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# A local-global question in automorphic forms

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Abstract

In this paper, we consider the SL(2) analogue of two well-known theorems about period integrals of automorphic forms on GL(2): one due to Harder–Langlands–Rapoport about non-vanishing of period integrals on $\mathrm{GL}_2(\mathbb{A}_F)$ of cuspidal automorphic representations on $\mathrm{GL}_2(\mathbb{A}_E)$ where $E$ is a quadratic extension of a number field $F$, and the other due to Waldspurger involving toric periods of automorphic forms on $\mathrm{GL}_2(\mathbb{A}_F)$. In both these cases, now involving SL(2), we analyze period integrals on global $L$-packets; we prove that under certain conditions, a global automorphic $L$-packet which at each place of a number field has a distinguished representation, contains globally distinguished representations, and further, an automorphic representation which is locally distinguished is globally distinguished.

**Contents**



## 1. Introduction

Let $F$ be a number field and $\mathbb{A}_F$ its adèle ring. Let $G$ be a reductive algebraic group over $F$ with center $Z$, and $H$ a reductive subgroup of $G$ over $F$ containing $Z$. For an automorphic form $\phi$ on $G(\mathbb{A}_F)$ on which $Z(\mathbb{A}_F)$ acts trivially, the period integral of $\phi$ with respect to $H$ is defined to be the integral (when convergent, which is the case if $\phi$ is cuspidal and $H(F)Z(\mathbb{A}_F)\backslash H(\mathbb{A}_F)$ has

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finite volume)

$$\mathcal{P}(\phi) = \int_{H(F)Z(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \phi(h)\, dh,$$

where $dh$ is the natural measure on $H(F)Z(\mathbb{A}_F)\backslash H(\mathbb{A}_F)$.

An automorphic representation $\Pi$ of $G(\mathbb{A}_F)$ is said to be globally distinguished with respect to $H$ if this period integral is nonzero for some $\phi \in \Pi$. More generally, if $\chi$ is a one-dimensional representation of $H(\mathbb{A}_F)$ trivial on $H(F)$ such that $Z(\mathbb{A}_F)$ acts trivially on $\phi(h)\chi^{-1}(h)$, and

$$\int_{H(F)Z(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \phi(h)\chi^{-1}(h)\, dh$$

is nonzero for some $\phi \in \Pi$, then $\Pi$ is said to be $\chi$-distinguished with respect to $H$.

The corresponding local notion is defined as follows. If $\Pi_v$ is an irreducible admissible representation of $G(F_v)$, $\Pi_v$ is said to be locally distinguished with respect to $H(F_v)$ if it admits a non-trivial $H(F_v)$-invariant linear form. Distinction with respect to $\chi_v$, a character of $H(F_v)$, is defined in a similar manner.

It is obvious that if $\Pi = \otimes_v \Pi_v$ is globally distinguished with respect to $H(\mathbb{A}_F)$, then each $\Pi_v$ is locally distinguished with respect to $H(F_v)$. Indeed, the period integral 'restricted' to $\Pi_v$ is a non-trivial $H(F_v)$-invariant linear form. The local-global question asks the converse: if $\Pi$ is such that each $\Pi_v$ is locally distinguished, is $\Pi$ globally distinguished? It seems best to break this question into several parts.

*Question* 1. Let $G$ be a reductive group over a local field $k$, and $H$ a closed subgroup. Then, is there a criterion in terms of Langlands parameters as to when a representation in an $L$-packet of $G(k)$ has an $H(k)$-invariant linear form? Similarly, if $G$ is a reductive group over a number field $F$, with a closed subgroup $H$, is there a criterion as to when a global $L$-packet of $G(\mathbb{A}_F)$ is globally distinguished by $H(\mathbb{A}_F)$?

It has been suggested by Jacquet, and corroborated in the work of Sakellaridis and Venkatesh [SV00], that in many cases, such as when $H$ is a spherical subgroup of $G$, representations of $G(k)$ which are distinguished by $H(k)$ arise as functorial lifts from a group $G_H$ to $G$ through a mapping of $L$-groups ${}^LG_H \longrightarrow {}^LG$; the complete picture of distinguished representations of $G(k)$ is then a refinement of this condition on $L$-parameters. For example, for the embedding $\mathrm{SO}_n \hookrightarrow \mathrm{SO}_n \times \mathrm{SO}_{n+1}$ as studied in [GGP12, GP92], there are no conditions on the parameters involved, i.e., ${}^LG_H = {}^LG$, but there are conditions on certain epsilon factors in the local case, and on $L$-values at $s = 1/2$ in the global case.

The work of Sakellaridis and Venkatesh in [SV00] is about the Plancherel decomposition of $L^2(H(k)\backslash G(k))$, so although it does not answer exactly this question about classifying irreducible admissible representations of $G(k)$ which are distinguished by $H(k)$, it is still closely related. There are global analogues too in these works.

Our work presumes an answer to Question 1, and indeed in the cases that we study, the answer to Question 1 has been known for a long time. We refer to § 2.2 for the case dealt with by Harder–Langlands–Rapoport in the local and global cases. In the case dealt with by Waldspurger for toric integrals for $\mathrm{GL}_2$, ${}^LG_H = {}^LG$, but there are finer arithmetic invariants, certain $L$ and $\epsilon$ factors, that we will come to later.

Given that one is supposed to know the answer to Question 1, we are trying in this paper to ask a local-global question.

*Question* 2. Let $\Pi = \otimes_v \Pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ such that each of the representations $\Pi_v$ of $G(F_v)$ is distinguished by $H(F_v)$. Is there an automorphic representation, say $\Pi'$, in the global $L$-packet of $G(\mathbb{A}_F)$ determined by $\Pi$ which is globally distinguished by $H(\mathbb{A}_F)$?

If we take it that the answers to the local and global parts in Question 1 are in terms of the Langlands parameters associated with $\Pi = \otimes \Pi_v$ to factor through ${}^LG_H \longrightarrow {}^LG$, we are led to questions about local versus global factoring of parameters through this mapping of $L$-groups.

The authors admit that they have not seen any general context, say for representations of an abstract group $W$, with subgroups $W_v$ which generate $W$, where one wants to force a representation of $W$ with values in ${}^LG$ to be conjugated to lie inside ${}^LG_H$, under the map ${}^LG_H \longrightarrow {}^LG$, given that the representation of $W$ restricted to $W_v$ can be conjugated to lie inside ${}^LG_H$; this is exactly what we will achieve in the SL(2) analogue of the case dealt with by Harder–Langlands–Rapoport, although, as there is no template for this work (of forcing representations to lie inside a subgroup through local conditions), we have to content ourselves with a sample theorem in which we restrict either the global representation to be non-CM, or the local representation to be a discrete series of a certain kind. In fact, this paper emphasizes the role that a discrete series local component of an automorphic representation might make to a global result: a local condition with a global effect, and we also know that the global result fails without having some local conditions [AP06, Theorem 8.2].

Since Question 2 is about an $L$-packet, one might expect, besides the parameter to factor through ${}^LG_H \longrightarrow {}^LG$, some $L$-value too to intervene in the answer to this question. (Some $L$-values, such as having a pole at $s = 1$, have an interpretation in terms of the Langlands parameters, whereas some other $L$-values, such as vanishing or non-vanishing at $s = 1/2$, do not!)

In the case studied by Harder–Langlands–Rapoport, there are conditions on the Langlands parameter, whereas in the work of Waldspurger [Wal85], as generalized in [GP92, GGP12], there are no conditions on the parameters involved, but there are conditions on $L$-values at $s = 1/2$.

If no automorphic member of the global $L$-packet determined by $\Pi$ is globally distinguished, say for reasons of an $L$-value, we do not need to proceed any further in this quest in the global $L$-packet determined by $\Pi$. So we assume that there is a member in the $L$-packet determined by $\Pi$ which is globally distinguished, which we can then assume to be $\Pi$ itself in our further study.

*Question* 3. Suppose $\Pi = \otimes_v \Pi_v$ is an automorphic representation of $G(\mathbb{A}_F)$ such that $\Pi$ is globally distinguished by $H(\mathbb{A}_F)$. Let $\Pi' = \otimes_v \Pi'_v$ be an automorphic representation of $G(\mathbb{A}_F)$ in the same $L$-packet as $\Pi$ such that $\Pi'_v$ is locally distinguished by $H(F_v)$ at all the places of $F$. Then is $\Pi'$ globally distinguished by $H(\mathbb{A}_F)$?

This is the local-global question referred to in the title of this paper, and which being a question about an individual automorphic representation, and not a question about an $L$-packet, is not governed by an $L$-value, but keeping the parametrization of automorphic representations in mind (due to Labesse–Langlands for $\mathrm{SL}_2$, and then Langlands, Kottwitz, and Arthur), should be related to a certain finite group of connected components of an appropriate representation (of the Langlands group). However, in the examples we deal with in this paper, the local-global principle turns out to be true.

The aim of this work is to initiate such a finer study in the global context of some low rank cases in detail, by varying the themes already studied in the literature. In this work we will consider Questions 2 and 3 above for two basic cases. These two cases will be variations on two rather well-studied examples where we change the groups involved slightly, allowing us to consider non-trivial local and global $L$-packets.

The first example is one of $(\mathrm{GL}_2(E), \mathrm{GL}_2(F))$ where $E/F$ is a quadratic extension of either local or global fields. This came up in the seminal work of Harder *et al.* [HLR86] which was later pursued by Flicker and Hakim [Fli88, Fli91, Hak91]. Global distinction here is characterized by an $L$-function, the Asai $L$-function, having a pole at $s=1$. We will analyze Questions 2 and 3 for the related pair $(\mathrm{SL}_2(E), \mathrm{SL}_2(F))$. The starting point of this investigation is an elementary observation that an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ has a non-trivial period integral on $\mathrm{SL}_2(\mathbb{A}_F)$ if and only if it is $\chi \circ \det$-distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$ for a Grössencharacter $\chi$ of $\mathbb{A}_F^\times/F^\times$, which is then a condition on the Asai $L$-function twisted by $\chi^{-1}$ to have a pole at $s=1$. This allows conclusions about $L$-packets of automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$, but making conclusions about individual automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ is subtler.

The second example that we will consider in this paper is related to the celebrated work of Waldspurger [Wal85]. Here $G=\mathrm{GL}_2(F)$, or more generally the invertible elements of a quaternion algebra over $F$, and $H$ is the torus defined by a quadratic algebra $E/F$. In this case, $\pi$ is globally $\chi$-distinguished for a Grössencharacter $\chi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ if and only if each $\pi_v$ is locally $\chi_v$-distinguished and $L(\frac{1}{2}, \mathrm{BC}(\pi) \otimes \chi^{-1}) \neq 0$, where $\mathrm{BC}(\pi)$ denotes the base change lift of $\pi$ to $\mathrm{GL}_2(\mathbb{A}_E)$. The local picture is well understood by the work of Saito and Tunnell [Sai93, Tun83], and involves certain local epsilon factors. We will analyze Questions 2 and 3 above for the related pair $(\mathrm{SL}_2(F), E^1)$ where $E^1$ is the subgroup of $E^\times$ of norm one elements. It may be noted that there are many non-conjugate embeddings of $E^1$ inside $\mathrm{SL}_2(F)$; we will fix one such embedding; our answers do not depend on this initial fixing of an embedding of $E^1$ inside $\mathrm{SL}_2(F)$.

In the first example, $(\mathrm{GL}_2(E), \mathrm{GL}_2(F))$, the local-global principle almost holds. If each $\pi_v$ is locally distinguished, then $\pi$ is either globally distinguished or is globally distinguished with respect to the quadratic character $\omega$ associated to $E/F$ [HLR86]. Thus, if each $\pi_v$ is distinguished and if at least one $\pi_v$ is not $\omega_v$-distinguished, then $\pi$ is globally distinguished. In particular, if $\pi_v$ is a square integrable representation at least at one place $v$ of $E$ which is inert over $F$, then $\pi$ is globally distinguished if and only if it is locally distinguished. This follows since a discrete series representation of $\mathrm{GL}_2(E_v)$, once distinguished by $\mathrm{GL}_2(F_v)$, cannot be $\omega_v$-distinguished.

In [AP06] we had constructed an example of an automorphic representation $\Pi$ on $\mathrm{SL}_2(\mathbb{A}_E)$ where each $\Pi_v$ is a locally distinguished representation of $\mathrm{SL}_2(E_v)$ but no member of the $L$-packet of $\Pi$ is globally distinguished. In this paper, we give a positive answer to Question 2 in some situations, but have not succeeded in getting a complete understanding of it.

Theorem 1.1. *Let $\Pi$ be a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_E)$. If $\Pi$ appears in the restriction of a CM representation of $\mathrm{GL}_2(\mathbb{A}_E)$, assume that there is at least one square integrable component at a place of $E$ which is inert over the corresponding place $v_0$ of $F$. In the CM case, assume that either $\Pi$ is CM by three distinct quadratic extensions of $E$, or alternatively if it is CM by a unique quadratic extension of $E$, then at the place $v_0$, the local component is also CM by a unique quadratic extension of $E_{v_0}$ (or more generally, it is CM only by quadratic extensions which are Galois over $F_{v_0}$). Suppose each $\Pi_v$ is distinguished by $\mathrm{SL}_2(F_v)$. Then there is a cuspidal representation in the $L$-packet of $\Pi$ which is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$.*

Question 3 has a complete answer in the following theorem.

Theorem 1.2. *Let $\Pi$ be a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_E)$ which is globally distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$. Let $\Pi' = \otimes_v \Pi'_v$ be an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_E)$ in the same $L$-packet*



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*as* $\Pi$ *such that* $\Pi'_v$ *is locally distinguished by* $\mathrm{SL}_2(F_v)$ *at all the places of* $F$*. Then* $\Pi'$ *is globally distinguished by* $\mathrm{SL}_2(\mathbb{A}_F)$*.*

A key ingredient in the proof of Theorem 1.1 is the multiplicity one theorem for automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ due to [Ram00], whereas Theorem 1.2 is proved via an exact determination of the fibers of the Asai lift from automorphic representations on $\mathrm{GL}_2(\mathbb{A}_E)$ to automorphic representations on $\mathrm{GL}_4(\mathbb{A}_F)$, completing an earlier work of Krishnamurthy in [Kri03].

