is N-equivalent to H, then $K \subset G_H$,*

(c) *$[G_H\colon H]$ is finite.*

Proof. Let $G_H = \{g | \, g \in G, gL_H = L_H\}$. Since

$$H \subset G_H, L_H \subset L_{G_H}.$$

Lemma 1 implies that $L_H \supset L_{G_H}$, so that $L_H = L_{G_H}$. Theorem 3 now implies that $[G_H\colon H]$ is finite. From this it follows that H has a finite number of conjugate subgroups in G_H. The intersection of these conjugate subgroups is a normal subgroup F of finite index in G_H. Since G_H is infinite, F is non-trivial. Therefore the sequence, G_H, F, H, is an N-chain, and G_H is N-equivalent to H. If K is N-equivalent to H, Lemma 2 implies that K leaves L_H invariant, so that $K \subset G_H$.

THEOREM 7. *Let H and K be finitely generated non-quasi-abelian subgroups of a discrete group. Then the following statements are equivalent:*

(a) *H and K are N-equivalent;*

(b) *there is a group J which is simultaneously normal and of finite index in H and K;*

(c) *$L_H = L_K$.*

Proof. If (a) *is true*, then $G_H = G_K$. (These are the groups introduced in Theorem 6.) Since H and K are of finite index in G_H, this is also true of $H \cap K$. Therefore $H \cap K$ contains a nontrivial subgroup J which is normal and of finite index in G_H. J is also normal and of finite index in H and K. This shows that (a) implies (b).

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If (b) is true, then H and K are N-equivalent. Therefore $L_H = L_K$. This shows that (b) implies (c).

Now suppose (c) is true, Then $G_H = G_K$. It follows that H and K are both N-equivalent to $G_H = G_K$, and hence to each other.

It would be interesting to determine whether there are algebraic conditions equivalent to the condition $L_H = L_K$, when H or K is infinitely generated.

The following is proved in (**9**, p. 76).

LEMMA 4. *Let U and V be two groups such that the isometric circles of U are contained in R_V and the isometric circles of V are contained in R_U. Then the group generated by U and V is the free product $U * V$, and $R_{U*V} = R_U \cap R_V$.*

THEOREM 8. *Let H be a finitely generated subgroup of a finitely generated non-quasi-abelian group G. Then $[G\colon H]$ is finite if and only if H is contained in no infinitely generated subgroup of G.*

Proof. If H is of finite index, then any larger group must also be of finite index, and so it is finitely generated.

Now suppose H is of infinite index. We shall find a subgroup of G which contains H and is infinitely generated. We may assume that G contains no limit-rotations. Then $\bar{R}_H$, which cannot be contained in D, contains intervals on E. We first show that one of these intervals contains points of L_G in its interior.

If $L_H = L_G$, then by Theorem 3 $[G\colon H]$ is finite. Thus there is a point $z_0 \in L_G - L_H$. Let $z \in R_H$; there is a sequence $\{g_n\} \subset G$ such that

$$\lim_{n\to\infty} g_n(z) = z_0.$$

Since R_H is a fundamental region for H, there is $h_n \in H$ so that $h_n g_n(z) \in R_H$. The sequence $\{h_n g_n(z)\}$ has a subsequence which converges to a point $z_1 \in \bar{R}_H \cap E$. The point z_1 is a limit point of G and belongs to an interval I_1 of $\bar{R}_H \cap E$. z_1 might possibly be an endpoint of I_1. Since G is not quasi-abelian, L_G consists of more than two points, and hence it is a perfect subset of E (see (**4**, p. 68)). Thus there is a sequence $\{x_n\}$ in L_G which converges to z_1. Suppose this sequence is outside I_1. z_1 is the endpoint of an isometric circle of an element $h \in H$. The transformation h maps I_1 outside $\bar{R}_H \cap E$, and maps a neighbouring interval, containing almost all of the sequence $\{x_n\}$, onto an interval I of $\bar{R}_H \cap E$. Therefore I contains points of L_G in its interior.

As is shown in (**10**), the fixed points of the translations of G are dense in L_G in the following sense. If $x, x' \in L_G$ and J and J' are intervals of E which contain x and x' respectively, then there is a translation $g \in G$, with a fixed point in each interval. A sufficiently high power g^n has isometric circles λ and λ' which intersect E inside J and J' respectively. Since the interval $I \subset \bar{R}_H \cap E$ contains points in L_G, it contains an infinite sequence of such points, which we denote by $\{y_1, y_1', y_2, y_2', \ldots ,\}$. Let I_k, I_k' be mutually disjoint subintervals of I which contain y_k and y_k' respectively. There is a translation $g_k \in G$ whose isometric circles λ_k and λ_k' intersect E inside I_k and I_k' respectively. By Lemma 4, the group generated by $\{g_1, g_2, \ldots ,\}$ is a free group F of infinite rank, whose fundamental region R_F is the region in D outside of all λ_k and λ_k'. Lemma 4 now implies that the group K generated by H and F is the free product $H * F$. Thus K is an infinitely generated group containing H.

THEOREM 9. *Let H be a finitely generated subgroup of a non-quasi-abelian group G. If H has a non-trivial intersection with every non-cyclic subgroup of G, then $[G\colon H]$ is finite.*

Proof. We shall show that $L_H = L_G$. Since G is not quasi-abelian, L_G is a perfect subset of E. Let $z \in L_G$, and let I be an open interval of E which contains z. I contains infinitely many points of L_G. Choose four of them z_1, z_1', z_2, z_2'. Let I_1, I_1', I_2, I_2' be non-intersecting subintervals of I, which contain z_1, z_1', z_2, z_2' respectively. As in the proof of Theorem 8, there are translations g_1 and $g_2 \in G$, such that the isometric circles λ_1 and λ_1' of g_1 intersect E inside I_1 and I_1' respectively, and the isometric circles λ_2 and λ_2' intersect E inside I_2 and I_2' respectively. The group K, generated by g_1 and g_2, is a free group of rank 2. By hypothesis, the intersection $H \cap K$ is non-trivial. It follows that H has an element whose fixed points are in I. Since this is true for any interval I containing z, it follows that $z \in L_H$ and $L_H = L_G$. Theorem 3 now implies the required result.

COROLLARY. *Let H and K be finitely generated non-quasi-abelian subgroups of a discrete group. If H has a non-trivial intersection with every non-cyclic subgroup of K, then $[K\colon H \cap K]$ is finite.*

Proof. By Theorem 2, $H \cap K$ is finitely generated. The Corollary now follows from Theorem 9.

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