In the example considered by Waldspurger, $(\mathrm{GL}_2(F), E^\times)$, unlike in the first example, there is a genuine global obstruction to global distinction, and this is the vanishing of the central value of the base change $L$-function. In our study of the pair $(\mathrm{SL}_2(F), E^1)$, we are naturally led to some questions about $L$-functions.

According to a well-known result of Friedberg and Hoffstein [FH95], for an automorphic representation $\pi$ on $\mathrm{GL}_2(\mathbb{A}_F)$, there are infinitely many quadratic characters $\eta$, with prescribed local behaviour at finitely many places, such that the twisted $L$-values, $L(\frac{1}{2}, \pi \otimes \eta)$, are nonzero, provided the global root number $\epsilon(\frac{1}{2}, \pi)$ is 1, possibly after twisting $\pi$ by a quadratic character (see also [Jac87]). This latter condition on the global root number of $\pi$ is automatic if $\pi$ has at least one square integrable component [Wal91]. For the analysis of the special linear analogue of the second example, one needs to understand (a special case) of the following simultaneous non-vanishing problem, stated as a conjecture.

CONJECTURE 1.3. Let $\pi_1$ and $\pi_2$ be two cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$. Let $\eta$ be a quadratic character such that

$$\epsilon(\tfrac{1}{2}, \pi_i \otimes \eta) = 1$$

for those $\pi_i$ which are self-dual among $\{\pi_1, \pi_2\}$. Then there are infinitely many quadratic characters $\eta'$, which agree with $\eta$ at any finitely many prescribed places of $F$, such that

$$L(\tfrac{1}{2}, \pi_1 \otimes \eta') \neq 0 \neq L(\tfrac{1}{2}, \pi_2 \otimes \eta').$$

Assuming the conjecture, we give a positive answer to Question 2 in this case, once again assuming that a local component is discrete series.

THEOREM 1.4. *Let* $D$ *be a quaternion algebra over a number field* $F$*, with* $E$ *a quadratic subfield of* $D$*. Let* $\Pi = \otimes_v \Pi_v$ *be a cuspidal representation of* $\mathrm{SL}_1(D)(\mathbb{A}_F)$ *with at least one square integrable component at a place* $v_0$ *of* $F$*; if* $E$ *is inert and* $D$ *is split at* $v_0$*, we further assume that* $v_0$ *is of odd residue characteristic and* $\Pi_{v_0}$ *is a supercuspidal representation if* $v_0$ *is a finite place of* $F$*. If each* $\Pi_v$ *is distinguished with respect to* $E_v^1$*, then assuming Conjecture 1.3 holds, there is a cuspidal representation in the* $L$*-packet of* $\Pi$ *which is distinguished with respect to* $\mathbb{A}_E^1$*.*

We have also achieved a positive answer to Question 3, assuming Conjecture 1.3, but only in the case when the global $L$-packet associated to the automorphic representation $\Pi$ is finite. In the more general case, we need a finer version of Conjecture 1.3, for which we refer the reader to § 11.

We end the introduction by noting the role played by analytic number theory (simultaneous non-vanishing of central $L$-values in this case) in questions on automorphic representations; whether one implies the other, or the other way around, only time will tell.



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## 2. Preliminaries

This section summarizes some of the key results that we make heavy use of throughout this paper.

### 2.1 Langlands correspondence, and CM forms

For a local field $F$, the local Langlands correspondence gives a bijection of the set of isomorphism classes of irreducible admissible complex representations of $\mathrm{GL}_n(F)$, and the set of isomorphism classes of $n$-dimensional semi-simple complex representations of the Weil–Deligne group $W_F' = W_F \times \mathrm{SL}_2(C)$; the bijection preserves $L$ and $\epsilon$-factors of pairs of representations, the details of which we will not go into. The local Langlands correspondence was established for $\mathrm{GL}_2(F)$ by Phil Kutzko, and in general by Harris and Taylor, and also by Henniart. If a representation $\Sigma$ of $W_F'$ of dimension $n$ is associated to an irreducible admissible representation $\pi$ of $\mathrm{GL}_n(F)$, then $\Sigma$ is often referred to as the Langlands parameter of $\pi$. For $n=1$, the local Langlands correspondence is nothing but the local classfield theory which identifies characters of $F^\times$ to characters of $W_F^{\mathrm{ab}}$, where $W_F^{\mathrm{ab}}$ is the maximal abelian quotient of $W_F$.

An irreducible admissible representation of $\mathrm{GL}_2(F)$ is said to be CM if its Langlands parameter is induced from a character of a quadratic extension of $F$; such representations are often referred to as automorphically induced from a character of a quadratic extension. A representation $\pi$ of $\mathrm{GL}_2(F)$ is CM if and only if it has a non-trivial self-twist by a character of $F^\times$; i.e., there is a character $\omega \neq 1$ of $F^\times$ such that $\pi \otimes \omega \cong \pi$.

For a number field $F$, an automorphic representation $\pi$ of $\mathrm{GL}_2(\mathbb{A}_F)$ is said to be CM if it has a non-trivial self-twist by a Grössencharacter, say $\omega$, of $\mathbb{A}_F^\times/F^\times$. The character $\omega$ is of order 2, and defines a quadratic extension $E$ of $F$. Associated to such a CM automorphic representation $\pi$ of $\mathrm{GL}_2(\mathbb{A}_F)$, there is a Grössencharacter $\chi$ of $\mathbb{A}_E^\times/E^\times$ (which is unique up to Galois conjugation) with $L(s,\chi) = L(s,\pi)$. The automorphic representation $\pi$ is said to be automorphically induced from this Grössencharacter $\chi$ of $\mathbb{A}_E^\times/E^\times$. It can be proved that the restriction of the two-dimensional representation of the global Weil group $W_F$ defined by $\mathrm{Ind}_{W_E}^{W_F}\chi$ to various decomposition groups is nothing but the Langlands parameters of various components of $\pi$.

In this paper, we will have many occasions to use the Asai lift of representations of $\mathrm{GL}_2(E)$, where $E$ is a quadratic extension of a local field $F$, to representations of $\mathrm{GL}_4(F)$. This is most easily defined using the local Langlands correspondence on the Weil group side where there is a group theoretic generality, called 'tensor induction', which constructs from an $n$-dimensional representation of a subgroup $H$ of index $m$ in a group $G$, a representation of $G$ of dimension $n^m$. This definition is recalled in § 6. The representation of $\mathrm{GL}_4(F)$ constructed using a representation $\pi$ of $\mathrm{GL}_2(E)$ is called the Asai lift of $\pi$, and denoted by $\mathrm{As}(\pi)$.

If $F$ is a number field, and $E$ a quadratic extension of $F$, and $\pi$ an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$, local Asai lifts from $\mathrm{GL}_2(E_v)$ to $\mathrm{GL}_4(F_v)$ patch together to give an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$, denoted again as $\mathrm{As}(\pi)$ [Kri03].

In the theory of automorphic representations, the notion of *base change*, which for representations of the Weil group corresponds to restriction to subgroups, plays a very central role; it does so in our work too. We will say nothing more about it, except that the notation will be $\mathrm{BC}(\pi)$ for the base change of a representation of $\mathrm{GL}_n(F)$ to $\mathrm{GL}_n(E)$; the notation BC will not specify $n$, or the fields $E$ and $F$, which can be either local or global; all these will be clear from the context.



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## 2.2 Distinction for $(\mathbf{GL_2}(E), \mathbf{GL_2}(F))$

Let $E/F$ be a quadratic extension of local or global fields. Let $\sigma$ be the non-trivial element of the Galois group $\mathrm{Gal}(E/F)$. The quadratic character of $F^\times$ (or of $\mathbb{A}_F^\times/F^\times$ in the global case) associated to $E/F$ by class field theory is denoted by $\omega_{E/F}$. Consider the symmetric space $(G, H)$ where $G$ is either the local group $\mathrm{GL}_2(E)$ (or the adelic group $\mathrm{GL}_2(\mathbb{A}_E)$) and $H$ is the fixed points of $\sigma$.

Let $\pi$ be an irreducible admissible representation of $\mathrm{GL}_2(E)$ (or on a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$). Let $\pi^\vee$ and $\pi^\sigma$ respectively denote the contragredient and the Galois conjugate of $\pi$. The central character of $\pi$ is denoted by $\omega_\pi$.

Suppose $\pi$ is distinguished with respect to $H$. Then $\omega_\pi$ is trivial on $F^\times$ (or on $\mathbb{A}_F^\times$). The following two propositions have their roots in the work of Harder–Langlands–Rapoport [HLR86].

Proposition 2.1. *Assume $\omega_\pi|_{F^\times} = 1$ in the local case. Then $\pi^\vee \cong \pi^\sigma$ if and only if $\pi$ is distinguished or $\omega_{E/F}$-distinguished with respect to $H$.*

The following is a closely related result.

Proposition 2.2. *Assume $\omega_\pi|_{F^\times} = 1$ in the local case. Then, $\pi$ is distinguished with respect to $H$ if and only if the Asai L-function $L(s, \mathrm{As}(\pi))$ has a pole at $s = 0$ ($s = 1$ in the global case).*

*Remark.* Both Propositions 2.1 and 2.2 are true for $\mathrm{GL}(n)$ with the added assumption in the local case that $\pi$ is a discrete series representation (in which case incidentally the condition $\omega_\pi|_{F^\times} = 1$ is also redundant) [AKT04, Fli88, Fli91, Hak91, Kab04].

Next we state a theorem of Flicker and Hakim [FH94, Theorem 0.3], which generalizes an earlier result of Jacquet-Lai [JL85], according to which the Jacquet–Langlands correspondence preserves distinction.

Proposition 2.3 (Flicker–Hakim). *Let $D$ be a quaternion algebra over a number field $F$. Let $\pi$ be a cuspidal representation of $D^\times(\mathbb{A}_E)$ and let $\pi^{JL}$ be its Jacquet–Langlands correspondent on $\mathrm{GL}_2(\mathbb{A}_E)$. Then $\pi$ is $D^\times(\mathbb{A}_F)$-distinguished if and only if $\pi^{JL}$ is $\mathrm{GL}_2(\mathbb{A}_F)$-distinguished and for each place $v$ of $F$ where $D$ is ramified, and $E$ is inert, if $\pi_v$ is a principal series, it is of the form $Ps(\mu^{-1}, \mu^\sigma)$.*

There is a local analogue of the above result proved independently by Hakim [Hak91] and the second author [Pra92], which we recall below.

Proposition 2.4. *Let $E$ be a quadratic extension of a local field $F$, and $D$ the quaternion division algebra over $F$. A discrete series representation $\pi$ of $\mathrm{GL}_2(E)$ is distinguished by $\mathrm{GL}_2(F)$ if and only if it is distinguished by $D^\times$.*

## 2.3 Multiplicity one for SL(2)

A crucial ingredient in several of our arguments in this paper is the following theorem of Ramakrishnan [Ram00, Theorem 4.1.2]. It is usually referred to in the literature as 'multiplicity one for SL(2)' since the multiplicity freeness of $L_0^2(\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F))$ is a consequence of this result by the work of Labesse–Langlands on the stable trace formula for SL(2) [LL79].

Theorem 2.5 (Multiplicity one for SL(2)). *Let $\pi$, $\pi'$ be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. Suppose for almost all $v$,*

$$\mathrm{Ad}(\pi_v) \cong \mathrm{Ad}(\pi'_v).$$

*Then there exists a Grössencharacter $\chi$ of $\mathbb{A}_F^\times/F^\times$, which is unique if $\pi$ is not CM, such that $\pi' \cong \pi \otimes \chi$.*

*Remark.* In this paper, we will mostly use the equivalent formulation: if $\pi_v$ and $\pi'_v$ are twists of each other by a character for almost all places $v$ of $F$, then they are twists of each other by a Grössencharacter.

### 2.4 Toric periods for GL(2)

In analyzing the toric periods for $\mathrm{SL}_1(D)$ where $D$ is a quaternion division algebra over $F$, our central tool is the well-known theorem of Waldspurger [Wal85, Theorem 2, p. 221], which we now state.

Theorem 2.6 (Waldspurger). *Let $D$ be a quaternion algebra over $F$, and let $\pi$ be a cuspidal automorphic representation of $D^\times(\mathbb{A}_F)$. Let $E$ be a quadratic extension of $F$ contained in $D$. Let $T = E^\times$ be the maximal torus of $D^\times$ defined over $F$ by $E$. Let $\Omega$ be a character of $T(F)\backslash T(\mathbb{A}_F)$ such that $\Omega|_{Z(\mathbb{A}_F)} = \omega_{\pi^{JL}}$. Then there exists $\phi \in \pi$ such that*

$$\int_{T(F)Z(\mathbb{A}_F)\backslash T(\mathbb{A}_F)} \phi(h)\Omega^{-1}(h)\, dh \neq 0$$

*if and only if:*

(i) *for every place $v$ of $F$, $\mathrm{Hom}_{T(F_v)}(\pi_v, \Omega_v) \neq 0$;*

(ii) *$L(\frac{1}{2}, BC(\pi) \otimes \Omega^{-1}) \neq 0$ (here $BC(\pi)$ denotes the base change of $\pi$ to $\mathrm{GL}_2(\mathbb{A}_E)$).*

Using a local theorem due to Saito–Tunnell to which we will turn soon, Theorem 2.6 can be reformulated as follows.

Theorem 2.7 (Waldspurger). *Let $E/F$ be a quadratic extension of number fields. Let $\pi$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let $\Omega$ be a Grössencharacter of $\mathbb{A}_E^\times/E^\times$ such that $\Omega|_{\mathbb{A}_F^\times} = \omega_\pi$. Then there exists a quaternion algebra $D$ over $F$ containing $E$ such that*

$$\int_{E^\times \mathbb{A}_F^\times\backslash \mathbb{A}_E^\times} \phi(h)\Omega^{-1}(h)\, dh \neq 0$$

*for a $\phi \in \pi^D$ if and only if $L(\frac{1}{2}, BC(\pi) \otimes \Omega^{-1}) \neq 0$. Here, $\pi^D$ is the Jacquet–Langlands correspondent of $\pi$.*

The local analogue of Waldspurger's result is the following theorem of Saito and Tunnell [Sai93, Tun83].

Proposition 2.8 (Saito–Tunnell). *Let $E/F$ be a quadratic extension of local fields. Let $\pi$ be an irreducible admissible representation of $\mathrm{GL}_2(F)$. If $\pi$ is in the discrete series, let $\pi'$ denote the corresponding irreducible representation of $D^\times$. Let $\Omega$ be a character of $E^\times$ such that $\Omega|_{F^\times} = \omega_\pi$. Then:*

(i) $\dim_{\mathbb{C}} \mathrm{Hom}_{E^\times}(\pi, \Omega) + \dim_{\mathbb{C}} \mathrm{Hom}_{E^\times}(\pi', \Omega) = 1$;

(ii) $\mathrm{Hom}_{E^\times}(\pi, \Omega) \neq 0 \iff \epsilon(\frac{1}{2}, BC(\pi) \otimes \Omega^{-1}, \psi') = 1$,
$\mathrm{Hom}_{E^\times}(\pi', \Omega) \neq 0 \iff \epsilon(\frac{1}{2}, BC(\pi) \otimes \Omega^{-1}, \psi') = -1$.

*Here, we take $\pi' = \{0\}$ if $\pi$ is a principal series representation, and $\psi'$ is any non-trivial character of $E$ which is trivial on $F$.*



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*Remark.* Suppose $\Omega$ factors through the norm, say $\Omega = \alpha \circ \mathbb{N}m$. Assume also that $\omega_\pi = 1$ (and thus $\alpha^2 = 1$). Then,

$$\epsilon(BC(\pi) \otimes \Omega^{-1}, \psi') = \omega_{E/F}(-1)\epsilon(\pi \otimes \alpha)\epsilon(\pi \otimes \alpha\omega_{E/F}). \tag{1}$$

Here and elsewhere, $\epsilon(\pi) = \epsilon(\frac{1}{2}, \pi, \psi)$, where $\psi$ is any non-trivial additive character of $F$.

## 3. Period integral for $\mathbf{GL_2}$ versus $\mathbf{SL_2}$

Suppose that $\widetilde{\pi}$ is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ where $E$ is a quadratic extension of a number field $F$. In this section, we write down an integral formula relating the period integral of automorphic functions in $\widetilde{\pi}$ along $\mathrm{SL}_2(\mathbb{A}_F)$ versus a similar period integral on $\mathrm{GL}_2(\mathbb{A}_F)$; this was considered in [AP06, § 3]. It allows one to prove that distinction by $\mathrm{SL}_2(\mathbb{A}_F)$ of an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ with trivial central character restricted to $\mathbb{A}_F^\times$ is the same as being $\omega$-distinguished for a quadratic character $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ (see [AP06, Proposition 3.3]). We note, as has been observed in [AP06, § 3], that an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ with non-trivial period integral on $\mathrm{SL}_2(\mathbb{A}_F)$ has a twist whose central character restricted to $\mathbb{A}_F^\times$ is trivial.

Before we proceed, note that if $\chi$ is a character of $k^\times$ we often abuse notation and continue to denote by $\chi$, the character $\chi \circ \det$ of $\mathrm{GL}_n(k)$.

The following is [AP06, Proposition 3.2], and is a simple consequence of elementary Fourier analysis.

Proposition 3.1. *Let $E$ be a quadratic extension of a number field $F$. Let $\phi$ be a cusp form on $\mathrm{GL}_2(\mathbb{A}_E)$ with central character which is trivial when restricted to $\mathbb{A}_F^\times$. Then*

$$\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \phi(g)\, dg = \sum_\omega \int_{\mathrm{GL}_2(F)\mathbb{A}_F^\times \backslash \mathrm{GL}_2(\mathbb{A}_F)} \phi(g)\omega(\det g)\, dg$$

*where the sum on the right-hand side of the equality sign is over all characters $\omega : \mathbb{A}_F^\times/F^\times \to \{\pm 1\}$.*

Consequently, we have the following proposition (see [AP06, Proposition 3.4]).

Proposition 3.2. *Let $\widetilde{\pi}$ be a cusp form on $\mathrm{GL}_2(\mathbb{A}_E)$ which is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$. Then there is a Grössencharacter $\eta$ of $\mathbb{A}_F^\times/F^\times$ such that $\widetilde{\pi}$ is $\eta$-distinguished for $\mathrm{GL}_2(\mathbb{A}_F)$. Conversely if $\widetilde{\pi}$ is $\eta$-distinguished for some Grössencharacter $\eta$ of $\mathbb{A}_F^\times/F^\times$, then $\widetilde{\pi}$ is $\mathrm{SL}_2(\mathbb{A}_F)$-distinguished. Hence there is a member of the $L$-packet of automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\widetilde{\pi}$ which is globally $\mathrm{SL}_2(\mathbb{A}_F)$-distinguished.*

The following proposition relates period integrals over $\mathbb{A}_E^1$ of automorphic forms of $\mathrm{GL}_2(\mathbb{A}_F)$ with period integrals over $\mathbb{A}_E^\times$. We omit the simple proof based on elementary Fourier analysis.

Proposition 3.3. *Let $E$ be a quadratic extension of a number field $F$. Let $\phi$ be a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central character. Then*

$$\int_{E^1\backslash \mathbb{A}_E^1} \phi(g)\, dg = \sum_\eta \int_{E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times} \phi(g)\eta(g)\, dg$$

*where the sum on the right-hand side of the equality sign is over all characters $\eta$ of the compact abelian group $E^\times \mathbb{A}_F^\times \mathbb{A}_E^1 \backslash \mathbb{A}_E^\times = E^\times \mathbb{A}_F^\times \mathbb{A}_E^{\times 2} \backslash \mathbb{A}_E^\times$.*

As a consequence, we have the following proposition.

Proposition 3.4. *If $\widetilde{\pi}$ is a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$ which is distinguished by $\mathbb{A}_E^1$, then there is a Grössencharacter $\eta$ of $\mathbb{A}_E^\times/E^\times$ such that $\widetilde{\pi}$ is $\eta$-distinguished for $\mathbb{A}_E^\times$. Conversely if $\widetilde{\pi}$ is $\eta$-distinguished for some Grössencharacter $\eta$ of $\mathbb{A}_E^\times/E^\times$, then $\widetilde{\pi}$ is $\mathbb{A}_E^1$-distinguished. Hence there is a member of the L-packet of automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ determined by $\widetilde{\pi}$ which is globally $\mathbb{A}_E^1$-distinguished.*

*Remark.* In Propositions 3.2 and 3.4, we have not assumed any condition on the central character of $\tilde{\pi}$. This is because the necessary condition on the central character in order to apply Propositions 3.1 and 3.3 respectively is automatic, after twisting by a Grössencharacter if necessary, by the assumption on distinction [AP06, Lemma 3.3].

## 4. Distinction as a functorial lift

In this section, we recast the well-known criterion (Proposition 2.1) about distinction of $\mathrm{GL}_2(E)$ representations to $\mathrm{SL}_2(E)$ in terms of the Langlands parameters. Because of Propositions 2.3 and 2.4, exactly the same criterion holds for a quaternion division algebra, but for the sake of simplicity of notation, we state the following theorem only for GL(2).

Theorem 4.1. *Let $E/F$ be a quadratic extension of non-Archimedean local fields. Then, an irreducible admissible representation $\pi$ of $\mathrm{GL}_2(E)$ is distinguished by $\mathrm{SL}_2(F)$ if and only if it belongs to the twisted base change map; i.e., a character twist of $\pi$ is a base change from $\mathrm{GL}_2(F)$. Exactly the same conclusion holds about global distinction of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$ with respect to $\mathrm{SL}_2(\mathbb{A}_F)$ when $E/F$ is a quadratic extension of number fields.*

*Proof.* We will write the argument below assuming $E/F$ is a quadratic extension of local fields, but the same argument works verbatim for number fields.

Let TBC denote the base change map from irreducible admissible representations of $\mathrm{GL}_2(F)$ to irreducible admissible representations of $\mathrm{GL}_2(E)$, both considered up to twists by characters. Thus, any representation in the image of TBC is of the form $\mathrm{BC}(\pi') \otimes \chi$ for a representation $\pi'$ of $\mathrm{GL}_2(F)$ and a character $\chi$ of $E^\times$.

We claim that the representation $\pi$ of $\mathrm{GL}_2(E)$ is distinguished by $\mathrm{GL}_2(F)$ with respect to a character $\eta$ of $F^\times$ if and only if $\pi$ is in the image of the twisted base change map. Since we are looking at representations modulo character twists, we can assume that $\eta = 1$, thus we assume that $\pi$ itself is distinguished, and therefore it follows that:

$$\omega_\pi|_{F^\times} = 1 \quad \text{and} \quad \pi^\vee \cong \pi^\sigma.$$

If we write $\omega_\pi = \mu^{-1}\mu^\sigma$ for a character $\mu$ of $E^\times$, then $\pi^\vee \cong \pi^\sigma$ implies that $\pi \otimes \mu$ is Galois stable and hence $\pi$ is in the image of TBC.

Conversely, if $\pi$ is of the form $\mathrm{BC}(\pi') \otimes \chi$, then we prove that $\pi$ is $\mathrm{SL}_2(F)$-distinguished. Without loss of generality, assume that $\pi = \mathrm{BC}(\pi')$.

Let $\omega'$ be the central character of $\pi'$, and let $\tilde{\omega}$ be an extension of $\omega'$ to $E^\times$. Then $\omega_\pi = \tilde{\omega} \cdot \tilde{\omega}^\sigma$, from which it can be checked that the representation $\pi'' = \pi \otimes \tilde{\omega}^{-1}$ has the property $\pi''^\vee \cong \pi''^\sigma$ and that the central character of $\pi''$ restricted to $F^\times$ is trivial. This shows that $\pi''$ is either distinguished or $\omega_{E/F}$-distinguished by $\mathrm{GL}_2(F)$, hence $\pi$ is $\eta$-distinguished for some character $\eta$ of $F^\times$. □



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This theorem allows one to interpret distinction in terms of existence of lifts,

$$\begin{array}{ccc} & & \mathrm{PGL}_2(\mathbb{C}) \times W_F \\ & \nearrow & \downarrow \\ W_F' & \longrightarrow & (\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})) \rtimes \mathrm{Gal}(E/F) \end{array}$$

where in the case that $F$ is a local field, $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$ is the Weil–Deligne group, and if $F$ is a global field, $W_F'$ needs to be replaced by the conjectural Langlands group whose irreducible $n$-dimensional complex representations classify cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$.

In the above picture, one could also ask the finer question about the number of distinct lifts of a given Langlands parameter, and it turns out that this question too has a nice answer: it is the dimension of the space of $\mathrm{SL}_2(F)$-invariant linear forms on an irreducible admissible representation of $\mathrm{SL}_2(E)$ with the given Langlands parameter [Pra00].

Observe that, in this language, the analogous question about lifts in the *untwisted* diagram

$$\begin{array}{ccc} & & \mathrm{PGL}_2(\mathbb{C}) \times W_F \\ & \nearrow & \downarrow \\ W_F' & \longrightarrow & (\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})) \times \mathrm{Gal}(E/F) \end{array}$$

locally asks if two representations of $\mathrm{GL}_2(F)$ are character twists of each other, and globally if they are twists of each other by a Grössencharacter of $\mathbb{A}_F^\times/F^\times$; thus in this case, a theorem of Dinakar Ramakrishnan (cf. Theorem 2.5) guarantees that local lifts in the above diagram imply a global lift, whereas a theorem, or rather a construction, of Blasius [Bla94] proves that existence of local lifts does not guarantee a global lift when $\mathrm{PGL}_2(\mathbb{C})$ is replaced by $\mathrm{PGL}_n(\mathbb{C})$.

## 5. Distinction in an $L$-packet for the pair $(\mathrm{SL}_2(E), \mathrm{SL}_2(F))$

In this section, we prove Theorem 1.1, which we restate here.

Theorem 1.1. *Let $\Pi$ be a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_E)$. If $\Pi$ appears in the restriction of a CM representation of $\mathrm{GL}_2(\mathbb{A}_E)$, assume that there is at least one square integrable component at a place of $E$ which is inert over the corresponding place $v_0$ of $F$. In the CM case, assume that either $\Pi$ is CM by three distinct quadratic extensions of $E$, or alternatively if it is CM by a unique quadratic extension of $E$, then at the place $v_0$, the local component is also CM by a unique quadratic extension of $E_{v_0}$ (or more generally, it is CM only by quadratic extensions which are Galois over $F_{v_0}$). Suppose each $\Pi_v$ is distinguished with respect to $\mathrm{SL}_2(F_v)$. Then there is a cuspidal representation in the $L$-packet of $\Pi$ which is distinguished with respect to $\mathrm{SL}_2(\mathbb{A}_F)$.*

*Proof of Theorem 1.1.* Let $\Pi = \otimes_v \Pi_v$ be a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_E)$ with each $\Pi_v$ distinguished by $\mathrm{SL}_2(F_v)$; here $v$ runs over the set of all places of $F$, and $\Pi_v$ are irreducible representations of $E_v = E \otimes_F F_v$. Let $\widetilde{\Pi} = \otimes_v \widetilde{\Pi}_v$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$ containing $\Pi$. We claim that there is a Grössencharacter $\chi$ of $\mathbb{A}_E^\times/E^\times$ such that

$$\widetilde{\Pi}^\sigma \cong \widetilde{\Pi}^\vee \otimes \chi.$$

To this end, observe that since for each $v$, the representation $\widetilde{\Pi}_v$ of $\mathrm{GL}_2(E_v)$ is given to be $\mathrm{SL}_2(F_v)$-distinguished, there is a character $\eta_v$ of $F_v^\times$ such that $\widetilde{\Pi}_v$ is $\eta_v^{-1}$-distinguished with respect to $\mathrm{GL}_2(F_v)$. Furthermore, if $\widetilde{\eta}_v$ denotes an extension of $\eta_v$ to $E_v^\times$, then $\widetilde{\Pi}_v \otimes \widetilde{\eta}_v$ is distinguished with respect to $\mathrm{GL}_2(F_v)$, and this implies that (cf. Proposition 2.1)

$$(\widetilde{\Pi}_v \otimes \widetilde{\eta}_v)^\vee \cong (\widetilde{\Pi}_v \otimes \widetilde{\eta}_v)^\sigma,$$

or

$$\widetilde{\Pi}_v^\sigma \cong \widetilde{\Pi}_v^\vee \otimes \eta_v \circ \mathbb{N}m.$$

By the theorem of Ramakrishnan (cf. Theorem 2.5), if two automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$ are locally twists of each other at all places of a number field $E$, then they are globally twists of each other by a Grössencharacter $\chi$ on $\mathbb{A}_E^\times/E^\times$, proving our claim that $\widetilde{\Pi}^\sigma \cong \widetilde{\Pi}^\vee \otimes \chi$.

For the proof of Theorem 1.1, it suffices to prove that there is a Grössencharacter $\chi$ on $\mathbb{A}_E^\times/E^\times$ with $\chi^\sigma = \chi$, and with $\widetilde{\Pi}^\sigma \cong \widetilde{\Pi}^\vee \otimes \chi$, because then one can write $\chi^{-1} = \mu\mu^\sigma$, which means that $(\widetilde{\Pi} \otimes \mu)^\sigma \cong (\widetilde{\Pi} \otimes \mu)^\vee$, and hence $\widetilde{\Pi} \otimes \mu$ is either $\mathrm{GL}_2(\mathbb{A}_F)$-distinguished, or $\omega_{E/F}$-distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$. This means that some member in the global $L$-packet determined by the automorphic representation $\Pi$ of $\mathrm{SL}_2(\mathbb{A}_E)$ has a non-trivial period integral on $\mathrm{SL}_2(\mathbb{A}_F)$.

We first treat the case when $\widetilde{\Pi}$ is non-CM. In this case, $\widetilde{\Pi}^\sigma \cong \widetilde{\Pi}^\vee \otimes \chi$ implies $\widetilde{\Pi}^\sigma \cong \widetilde{\Pi}^\vee \otimes \chi^\sigma$, and therefore since $\widetilde{\Pi}$ does not have CM, $\chi^\sigma = \chi$.

We now assume that $\widetilde{\Pi}$ has CM and that there is a place of $F$ inert in $E$, say $v_0$ in $E$, such that $\widetilde{\Pi}_{v_0}$ is square integrable. Suppose the assertion of the theorem is not true. Then $\widetilde{\Pi}^\sigma \cong \widetilde{\Pi}^\vee \otimes \chi$ with $\chi \neq \chi^\sigma$. Let $\Sigma$ be the two-dimensional representation of $W_E$ associated to the CM representation $\widetilde{\Pi}$. The isomorphism $\widetilde{\Pi}^\sigma \cong \widetilde{\Pi}^\vee \otimes \chi$ translates into $\widetilde{\Sigma}^\sigma \cong \widetilde{\Sigma}^\vee \otimes \chi$, and it is this isomorphism of two-dimensional representations of $W_E$ that we will analyze. This isomorphism gives us,

$$\Sigma \otimes \Sigma^\sigma \cong \chi \oplus \chi^\sigma \oplus \rho,$$

for a certain two-dimensional representation $\rho$ of $W_E$ which is invariant under $\sigma$. We will now look at the above decomposition at the place $v_0$ of $E$:

$$\Sigma_{v_0} \otimes \Sigma_{v_0}^\sigma \cong \chi_{v_0} \oplus \chi_{v_0}^\sigma \oplus \rho_{v_0}.$$

Note that $\chi_{v_0} \neq \chi_{v_0}^\sigma$, since $\widetilde{\Pi}_{v_0}$ is square integrable and thus corresponds to an irreducible representation $\Sigma_{v_0}$ of $W_{E_{v_0}}$, and therefore each character in the decomposition of $\Sigma_{v_0} \otimes \Sigma_{v_0}^\sigma$ appears with multiplicity 1 by Schur's lemma, forcing $\chi_{v_0} \neq \chi_{v_0}^\sigma$.

If $\widetilde{\Pi}$ has CM by three quadratic extensions, then $\Sigma$ has self-twists by three quadratic characters, forcing $\Sigma \otimes \Sigma^\sigma$, which contains a character, to be a sum of four characters permuted amongst themselves by $\sigma$. Therefore, $\rho$ is a sum of two characters which we assume is of the form $\mu \oplus \mu^\sigma$ with $\mu \neq \mu^\sigma$, as the other possibilities create a $\sigma$-invariant character of $\mathbb{A}_E^\times/E^\times$. But again by Schur's lemma, we must have $\mu_{v_0} \neq \mu_{v_0}^\sigma$, which is contradictory to our assumption of having a $\sigma$-invariant character inside $\Sigma_v \otimes \Sigma_v^\sigma$ at all places $v$ of $E$, completing the proof of the theorem when $\widetilde{\Pi}$ has CM by three quadratic extensions.

Since

$$\Sigma_{v_0} \otimes \Sigma_{v_0}^\sigma$$

contains a $\sigma$-invariant character, and $\chi_{v_0} \neq \chi_{v_0}^\sigma$, in the decomposition

$$\Sigma_{v_0} \otimes \Sigma_{v_0}^\sigma \cong \chi_{v_0} \oplus \chi_{v_0}^\sigma \oplus \rho_{v_0}$$

$\rho_{v_0}$ must be a sum of two $\sigma$-invariant characters, say $\rho_{v_0} = \mu_1 + \mu_2$, thus,

$$\Sigma_{v_0} \otimes \Sigma_{v_0}^\sigma \cong \chi_{v_0} \oplus \chi_{v_0}^\sigma \oplus \mu_1 \oplus \mu_2.$$

This clearly implies that $\Sigma_{v_0}$ has self-twist by $\mu_1/\mu_2$ which is $\sigma$-invariant, as well as by $\chi_{v_0}/\mu_1$ which is not $\sigma$-invariant.



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This restricts possibilities regarding $\Pi_{v_0}$, proving the theorem if $\Pi_{v_0}$ has CM either by a unique quadratic extension of $E_{v_0}$, or by three quadratic extensions of $E_{v_0}$ which are all Galois over $F_{v_0}$. $\square$

## 6. Tensor induction, or Asai lift

In the study of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$ which are distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$, the Asai lift plays an important role, and it does so in our work on the corresponding questions for $\mathrm{SL}_2$. The specific aim of this section will be to determine the fibers of the Asai lift $\widetilde{\pi} \to \mathrm{As}(\widetilde{\pi})$ from automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$ to automorphic representations of $\mathrm{GL}_4(\mathbb{A}_F)$. This question was discussed by Krishnamurthy in [Kri03]; however, in the case where it really concerns us, the case of CM representations of $\mathrm{GL}_2(\mathbb{A}_E)$, his result was incomplete exactly in the place where it matters to us. We have completed his work in this section.[1]

We begin this section by carefully recalling the notion of *tensor induction*, also called Asai lift in a particular case (when the subgroup involved is of index 2), which is a purely group theoretic notion.

Let $H$ be a subgroup of a group $G$ of finite index $n$, and $\mathcal{G}$ an arbitrary group. Define $\mathcal{G}^{G/H}$ to be the set of all set theoretic maps $\phi : G \to \mathcal{G}$ such that $\phi(gh) = \phi(g)$ for all $g \in G, h \in H$. Clearly $\mathcal{G}^{G/H}$ is a group with a natural action of $G$ on the left, so we can form the semi-direct product $\mathcal{G}^{G/H} \rtimes G$.

It is easy to prove the following lemma, which is nothing but a form of Frobenius reciprocity for induced representations in this context.

Lemma 6.1. *There exists a natural bijection*

$$\mathrm{Hom}\,(H, \mathcal{G})/{\sim} \longleftrightarrow \mathrm{Hom}\,(G, \mathcal{G}^{G/H} \rtimes G)/{\sim},$$

*where we consider only those homomorphisms in* $\mathrm{Hom}\,(G, \mathcal{G}^{G/H} \rtimes G)$, *whose composition with the natural map from* $\mathcal{G}^{G/H} \rtimes G$ *to* $G$ *is the identity map from* $G$ *to* $G$*; the equivalence relation on the left-hand side is conjugation by* $\mathcal{G}$, *and on the right is conjugation by* $\mathcal{G}^{G/H}$.

Now given a representation $(\pi, V)$ of $\mathcal{G}$, it gives rise to a representation $\otimes^{G/H} V$ of $\mathcal{G}^{G/H}$ which clearly extends to one of the semi-direct product $\mathcal{G}^{G/H} \rtimes G$. Taking $\mathcal{G}$ to be $\mathrm{GL}(V)$ with its natural representation on $V$, the previous lemma allows one to associate to a representation $(\pi, V)$ of $H$ of dimension $d$, a representation of $G$, to be denoted by $\mathrm{As}(V)$, of dimension $d^n$, called the tensor induction, or the Asai lift of the representation $(\pi, V)$ of $H$.

For a vector space $W$ over $\mathbb{C}$ equipped with a quadratic form $q$ on it, there is the notion of the orthogonal similitude group $\mathrm{GO}(W)$, defined by

$$\mathrm{GO}(W) = \{g \in \mathrm{GL}(W) \mid q(gw) = \lambda(g) q(w) \forall w \in W\};$$

the map $g \to \lambda(g) \in \mathbb{C}^\times$ is a character on $\mathrm{GO}(W)$, called the similitude character. If $W$ is of even dimension, the special orthogonal similitude group, denoted by $\mathrm{GSO}(W)$, which is the connected component of identity of $\mathrm{GO}(W)$, is defined by

$$\mathrm{GSO}(W) = \{g \in \mathrm{GO}(W) \mid \lambda(g)^{\dim W/2} = \det g\}.$$

The following well-known result lies at the basis of our proof. It can itself be considered as a local-global principle for orthogonal groups, eventually responsible for multiplicity one

[1] There is a recent paper of Krishnamurthy where he too completes his earlier work [Kri12].

(conjecture) for automorphic representations of orthogonal groups, or more generally any classical group [Lar94].

Lemma 6.2. *Let $W$ be a finite-dimensional vector space over $\mathbb{C}$ together with a quadratic form $q$ on it. Suppose $\pi_1$ and $\pi_2$ are two representations of a group $G$ into $\mathrm{GO}(W)$ such that the similitude characters $\lambda_1$ and $\lambda_2$ of $\pi_1$ and $\pi_2$ are the same. Then the representations $\pi_1$ and $\pi_2$ of $G$ with values in $\mathrm{GO}(W)$ are equivalent, i.e., conjugate in $\mathrm{GO}(W)$, if and only if they are equivalent in $\mathrm{GL}(W)$.*

With these generalities in place, we now come to the special situation afforded by two-dimensional representations of a subgroup $N$ of index 2 in a group $G$. In this case, we find it more convenient to use a concrete realization of $\mathrm{GO}(4, \mathbb{C})$, which we realize on the space $M(2, \mathbb{C})$ of $2 \times 2$ matrices with $X \to \det X$ as the quadratic form. Clearly, $(A, B) \in \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ acting on $M(2, \mathbb{C})$ as $X \to A \cdot X \cdot {}^tB$ defines a mapping from $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ onto $\mathrm{GSO}(4, \mathbb{C})$, and the involution $X \to {}^tX$ belongs to $\mathrm{O}(4, \mathbb{C})$ but not to $\mathrm{SO}(4, \mathbb{C})$. Thus, we have an exact sequence,

$$1 \to \mathbb{C}^\times \to [\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})] \rtimes \mathbb{Z}/2 \to \mathrm{GO}(4, \mathbb{C}) \to 1,$$

where $\mathbb{C}^\times$ sits inside $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ as scalar matrices $(z, z^{-1})$.

From Lemma 6.1, a representation $\pi_1$ of $N$ into $\mathrm{GL}_2(\mathbb{C})$ gives rise to a homomorphism of $G$ into $[\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})] \rtimes \mathbb{Z}/2$ whose projection to $\mathbb{Z}/2$ is nothing but the natural projection from $G$ to $G/N = \mathbb{Z}/2$ (cf. Lemma 6.1 and the definition of the Asai lift). It will be convenient at this point to use the language of cohomology of groups (with non-abelian coefficients). In this language, we have an exact sequence of $G$-groups:

$$1 \to \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GSO}(4, \mathbb{C}) \to 1,$$

where $\mathbb{C}^\times$ is the $G$-module on which $N$ operates trivially, and the non-trivial element of $G/N$ operates on $\mathbb{C}^\times$ by $z \to z^{-1}$; the group $G$ operates on $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$, and $\mathrm{GSO}(4, \mathbb{C})$ via $G/N$ which acts by permuting the factors in $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$, and by the automorphism $X \to {}^tX$ of $M_2(\mathbb{C})$ acting on $\mathrm{GSO}(4, \mathbb{C})$ by conjugation. This exact sequence of $G$-groups gives rise to an exact sequence of pointed sets:

$$H^1(G, \mathbb{C}^\times) \to H^1(G, \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})) \to H^1(G, \mathrm{GSO}(4, \mathbb{C})).$$

Since $\mathbb{C}^\times$ is a central subgroup of $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ (sitting as scalar matrices $(z, z^{-1})$), it follows that two elements of $H^1(G, \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))$ have the same image in $H^1(G, \mathrm{GSO}(4, \mathbb{C}))$ if and only if they differ by an element of $H^1(G, \mathbb{C}^\times)$ (see [Ser02, Part I, § 5, Proposition 42]).

In terms of group cohomology, we have the identifications

$$\frac{\mathrm{Hom}\,[G, [\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})] \rtimes \mathbb{Z}/2]}{\sim} \longleftrightarrow H^1(G, \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})),$$

and

$$\frac{\mathrm{Hom}\,[G, \mathrm{GO}(4, \mathbb{C})]}{\sim} \longleftrightarrow H^1(G, \mathrm{GSO}(4, \mathbb{C})),$$

where $\sim$ denotes the equivalence relation on the set of homomorphisms given by conjugation by $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ (respectively $\mathrm{GSO}(4, \mathbb{C})$).

It follows that two homomorphisms $\phi_1$ and $\phi_2$ of $G$ to $(\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})) \rtimes \mathbb{Z}/2$ which give rise to the same representation of $G$ with values in $\mathrm{GO}(4, \mathbb{C})$ differ by an element of $H^1(G, \mathbb{C}^\times)$ which we calculate in the following lemma.

Lemma 6.3. *Let $N$ be an index 2 subgroup of a group $G$. Let $\mathbb{C}^\times$ be the $G$ module on which $N$ operates trivially, and the non-trivial element of $G/N$ acts on $\mathbb{C}^\times$ by $z \to z^{-1}$. Then $H^1(G, \mathbb{C}^\times)$*

*can be identified to the set of those characters of $N$ with values in $\mathbb{C}^\times$ whose transfer to $G$ is trivial.*

*Proof.* Note the exact sequence,

$$0 \longrightarrow H^1(G/N, \mathbb{C}^\times) \longrightarrow H^1(G, \mathbb{C}^\times) \longrightarrow H^1(N, \mathbb{C}^\times)^{G/N} \longrightarrow H^2(G/N, \mathbb{C}^\times),$$

in which $G/N = \mathbb{Z}/2$. From well-known calculations on cohomology of cyclic groups, it is easy to see that $H^1(\mathbb{Z}/2, \mathbb{C}^\times) = 0$, and $H^2(\mathbb{Z}/2, \mathbb{C}^\times) = \mathbb{Z}/2$. (In this lemma, $\mathbb{C}^\times$ comes equipped with the action of $G/N = \mathbb{Z}/2$ by $z \to z^{-1}$.) So, the above exact sequence can be written as:

$$0 \longrightarrow H^1(G, \mathbb{C}^\times) \longrightarrow H^1(N, \mathbb{C}^\times)^{G/N} \longrightarrow H^2(G/N, \mathbb{C}^\times).$$

Since $N$ operates trivially on $\mathbb{C}^\times$, $H^1(N, \mathbb{C}^\times)$ is simply the character group of $N$. The group $G/N$ operates on $H^1(N, \mathbb{C}^\times)$ by sending a character $\phi \in H^1(N, \mathbb{C}^\times)$ to the character $\phi^g(n) = g^{-1}\phi(gng^{-1})$ of $N$. It follows that $H^1(N, \mathbb{C}^\times)^{G/N}$ can be identified to the group of characters $\phi$ of $N$ for which $\phi^{-1}(n) = \phi(g_0 n g_0^{-1})$ where $g_0$ is any element of $G$ not in $N$; these are simply the characters of $N$ which when composed with the transfer map from $G/[G, G]$ to $N/[N, N]$ are trivial on $N$. To get the conclusion of the lemma, we need to prove that among these characters of $N$, those which go to 0 under the boundary map $H^1(N, \mathbb{C}^\times)^{G/N} \longrightarrow H^2(G/N, \mathbb{C}^\times)$, are exactly those whose transfer to $G$ is trivial (and not just restriction to the subgroup $N$ which is of index 2). Observe that the transfer map from $G$ to $N$ on elements of $G$ outside $N$ is simply the squaring map $g \to g^2$. So we need to prove that if a character $\phi$ in $H^1(N, \mathbb{C}^\times)^{G/N}$ goes to zero in $H^2(G/N, \mathbb{C}^\times)$, then $\phi(g_0^2) = 1$ where $g_0$ is any element of $G$ not in $N$. For this we need to interpret this boundary map, which is nothing but the push-out diagram under the homomorphism $\phi : N \to \mathbb{C}^\times$ of the exact sequence $0 \to N \to G \to \mathbb{Z}/2 \to 0$, thus fits in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \square & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \end{array}$$

To say that the push-out diagram is trivial, i.e., the short exact sequence

$$0 \longrightarrow \mathbb{C}^\times \longrightarrow \square \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

splits, is clearly equivalent to saying that $\phi(g_0)^2 = 1$, so the proof of the lemma is completed. □

In the following proposition, for a character $\chi$ of $N$, let $r(\chi)$ be the character of $G$ obtained from $\chi$ using the transfer map from $G/[G, G]$ to $N/[N, N]$. (Note that $r(\chi)$ is the special case of the tensor induction corresponding to one-dimensional representations.) The previous lemma proves the following proposition which is the main result of this section.

Proposition 6.4. *Let $N$ be an index 2 subgroup of a group $G$, and let $\pi_1$ and $\pi_2$ be two two-dimensional representations of $N$, with $\mathrm{As}(\pi_1)$ and $\mathrm{As}(\pi_2)$ of dimension 4 their tensor induction to $G$. Assume that $r(\det \pi_1) = r(\det \pi_2)$. Then $\mathrm{As}(\pi_1) \cong \mathrm{As}(\pi_2)$ if and only if $\pi_1 \cong \pi_2 \otimes \chi$, or $\pi_1^\sigma \cong \pi_2 \otimes \chi$, for a character $\chi$ of $N$ with $r(\chi) = 1$.*

*Proof.* By our assumption on $r \circ \det$ we can appeal to Lemma 6.2 and thus it is enough to check $\mathrm{As}(\pi_1) = \mathrm{As}(\pi_2)$ inside $\mathrm{GO}(4, \mathbb{C})$. Thus, $\pi_2$ differs from $\pi_1$ (or $\pi_1^\sigma$, with $G/N = \langle\sigma\rangle$) by an element of $H^1(G, \mathbb{C}^\times)$; the ambiguity in $\pi_1$ and $\pi_1^\sigma$ arises since $\sim$ in the identification of $H^1(G, \mathrm{GSO}(4, \mathbb{C}))$

with Hom $[G, \mathrm{GO}(4,\mathbb{C})]/\sim$ does not capture conjugacy of homomorphisms from $G$ to $\mathrm{GO}(4,\mathbb{C})$ by $\mathrm{GO}(4,\mathbb{C})$ but only conjugacy by $\mathrm{GSO}(4,\mathbb{C})$. By Lemma 6.3, $H^1(G,\mathbb{C}^\times)$ corresponds to a character $\chi$ of $N$ with $r(\chi)=1$. □

The abstract group theoretic proof given above for the fibers of the map $\pi \to \mathrm{As}(\pi)$, yields an exact description of the fibers of the Asai lift from automorphic forms on $\mathrm{GL}_2(\mathbb{A}_E)$ to automorphic forms on $\mathrm{GL}_4(\mathbb{A}_F)$ for CM representations of $\mathrm{GL}_2(\mathbb{A}_E)$. Luckily, non-CM representations were already handled by Krishnamurthy in [Kri03], so this description of the fibers holds in all cases. Thus, we have the following theorem.

Theorem 6.5. *Let $\pi_1$ and $\pi_2$ be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$ such that their central characters agree on $\mathbb{A}_F^\times$. Then they have isomorphic Asai lifts to $\mathrm{GL}_4(\mathbb{A}_F)$ if and only if either $\pi_1 \cong \pi_2 \otimes \chi$ or $\pi_1^\sigma \cong \pi_2 \otimes \chi$, for a Grössencharacter $\chi$ of $\mathbb{A}_E^\times/E^\times$ with $\chi|_{\mathbb{A}_F^\times} = 1$.*

The proof of Proposition 6.4 also gives a proof of the following proposition which, however, we will not have occasion to use.

Proposition 6.6. *Let $V_1, V_2, W_1, W_2$ be two-dimensional representations of a group $G$ such that*

$$V_1 \otimes V_2 \cong W_1 \otimes W_2,$$

*and*

$$\det(V_1)\det(V_2) = \det(W_1)\det(W_2).$$

*Then there exists a character $\chi$ of $G$ such that*

$$V_1 \cong \chi \otimes W_1, \quad V_2 \cong \chi^{-1} \otimes W_2,$$

*or,*

$$V_2 \cong \chi \otimes W_1, \quad V_1 \cong \chi^{-1} \otimes W_2.$$

*Remark.* A weaker version of this proposition was proved in [MP00] in which $V_1$ and $V_2$ were assumed to be non-CM representations, which went into the proof of [Kri03].

*Question.* Since $(U_1 \otimes U_2) \otimes U_3 \cong U_1 \otimes (U_2 \otimes U_3)$, there is no simple way to generalize the previous proposition for larger dimensional representations, except possibly when, in the notation of the proposition, $\dim V_1$ and $\dim V_2$ are prime. Similarly, since $\mathrm{As}(U_1 \otimes U_2^\sigma) \cong \mathrm{As}(U_1 \otimes U_2)$, there is no simple generalization of the proposition about fibers of the Asai lift of two-dimensional representations except possibly when dealing with representations of prime dimension. We do not know if in these special cases in which representations involved are of prime dimension, fibers of Asai lift or of tensor product are as described in Propositions 6.4 and 6.6.

## 7. Local-global principle for the pair $(\mathrm{SL}_2(E), \mathrm{SL}_2(F))$

In this section, we work inside an $L$-packet to prove the local-global principle for automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ with respect to $\mathrm{SL}_2(\mathbb{A}_F)$. This is Theorem 1.2, recalled here for the convenience of the reader.

Theorem 1.2. *Let $\Pi$ be a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_E)$ which is globally distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$. Let $\Pi' = \otimes_v \Pi'_v$ be an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_E)$ in the same $L$-packet as $\Pi$ such that $\Pi'_v$ is locally distinguished by $\mathrm{SL}_2(F_v)$ at all the places of $F$. Then $\Pi'$ is globally distinguished.*

Before we begin the proof of this theorem, we make a review of the theory of $L$-packets, both locally as well as globally for $\mathrm{SL}_2$, which is due to Labesse–Langlands [LL79], and also review some of our own work about distinguished representations relevant to this study [AP03, AP06].

We deal with the pair $(\mathrm{SL}_2(\mathbb{A}_E), \mathrm{SL}_2(\mathbb{A}_F))$ in this section, making an essential use of the theory of Whittaker models, and then in a later section (cf. § 9) observe that some of our work carries over to the more general situation of the group of norm one elements of a quaternion algebra.

Note that the group $\mathbb{A}_E^\times$ sitting inside $\mathrm{GL}_2(\mathbb{A}_E)$ as

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

operates on $\mathrm{SL}_2(\mathbb{A}_E)$ via conjugation action, and therefore on the set of isomorphism classes of representations of $\mathrm{SL}_2(\mathbb{A}_E)$. The orbit of $\Pi = \otimes \Pi_v$, an irreducible representation of $\mathrm{SL}_2(\mathbb{A}_E)$, under the action of $\mathbb{A}_E^\times$ is precisely the global $L$-packet of representations of $\mathrm{SL}_2(\mathbb{A}_E)$ containing $\Pi$. Let $G_\Pi \subset \mathbb{A}_E^\times$, $G_\Pi = \prod G_v$ be the stabilizer of the isomorphism class of the representation $\Pi = \otimes \Pi_v$, where $G_v$ is the stabilizer inside $E_v^\times$ of the isomorphism class of the representation $\Pi_v$ of $\mathrm{SL}_2(E_v)$. It can be seen that $G_v$ contains $\mathcal{O}_v^\times$ for almost all primes $v$ of $E$, where $\mathcal{O}_v$ is the ring of integers of $E_v$, and so $G_\Pi$ is an open (and hence closed) subgroup of $\mathbb{A}_E^\times$.

Clearly, the action of $E^\times$ on $\mathrm{SL}_2(\mathbb{A}_E)$ takes automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ to automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$. Since every cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A}_E)$ must have a Whittaker model for a non-trivial character of $\mathbb{A}_E/E$, and any two non-trivial characters of $\mathbb{A}_E/E$ are conjugate by $E^\times$, it follows from the uniqueness of Whittaker models (for $\mathrm{GL}_2(\mathbb{A}_E)$!) that $E^\times$ acts transitively on the set of automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ which are in the same $L$-packet as $\Pi$.

There is another way of interpreting $G_\Pi = \prod G_v$. For this, let $\widetilde{\Pi}$ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ containing $\Pi$. Then, for a character $\omega : \mathbb{A}_E^\times \to \mathbb{C}^\times$, $\widetilde{\Pi} \otimes \omega \cong \widetilde{\Pi}$ if and only if $\omega$ is trivial on $G_\Pi$. This implies that $\mathbb{A}_E^\times/(E^\times G_\Pi)$ is a finite group whose character group is nothing but the finite group of Grössencharacters $\omega$ of $\mathbb{A}_E^\times/E^\times$ such that $\widetilde{\Pi} \otimes \omega \cong \widetilde{\Pi}$.

From the previous observations, we note that a representation of $\mathrm{SL}_2(\mathbb{A}_E)$ which belongs to the $L$-packet determined by $\Pi$ determines an element of the finite group $\mathbb{A}_E^\times/(E^\times G_\Pi)$ (which is known to be either $\{1\}$, $\mathbb{Z}/2$, or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$), which is trivial if and only if the corresponding representation is automorphic. This result due to Labesse and Langlands [LL79] remains true for division algebras, but this simple proof does not work.

We next review the work in [AP03, AP06] relevant to the local-global study of the pair $(\mathrm{SL}_2(\mathbb{A}_E), \mathrm{SL}_2(\mathbb{A}_F))$.

It follows from [AP06, Theorem 4.2] that if $\Pi$ is globally distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$, then $\Pi$ has a Whittaker model with respect to a character $\psi : \mathbb{A}_E/(E\mathbb{A}_F) \to \mathbb{C}^\times$. Conversely, by the same theorem, if $\Pi$ has a Whittaker model with respect to $\psi : \mathbb{A}_E/(E\mathbb{A}_F) \to \mathbb{C}^\times$, and some member in the $L$-packet determined by $\Pi$ is globally distinguished, then such a $\Pi$ is itself globally distinguished. A similar local result also holds: in a local $L$-packet of $\mathrm{SL}_2(E_v)$ where at least one representation is distinguished by $\mathrm{SL}_2(F_v)$, the $\mathrm{SL}_2(F_v)$-distinguished representations are precisely those which have a Whittaker model with respect to a non-trivial character of $E_v/F_v$ (see [AP03, Lemmas 3.1 and 3.2]), hence since $F_v^\times$ acts transitively on non-trivial characters of $E_v/F_v$, $F_v^\times$ acts transitively on distinguished members of an $L$-packet of representations of $\mathrm{SL}_2(E_v)$.



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Define groups,

$$H_0 = \mathbb{A}_E^\times,$$
$$H_1 = \mathbb{A}_F^\times G_\Pi,$$
$$H_2 = E^\times G_\Pi,$$
$$H_3 = F^\times G_\Pi.$$

From these theorems due to Labesse–Langlands [LL79], and the theorems due to the authors in [AP03, AP06] we deduce the following.

(i) The set $H_0 \cdot \Pi$ is the $L$-packet of representations of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\Pi$.

(ii) The set $H_1 \cdot \Pi$ is the set of locally distinguished representations in the $L$-packet of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\Pi$.

(iii) The set $H_2 \cdot \Pi$ is the set of automorphic representations in the $L$-packet of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\Pi$.

(iv) The set $H_3 \cdot \Pi$ is the set of globally distinguished representations in the $L$-packet of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\Pi$.

Clearly $H_1 \cap H_2$ contains $H_3$, and $(H_1 \cap H_2)/H_3$ measures the obstruction to locally distinguished automorphic representations to be globally distinguished; equivalently, a locally distinguished automorphic representation $\Pi$ of $\mathrm{SL}_2(\mathbb{A}_E)$ determines an element $h_\Pi$ of $H_1 \cap H_2$ such that $\Pi$ is globally distinguished if and only if $h_\Pi \in H_3$. We will in fact prove that $(H_1 \cap H_2)/H_3$ is trivial by proving that its character group is trivial.

Let $X(A)$ denote the character group of a locally compact abelian group $A$.

Noting that $(H_1 \cap H_2)/H_3$ is nothing but the kernel of the map,

$$H_1/H_3 \to H_0/H_2,$$

the character group of $(H_1 \cap H_2)/H_3$ is the cokernel of the natural map

$$X(H_0/H_2) \to X(H_1/H_3).$$

We note that the mapping of the character groups is simply the map taking a character $\alpha$ of $H_0$ which is trivial on $H_2$ to its restriction to $H_1$; note that since $\alpha$ is trivial on $H_2$, it is in particular trivial on $H_3$ which is a subgroup of $H_2$.

Theorem 7.1. *The group $(H_1 \cap H_2)/H_3$, which measures the difference between locally distinguished automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ and globally distinguished automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$, is trivial.*

*Proof.* We will prove that $(H_1 \cap H_2)/H_3$ is trivial by proving that its character group is trivial. From the analysis above, it suffices to prove the surjectivity of the natural map

$$X(H_0/H_2) \to X(H_1/H_3).$$

Equivalently, we need to prove that a character of $\mathbb{A}_F^\times/[F^\times(\mathbb{A}_F^\times \cap G_\Pi)]$, can be extended to a Grössencharacter of $\mathbb{A}_E^\times/E^\times$, which is a self-twist of $\widetilde{\Pi}$, where $\widetilde{\Pi}$ is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$ such that $\Pi$ appears in its restriction to $\mathrm{SL}_2(\mathbb{A}_E)$. Without loss of generality, we may also assume that $\widetilde{\Pi}$ is globally distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$ (cf. Proposition 3.2).

Let $\chi$ be a character of $\mathbb{A}_F^\times/[F^\times(\mathbb{A}_F^\times \cap G_\Pi)]$, thought of as a character of $\mathbb{A}_F^\times/[\mathbb{A}_F^\times \cap G_\Pi]$. Since $\mathbb{A}_F^\times/[\mathbb{A}_F^\times \cap G_\Pi]$ is a subgroup of the discrete group $\mathbb{A}_E^\times/G_\Pi$, there is a character $\widetilde{\chi}$ of $\mathbb{A}_E^\times$ trivial on $G_\Pi$ extending $\chi$. (We will eventually try to get one which is a Grössencharacter.)

Let $\mathrm{As}(\widetilde{\Pi})$ denote the Asai lift of a representation of $\mathrm{GL}_2(\mathbb{A}_E)$ to $\mathrm{GL}_4(\mathbb{A}_F)$. By local considerations, it is clear that

$$\mathrm{As}(\widetilde{\Pi} \otimes \widetilde{\chi}) \cong \mathrm{As}(\widetilde{\Pi}) \otimes \chi.$$

Since $\widetilde{\chi}$ is trivial on $G_\Pi$, $\widetilde{\Pi} \otimes \widetilde{\chi} = \widetilde{\Pi}$, and hence

$$\mathrm{As}(\widetilde{\Pi} \otimes \widetilde{\chi}) \cong \mathrm{As}(\widetilde{\Pi}) \cong \mathrm{As}(\widetilde{\Pi}) \otimes \chi.$$

Now let $\widehat{\chi}$ be a character of $\mathbb{A}_E^\times/E^\times$ extending the character $\chi$ of $\mathbb{A}_F^\times/F^\times$. We have,

$$\begin{aligned}\mathrm{As}(\widetilde{\Pi} \otimes \widehat{\chi}) &\cong \mathrm{As}(\widetilde{\Pi}) \otimes \chi \\ &\cong \mathrm{As}(\widetilde{\Pi}).\end{aligned}$$

Since the Asai lifts of the two automorphic representations $\widetilde{\Pi}$ and $\widetilde{\Pi} \otimes \widehat{\chi}$ of $\mathrm{GL}_2(\mathbb{A}_E)$ to $\mathrm{GL}_4(\mathbb{A}_F)$ are isomorphic, we can use Theorem 6.5 to conclude a relationship between $\widetilde{\Pi}$ and $\widetilde{\Pi} \otimes \widehat{\chi}$. Before we can apply this proposition, we need to check that the two representations $\widetilde{\Pi}$ and $\widetilde{\Pi} \otimes \widehat{\chi}$ have the same central characters restricted to $\mathbb{A}_F^\times$. But this follows as $\mathbb{A}_E^{\times 2} \subset G_\Pi$, and hence $\chi^2 = 1$.

By Theorem 6.5, there are two cases.

*Case* 1. There is a character $\chi_1$ of $\mathbb{A}_E^\times/E^\times$ trivial on $\mathbb{A}_F^\times/F^\times$ such that

$$\widetilde{\Pi} \otimes \widehat{\chi} \cong \widetilde{\Pi} \otimes \chi_1.$$

Therefore, $\widetilde{\Pi} \cong \widetilde{\Pi} \otimes (\chi_1^{-1}\widehat{\chi})$. Since $\chi_1$ is trivial on $\mathbb{A}_F^\times/F^\times$, the character $\chi_1^{-1}\widehat{\chi}$ is an extension of $\chi$ to a Grössencharacter on $\mathbb{A}_E^\times/E^\times$, which is a self-twist of $\widetilde{\Pi}$, proving the desired statement in this case.

*Case* 2. There is a character $\chi_1$ of $\mathbb{A}_E^\times/E^\times$ trivial on $\mathbb{A}_F^\times/F^\times$ such that

$$\begin{aligned}\widetilde{\Pi} \otimes \widehat{\chi} &\cong \widetilde{\Pi}^\sigma \otimes \chi_1 \\ &\cong \widetilde{\Pi}^\vee \otimes \chi_1 \\ &\cong \widetilde{\Pi} \otimes (\chi_1 \omega_{\widetilde{\Pi}}^{-1}),\end{aligned}$$

which again proves the desired statement since $\omega_{\widetilde{\Pi}}$ restricted to $\mathbb{A}_F^\times$ is trivial. Note that in the above we have also used the fact that $\widetilde{\Pi}$ is assumed to be distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$. □

*Remark.* Although Asai lift naturally comes up in questions about distinguished representations for the pair $(\mathrm{GL}_2(\mathbb{A}_E), \mathrm{GL}_2(\mathbb{A}_F))$, its use in the previous theorem is for an entirely different purpose: to prove that a certain character of $\mathbb{A}_E^\times$ can be assumed to be a Grössencharacter when its restriction to $\mathbb{A}_F^\times/F^\times$ is known to be a Grössencharacter. In this, the crucial property of the Asai lift used is the fact that $\mathrm{As}(\Pi \otimes \chi) \cong \mathrm{As}(\Pi) \otimes \chi|_{\mathbb{A}_F^\times}$, so even if $\chi$ is not a Grössencharacter, since its restriction to $\mathbb{A}_F^\times$ is, the Asai lift is an automorphic representation. This is then combined with the knowledge about fibers of the Asai lift to conclude that $\chi$, or a variant of it, is automorphic. Later, when we deal with toric period integrals, we will use very similar arguments using the base change map for a similar effect. Although base change does appear in toric period questions, it is put to an unrelated use in this paper!

## 8. Examples

It may be useful to enumerate all the possibilities for the groups which appear in the previous section, which we do here.

According to the notation introduced in [AP03, AP06], and the proof of the previous theorem, we have,

$$\begin{aligned} X(H_1/H_3) \subset X_{\widetilde{\Pi}} &= \{\chi \in \widehat{\mathbb{A}_F^\times/F^\times} \mid \widetilde{\Pi} \text{ is } \chi\text{-distinguished}\} \\ X(H_0/H_1H_2) = Y_{\widetilde{\Pi}} &= \{\chi \in \widehat{\mathbb{A}_E^\times/E^\times} \mid \widetilde{\Pi} \otimes \chi = \widetilde{\Pi}, \chi|_{\mathbb{A}_F^\times} = 1\} \\ X(H_0/H_2) = Z_{\widetilde{\Pi}} &= \{\chi \in \widehat{\mathbb{A}_E^\times/E^\times} \mid \widetilde{\Pi} \otimes \chi = \widetilde{\Pi}\}. \end{aligned}$$

Further, there is an isomorphism of groups $X_{\widetilde{\Pi}} \cong Y_{\widetilde{\Pi}}$.

We now enumerate all the possibilities for the groups $X_{\widetilde{\Pi}}, Y_{\widetilde{\Pi}}, Z_{\widetilde{\Pi}}$, and refer the reader to [AP06, proof of Theorem 6.9].

(i) The representation $\widetilde{\Pi}$ is not CM. In this case, $X_{\widetilde{\Pi}} = Y_{\widetilde{\Pi}} = Z_{\widetilde{\Pi}} = \{1\}$.

(ii) The representation $\widetilde{\Pi}$ is CM by exactly one quadratic extension of $E$. In this case, $X_{\widetilde{\Pi}} \cong Y_{\widetilde{\Pi}} = Z_{\widetilde{\Pi}} \cong \mathbb{Z}/2$, and therefore,

$$\frac{X_{\widetilde{\Pi}}}{Z_{\widetilde{\Pi}}/Y_{\widetilde{\Pi}}} \cong \mathbb{Z}/2.$$

(iii) The representation $\widetilde{\Pi}$ is CM by three quadratic extensions of $E$, with exactly one Galois over $F$. In this case, $X_{\widetilde{\Pi}} \cong Y_{\widetilde{\Pi}} \cong \mathbb{Z}/2$, and $Z_{\widetilde{\Pi}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and therefore,

$$\frac{X_{\widetilde{\Pi}}}{Z_{\widetilde{\Pi}}/Y_{\widetilde{\Pi}}} = \{1\}.$$

(iv) The representation $\widetilde{\Pi}$ is CM by three quadratic extensions of $E$, all Galois over $F$. In this case, $X_{\widetilde{\Pi}} \cong Y_{\widetilde{\Pi}} = Z_{\widetilde{\Pi}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and therefore,

$$\frac{X_{\widetilde{\Pi}}}{Z_{\widetilde{\Pi}}/Y_{\widetilde{\Pi}}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

## 9. A more general situation

In the context of distinction for the pair $(\mathrm{GL}_2(E), \mathrm{GL}_2(F))$, the most general pair of this kind that one could consider is $(\mathrm{GL}_1(D)(E), \mathrm{GL}_1(D)(F))$ where $D$ is a quaternion algebra over a number field $F$, and $E$ is a quadratic extension of $F$. In fact it was in the study of distinction for this pair that the relative trace formula was introduced by Jacquet in collaboration with K. Lai in [JL85], who dealt with only those quaternion algebras $D$ over $F$ for which $D \otimes_F E \cong \mathrm{M}_2(E)$; the more general situation was considered in the paper [FH94]. These papers prove that a cuspidal representation $\pi$ of $\mathrm{GL}_1(D)(\mathbb{A}_E)$ is globally distinguished by $\mathrm{GL}_1(D)(\mathbb{A}_F)$ if and only if $\pi^{JL}$, the Jacquet–Langlands lift of $\pi$ to $\mathrm{GL}_2(\mathbb{A}_E)$, is globally distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$, together with the necessary local conditions at places of $F$ where $D$ is ramified, $E$ is inert, and $\pi$ is a principal series representation. Thus distinction for these pairs also is dictated by the existence of a pole at $s = 1$ of the Asai L-function.

Our work in the previous two sections for the pair $(\mathrm{SL}_2(E), \mathrm{SL}_2(F))$ was a consequence of this characterization of distinction for $\mathrm{GL}_2(E)$ representations in terms of the Asai $L$-function, and an input on distinction for the pair $(\mathrm{SL}_2(E), \mathrm{SL}_2(F))$ in terms of the Whittaker model with respect to a character of $E$ trivial on $F$ which was proved in [AP03] in the local case, and [AP06] in the global case.

In this section we consider distinction for the pair $(\mathrm{SL}_1(D)(\mathbb{A}_E), \mathrm{SL}_1(D)(\mathbb{A}_F))$ where $D$ is a quaternion algebra over a number field $F$, and $E$ is a quadratic extension of $F$. At places of $F$

where $E$ is inert and $D$ is ramified, we will be dealing with distinction properties for the pair $(\mathrm{SL}_2(E_v), \mathrm{SL}_1(D_v))$; an added subtlety here is that the embedding of $\mathrm{SL}_1(D_v)$ in $\mathrm{SL}_2(E_v)$ is unique only up to conjugation by $\mathrm{GL}_2(E_v)$, and there seems no preferred embedding of $\mathrm{SL}_1(D_v)$ in $\mathrm{SL}_2(E_v)$. Thus one must keep in mind that the question about classifying representations of $\mathrm{SL}_2(E_v)$ distinguished by $\mathrm{SL}_1(D_v)$ is meaningless unless there is a way of fixing an embedding of $\mathrm{SL}_1(D_v)$ inside $\mathrm{SL}_2(E_v)$.

Recall that for the pair $(\mathrm{SL}_2(\mathbb{A}_E), \mathrm{SL}_2(\mathbb{A}_F))$, our proof of the local-global property depended on defining the groups,

$$\begin{aligned} H_0 &= \mathbb{A}_E^\times, \\ H_1 &= \mathbb{A}_F^\times G_\Pi, \\ H_2 &= E^\times G_\Pi, \\ H_3 &= F^\times G_\Pi, \end{aligned}$$

and noting the following.

(i) The set $H_0 \cdot \Pi$ is the $L$-packet of representations of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\Pi$.

(ii) The set $H_1 \cdot \Pi$ is the set of locally distinguished representations in the $L$-packet of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\Pi$.

(iii) The set $H_2 \cdot \Pi$ is the set of automorphic representations in the $L$-packet of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\Pi$.

(iv) The set $H_3 \cdot \Pi$ is the set of globally distinguished representations in the $L$-packet of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\Pi$.

Then we proved, via considerations with the Asai lift, specifically determination of the fibers of the Asai lift, that $(H_1 \cap H_2)/H_3 = 1$, for which we did not need the interpretation of $H_3 \cdot \Pi$ as the set of globally distinguished representations in the $L$-packet of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by $\Pi$; we only needed to know that members of $H_3 \cdot \Pi$ are globally distinguished.

The groups $H_0, H_1, H_2, H_3$ were defined in the context of $(\mathrm{SL}_2(\mathbb{A}_E), \mathrm{SL}_2(\mathbb{A}_F))$ using the embedding of $E^\times$, or of $\mathbb{A}_E^\times$, inside $\mathrm{GL}_2(\mathbb{A}_E)$ as the group of diagonal matrices:

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

When dealing with $\mathrm{SL}_1(D)$, this diagonal subgroup does not make sense, but we can instead replace these by the image of $D^\times(E)$, respectively $D^\times(E_v)$, respectively $D^\times(\mathbb{A}_E)$ in $E^\times$, respectively $E_v^\times$, respectively $\mathbb{A}_E^\times$, via the reduced norm mapping. The group $G_\Pi$ itself may be defined as the image of the reduced norm mapping of the stabilizer in $D^\times(\mathbb{A}_E)$ of a representation $\Pi$ of $\mathrm{SL}_1(D)(\mathbb{A}_E)$. Thus $H_0$ is the image of $D^\times(\mathbb{A}_E)$ in $\mathbb{A}_E^\times$ under the reduced norm mapping, denoted here by det, or rather the image of $D^\times(\mathbb{A}_E)$ in $\mathbb{A}_E^\times$ multiplied by $G_\Pi$ inside $\mathbb{A}_E^\times$; similarly, $H_1 = \det(D^\times(\mathbb{A}_F)) \cdot G_\Pi \subset \mathbb{A}_E^\times$, $H_2 = \det(D^\times(E)) \cdot G_\Pi \subset \mathbb{A}_E^\times$, and $H_3 = \det(D^\times(F)) \cdot G_\Pi \subset \mathbb{A}_E^\times$.

In order to analyse the local-global question for $(\mathrm{SL}_1(D)(\mathbb{A}_E), \mathrm{SL}_1(D)(\mathbb{A}_F))$ inside an $L$-packet containing a globally distinguished representation, we can adopt more or less the same strategy. But we see from Lemma 9.2 below that $H_1 \cdot \Pi$ does not capture all the locally distinguished representations, and thus we cannot proceed along exactly the same lines. However, we note that $(H_1 \cap H_2)/H_3 = 1$ proves that locally distinguished representations



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of $(\mathrm{SL}_1(D)(\mathbb{A}_E), \mathrm{SL}_1(D)(\mathbb{A}_F))$ appearing in the restriction of a globally distinguished representation of

$$D^+(\mathbb{A}_E) = \{g \in D(\mathbb{A}_E) \mid \det g \in \mathbb{A}_F^\times \mathbb{A}_E^{\times 2}\}$$

are globally distinguished.

Let $\Pi^+$ be an automorphic representation of $D^+(\mathbb{A}_E)$ which is globally distinguished with respect to $\mathrm{SL}_1(D)(\mathbb{A}_F)$. Let $\Pi$ be an automorphic representation of $\mathrm{SL}_1(D)(\mathbb{A}_E)$ which comes in the restriction of $\Pi^+$. Observe that:

(i) the set $H_1 \cdot \Pi$ is the set of all the irreducible components of the restriction of $\Pi^+$ to $\mathrm{SL}_1(D)(\mathbb{A}_E)$;

(ii) the set $H_2 \cdot \Pi$ is the set of all automorphic representations belonging to the $L$-packet of $\mathrm{SL}_1(D)(\mathbb{A}_E)$ determined by $\Pi$;

(iii) $(H_1 \cap H_2)/H_3 = 1$.

The statement (ii) is part of the work of Labesse–Langlands mentioned earlier [LL79].

The proof of (iii) follows the same lines as given earlier for $\mathrm{SL}_2(\mathbb{A}_E)$ using the fibers of the Asai lift, which this time can be considered to be lifting of automorphic representations of $D^\times(\mathbb{A}_E)$ to $\mathrm{GL}_4(\mathbb{A}_F)$ via the intermediary of the Jacquet–Langlands correspondence to $\mathrm{GL}_2(\mathbb{A}_E)$. We note that we also need to use the standard local-global theorem for norms of quaternion division algebra: an element of $F^\times$ arises as a norm from $D^\times$ if and only if it does so locally at all places of $F$.

We summarize the above discussion in the following theorem.

Theorem 9.1. *Suppose $\Pi^+$ is an irreducible cuspidal representation of $D^+(\mathbb{A}_E)$ which is globally $\mathrm{SL}_1(D)(\mathbb{A}_F)$ distinguished. Then the part of the $L$-packet of $\mathrm{SL}_1(D)(\mathbb{A}_E)$ determined by the restriction of $\Pi^+$ has the local-global property for $\mathrm{SL}_1(D)(\mathbb{A}_F)$; more precisely, automorphic representations of $\mathrm{SL}_1(D)(\mathbb{A}_E)$ contained in $\Pi^+$ which are locally distinguished by $\mathrm{SL}_1(D)(\mathbb{A}_F)$, belong to one orbit under the action of $D^\times(F)$.*

Now we would like to understand the local-global question for a cuspidal representation $\Pi$ of $\mathrm{SL}_1(D)(\mathbb{A}_E)$. One important fact which went into our analysis of local-global distinction for the pair $(\mathrm{SL}_2(\mathbb{A}_E), \mathrm{SL}_2(\mathbb{A}_F))$ was that if a representation of $\mathrm{GL}_2^+(E_v)$ is distinguished by $\mathrm{GL}_2(F_v)$ then it must have a Whittaker model for a character of $E_v$ which is trivial on $F_v$. This has the corollary that if two irreducible representations $\Pi_1$ and $\Pi_2$ of $\mathrm{GL}_2^+(E_v)$ belonging to the same $L$-packet are respectively $\omega_1$- and $\omega_2$-distinguished by $\mathrm{GL}_2(F_v)$ for two characters $\omega_1, \omega_2 : F_v^\times \to \mathbb{C}^\times$, then $\Pi_1 \cong \Pi_2$ (although $\omega_1$ may not be the same as $\omega_2$). This is what allowed us to prove that representations of $\mathrm{SL}_2(E_v)$ distinguished by $\mathrm{SL}_2(F_v)$ belonging to one $L$-packet are in a single orbit for the action of $\mathrm{GL}_2(F_v)$. This property fails for the pair $(\mathrm{SL}_2(E_v), \mathrm{SL}_1(D_v))$ because of the following lemma.

Lemma 9.2. *Let $K$ be a quadratic ramified extension of a non-Archimedean local field $k$ of odd residue characteristic, and $D$ a quaternion division algebra over $k$. Let $\mu$ be an unramified character of $K^\times$ of order 4 with $\mu^2 =: \omega$. Let $\mathrm{GL}_2^+(K) = \{g \in \mathrm{GL}_2(K) \mid \omega(\det g) = 1\} = \{g \in \mathrm{GL}_2(K) \mid \det g \in k^\times K^{2\times}\}$. Then the principal series representation $\pi = \mathrm{Ps}(\mu, \mu\omega)$ of $\mathrm{GL}_2(K)$ decomposes as a sum of two irreducible representations $\pi^+$ and $\pi^-$ when restricted to $\mathrm{GL}_2^+(K)$ with $\pi^+$ spherical, i.e., the one which contains a vector fixed under $\mathrm{GL}_2(\mathcal{O}_K)$. Fix an embedding of $D^\times$ in $\mathrm{GL}_2^+(K)$ such that $D^\times \subset K^\times \cdot \mathrm{GL}_2(\mathcal{O}_K)$, then the trivial representation of $D^\times$ appears in $\pi^+$, and the character $\omega_{K/k}$ of order 2 of $k^\times$ associated with $K/k$, considered as a character of $D^\times$ through the reduced norm mapping, appears in $\pi^-$.*



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*Proof.* It is easy to see that $D^\times$ operates transitively on $\mathbb{P}^1(K)$ such that the stabilizer of the point $\infty$ in $\mathbb{P}^1(K)$ is isomorphic to $K^\times$, hence by Mackey theory we easily deduce that there are exactly two one-dimensional representations of $D^\times$ contained in $\pi$, one the trivial character, and the other which is the character $\omega_{K/k}$ of $k^\times$ considered as a character of $D^\times$. We need to decide which of these characters of $D^\times$ appear in $\pi^+$, and which of these characters appear in $\pi^-$.

Our first task will be to construct an embedding of $D^\times$ in $K^\times \cdot \mathrm{GL}_2(\mathcal{O}_K)$ (for $K$ a quadratic ramified extension of $k$). For this we fix some notation.

Let $\varpi_K$ be a uniformizing element in $K$, and $\mathcal{O}_k, \mathcal{O}_K, \mathcal{O}_D$ be respectively the maximal compact subrings of $k, K, D$. Fix an embedding $\iota : K \hookrightarrow D$. We will consider $\mathcal{O}_D$ as a free rank 2 module over $\mathcal{O}_K$ from the right, and as an $\mathcal{O}_D$-module from the left. This gives an embedding $\mathcal{O}_D \hookrightarrow \mathrm{End}_{\mathcal{O}_K}(\mathcal{O}_D)$. Since $\mathcal{O}_D$ is invariant under conjugation by $D^\times$, and hence by $K^\times$, left multiplication by $K^\times$ on $\mathcal{O}_D$ can be considered to be an inner-conjugation by $K^\times$ up to an action of $K^\times$ on the right:

$$x \cdot \mathcal{O}_D = x\mathcal{O}_d x^{-1} \cdot x,$$

therefore $K^\times \subset D^\times$ is contained in $K^\times \cdot \mathrm{End}_{\mathcal{O}_K}(\mathcal{O}_D)$, and since $D^\times = K^\times \mathcal{O}_D^\times$, $D^\times$ is contained in $K^\times \cdot \mathrm{End}_{\mathcal{O}_K}(\mathcal{O}_D)$.

Observe that $\mathcal{O}_D$ comes equipped with a natural filtration consisting of two-sided ideals: $\mathcal{O}_D \supset \varpi_K \mathcal{O}_D \supset \varpi_K^2 \mathcal{O}_D \supset \cdots$ such that the successive quotients are modules for $\mathcal{O}_D/\varpi_K\mathcal{O}_D \cong \mathbb{F}_{q^2}$, if $\mathbb{F}_q$ is the residue field of $k$. We thus have natural maps,

$$\mathcal{O}_D^\times \hookrightarrow \mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_D) \longrightarrow \mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_D/\varpi_K\mathcal{O}_D) = \mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^2}).$$

Under the composite map from $\mathcal{O}_D^\times$ to $\mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^2}) = \mathrm{GL}_2(\mathbb{F}_q)$, the image of $\mathcal{O}_D^\times$ is clearly $\mathbb{F}_{q^2}^\times$ acting on $\mathbb{F}_{q^2}$, giving rise to an embedding $\mathbb{F}_{q^2}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{F}_q)$. Further, since multiplication by $x \in K^\times$ on $\mathcal{O}_D$ on the left is up to a central element conjugation by $x$ on $\mathcal{O}_D$, the action of $K^\times$ on $\mathcal{O}_D/\pi_K\mathcal{O}_D$ is an automorphism of algebras, i.e., an element of the Galois group of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Thus the image of $D^\times$ is contained in the normalizer of $\mathbb{F}_{q^2}^\times$ inside $\mathrm{GL}_2(\mathbb{F}_q)$.

Since $D^\times \subset K^\times \cdot \mathrm{GL}_2(\mathcal{O}_K)$, given that $\pi$ has trivial central character and $\pi^+$ has a fixed vector under $\mathrm{GL}_2(\mathcal{O}_k)$, the trivial representation of $D^\times$ appears in $\pi^+$. The representation $\pi^-$ is obtained from $\pi^+$ by conjugating by the matrix,

$$\begin{pmatrix} \varpi_K & 0 \\ 0 & 1 \end{pmatrix},$$

hence it is clear that $\pi^-$ has a subrepresentation on which

$$\Gamma_0(\varpi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_K) \,\middle|\, \varpi_K | c \right\}$$

acts trivially. This means that $\pi^-$ must contain the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the residue field of $K$, as the Steinberg is the only non-trivial irreducible representation of $\mathrm{PGL}_2(\mathbb{F}_q)$ with a vector fixed under the group of upper triangular matrices. Since the Steinberg representation contains all non-trivial characters of $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$, the unique non-trivial character of $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$ of order 2 appears in $\pi^-$. (This is where we use that $q$ is odd to ensure that $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$ has a character of order 2.) Since the unique non-trivial character of $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$ of order 2 is left invariant by the normalizer of $\mathbb{F}_{q^2}^\times$ inside $\mathrm{GL}_2(\mathbb{F}_q)$, we conclude that there is a character of order 2 of $D^\times/k^\times$ appearing in $\pi^-$, which cannot be anything else but $\omega_{K/k}$. □

Because of this lemma, the local-global property fails, which we record in the following proposition.

Proposition 9.3. *Let $E$ be a quadratic extension of a number field $F$, and $D$ a quaternion division algebra over $F$. Then there exists an automorphic representation $\pi$ of $D^+(\mathbb{A}_E)$ which is locally distinguished by $\mathrm{SL}_1(D)(\mathbb{A}_F)$, but not globally distinguished in terms of having nonzero period integral on this subgroup; more precisely, on the $\pi$-isotypic piece of the automorphic representations of $D^+(\mathbb{A}_E)$, the $\mathrm{SL}_1(D)(\mathbb{A}_F)$-period integral is identically zero.*

*Proof.* Let $\tilde{\Pi}$ be a non-CM cuspidal representation of $D^\times(\mathbb{A}_E)$ with unramified principal series local components at many places where $E/F$ is ramified so that we are in the context of Lemma 9.2, and the restriction of $\tilde{\Pi}$ to $D^+(\mathbb{A}_E)$ has more than four direct summands which are locally distinguished by $\mathrm{SL}_1(D)(\mathbb{A}_F)$. Since a non-CM $L$-packet is stable, all these direct summands are automorphic as well. If all these representations were globally distinguished with respect to $\mathrm{SL}_1(D)(\mathbb{A}_F)$, then, in particular, they will be globally $\omega$-distinguished with respect to $D^\times(\mathbb{A}_F)$ for certain quadratic characters $\omega : \mathbb{A}_F^\times/F^\times \to \{\pm 1\}$; these quadratic characters $\omega$ are necessarily distinct by multiplicity one theorem regarding the space of $D^\times(\mathbb{A}_F)$-invariant linear forms on an irreducible representation of $D^\times(\mathbb{A}_E)$. It would then follow that $\tilde{\Pi}$ is distinguished by $D^\times(\mathbb{A}_F)$ for more than four Grössencharacters, which is not possible as global distinction is characterized in terms of the Asai lift of $\tilde{\Pi}$ to $\mathrm{GL}_4(\mathbb{A}_F)$ to contain a Grössencharacter as a direct summand, and so $\tilde{\Pi}$ can be $\omega$-distinguished for at most four Grössencharacters $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$. ◻

On the positive side, we have the following result.

Proposition 9.4. *Let $E$ be a quadratic extension of a number field $F$, and $D$ a quaternion division algebra over $F$. Let $\Pi$ be an automorphic representation of $D^+(\mathbb{A}_E)$ which is locally distinguished by $D^\times(\mathbb{A}_F)$ by a Grössencharacter $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$. Then if $\Pi$ has a discrete series local component, it is globally distinguished in terms of having nonzero period integral on this subgroup with respect to the character $\omega$.*

*Proof.* Let $\widetilde{\Pi}$ be an automorphic representation of $D^\times(\mathbb{A}_E)$ containing $\Pi$. Since $\Pi$ and hence $\widetilde{\Pi}$ are locally $\omega$-distinguished, by the local result due to Hakim [Hak91], and the second author [Pra92] (cf. Proposition 2.4), $\widetilde{\Pi}^{JL}$ is locally $\omega$-distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$ as well. This means that $\widetilde{\Pi}^{JL}$ is (globally) $\omega$-distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$ (this is where we use $\Pi$ having a discrete series local component, otherwise the conclusion is either $\omega$-distinguished or $\omega \cdot \omega_{E/F}$-distinguished), and hence, by the global result (cf. Proposition 2.3) $\widetilde{\Pi}$ is $\omega$-distinguished with respect to $D^\times(\mathbb{A}_F)$. (Since $\widetilde{\Pi}$ is locally $\omega$-distinguished, the necessary local condition to apply Proposition 2.3 on principal series component of $\widetilde{\Pi}^{JL}$ to be $Ps(\mu^{-1}, \mu^\sigma)$ at places of $F$ at which $D$ is ramified and $E$ is inert is satisfied.)

Now by the multiplicity one theorem about the space of $D^\times(F_v)$-invariant forms on an irreducible representation of $D^\times(E_v)$, it follows that $\Pi$ itself is $\omega$-distinguished, completing the proof of the proposition. ◻

Thus the proof of Proposition 9.4 says that the problem in the failure of the local-global principle in Proposition 9.3 is one of patching local characters of $F_v^\times$ into a Grössencharacter on $\mathbb{A}_F^\times/F^\times$. We can capture this more precisely as follows.

To an automorphic representation $\Pi = \otimes \Pi_v$ of $D^\times(\mathbb{A}_E)$, define local groups $\mathcal{S}_v$ consisting of characters $\omega_v$ of $F_v^\times$ such that $\Pi_v$ is $\omega_v$-distinguished with respect to the subgroup $D^\times(F_v)$. We know that $\mathcal{S}_v$ is a finite set consisting of at most four elements, and that for most places $v$ of $F$, there is an unramified character in $\mathcal{S}_v$, and there are at most two unramified characters in $\mathcal{S}_v$ which, if there are two, are twists of each other by $\omega_{E_v/F_v}$. So the previous proposition can also be stated as saying that an automorphic representation $\Pi$ of $D^\times(\mathbb{A}_E)$ is globally distinguished by $\mathrm{SL}_1(D)(\mathbb{A}_F)$ if and only if there is a Grössencharacter $\omega = \prod \omega_v : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ such that $\omega_v$ belongs to $\mathcal{S}_v$ for all places $v$ of $F$.

## 10. Distinction in an $L$-packet for the toric period

In this section we prove Theorem 1.4 which we recall again here.

Since multiplicity one theorem is not true for automorphic representations of $\mathrm{SL}_1(D)(\mathbb{A}_F)$, a small care is needed in the following theorem in defining 'an $L$-packet of automorphic representations on $\mathrm{SL}_1(D)(\mathbb{A}_F)$' by which we will mean the automorphic representations on $\mathrm{SL}_1(D)(\mathbb{A}_F)$ obtained by restricting one from $D^\times(\mathbb{A}_F)$.

Theorem 1.4. *Let $D$ be a quaternion algebra over a number field $F$, with $E$ a quadratic subfield of $D$. Let $\Pi = \otimes_v \Pi_v$ be a cuspidal representation of $\mathrm{SL}_1(D)(\mathbb{A}_F)$ with at least one square integrable component at a place $v_0$ of $F$; if $E$ is inert and $D$ is split at $v_0$, we further assume that $v_0$ is of odd residue characteristic and $\Pi_{v_0}$ is a supercuspidal representation if $v_0$ is a finite place of $F$. If each $\Pi_v$ is distinguished with respect to $E_v^1$, then assuming Conjecture 1.3 holds, there is a cuspidal representation in the $L$-packet of $\Pi$ which is distinguished with respect to $\mathbb{A}_E^1$.*

Let $\widetilde{\Pi} = \otimes \widetilde{\Pi}_v$ be a cuspidal representation of $D^\times(\mathbb{A}_F)$ containing $\Pi = \otimes \Pi_v$. We assume that each $\Pi_v$ is distinguished with respect to $E_v^1$, the group of norm one elements of $E_v^\times$. We assume, without loss of generality, that the central character of $\widetilde{\Pi}$ is trivial. Since $\Pi_v$ and hence $\widetilde{\Pi}_v$ is distinguished with respect to $E_v^1$, $\widetilde{\Pi}_v$ is $\alpha_v$-distinguished with respect to $E_v^\times$, where $\alpha_v$ is a character of $E_v^\times/F_v^\times E_v^1$, hence there is a quadratic character $\beta_v$ of $F_v^\times$ such that $\widetilde{\Pi}_v$ is $\beta_v \circ \mathbb{N}m$-distinguished with respect to $E_v^\times$. Since being distinguished is no condition at places $v$ of $F$ where $\widetilde{\Pi}_v$ is a principal series representation, so at all but finitely many places of $F$, we can assume that $\widetilde{\Pi}_v$ is $\beta_v \circ \mathbb{N}m$-distinguished at all places $v$ of $F$ for a Grössencharacter $\beta$ of $\mathbb{A}_F^\times/F^\times$ with $\beta^2 = 1$.

The proof of Theorem 1.4 will depend on two technical lemmas, one local and the other global. The local lemma allows one to twist a representation $\widetilde{\Pi}_0$ of $\mathrm{GL}_2(F_0)$ by a quadratic character $\chi_0$ to change epsilon factor $\epsilon(\widetilde{\Pi}_0)$ to $\epsilon(\widetilde{\Pi}_0 \otimes \chi_0)$ so that the global epsilon factor $\epsilon(\widetilde{\Pi} \otimes \chi)$ can be assumed to be 1 if the original $\epsilon(\widetilde{\Pi})$ was $-1$; this then allows one to appeal to Conjecture 1.3 about simultaneous non-vanishing of two $L$-values. An added subtlety that we must deal with is that in changing the sign of $\epsilon(\widetilde{\Pi}_0)$ to $\epsilon(\widetilde{\Pi}_0 \otimes \chi_0)$, such quadratic characters must appear in the representation $\widetilde{\Pi}_0$ at that place, so by the theorem of Saito–Tunnell, some other epsilons must be controlled. This is where existence of a discrete series component of the automorphic representation is used.

Lemma 10.1. *Let $\widetilde{\Pi} = \otimes \widetilde{\Pi}_v$ be an automorphic representation of $D^\times(\mathbb{A}_F)$ with trivial central character. Let $E$ be a quadratic extension of $F$ contained in $D$ with $\omega_{E/F} = \prod \omega_v$ the associated Grössencharacter of $\mathbb{A}_F^\times/F^\times$. Assume that $\widetilde{\Pi}$ has at least one square integrable component at a place $v_0$ of $F$; if $E$ is inert and $D$ is split at $v_0$, we further assume that $v_0$ is of odd residue characteristic and $\widetilde{\Pi}_{v_0}$ is a supercuspidal representation if $v_0$ is a finite place of $F$. Assume that $\widetilde{\Pi}$ is locally distinguished by the character $\beta \circ \mathbb{N}m$ of $\mathbb{A}_E^\times$ for a quadratic Grössencharacter $\beta$ of $\mathbb{A}_F^\times/F^\times$. Then there is a Grössencharacter $\eta$ of $\mathbb{A}_F^\times/F^\times$ with $\eta^2 = 1$, such that*

$$\epsilon(\widetilde{\Pi} \otimes \eta) = 1 = \epsilon(\widetilde{\Pi} \otimes \omega_{E/F}\eta), \tag{2}$$

*and furthermore,*

$$\epsilon(\widetilde{\Pi}_v \otimes \eta_v)\epsilon(\widetilde{\Pi}_v \otimes \omega_v\eta_v) = \epsilon(\widetilde{\Pi}_v \otimes \beta_v)\epsilon(\widetilde{\Pi}_v \otimes \omega_v\beta_v) \tag{3}$$

*for all $v$. Moreover, $\eta$ can be made to agree with $\beta$ at finitely many prescribed places other than $v_0$.*

Before we proceed to prove this lemma, we fix some more notation, and prove a few intermediate results. Some of this notation, as well as the proofs that follow, is due to one of the referees of this paper.

Let $F_0$ be a local field, and $E_0$ a separable quadratic extension of $F_0$, with $\omega_{E_0/F_0}$ the corresponding quadratic character of $F_0^\times$. Let $\pi$ be a discrete series representation of $\mathrm{GL}_2(F_0)$ with $\omega_\pi = 1$. For $\nu = \pm 1$, let

$$X^\nu = X_\pi^\nu = \{\alpha : F_0^\times \to \{\pm 1\} \mid \epsilon(\pi \otimes \alpha) = \nu\alpha(-1)\epsilon(\pi)\}.$$

Then, we have the following proposition (see [Wal91]).

PROPOSITION 10.2. *If $\pi$ is a discrete series representation of $\mathrm{PGL}_2(F_0)$, then $X^\nu \neq \emptyset$ for $\nu = \pm 1$.*

Now, for $\nu, \nu' \in \{\pm 1\}$, let

$$X^{\nu,\nu'} = X_\pi^{\nu,\nu'} = \{\alpha \in X^\nu \mid \alpha\omega_{E_0/F_0} \in X^{\nu'}\}.$$

We have the following proposition.

PROPOSITION 10.3. *The sets $X^{\nu,\nu'}$ satisfy the following properties.*

(i) *The map $\alpha \mapsto \alpha\omega_{E_0/F_0}$ is a bijection between $X^{+-}$ and $X^{-+}$.*

(ii) *If $X^{+-} = \emptyset$, then both $X^{++}$ and $X^{--}$ are non-empty.*

(iii) *Let $\alpha \in X^{\nu\nu'}$. Then $\pi$ is $\alpha \circ \mathbb{N}m$-distinguished with respect to $E^\times$ if and only if $\nu\nu' = 1$, and $\pi'$, the representation of $D_0^\times$ associated to $\pi$ by the Jacquet–Langlands correspondence where $D_0$ is the unique quaternion division algebra over $F_0$, is $\alpha \circ \mathbb{N}m$-distinguished with respect to $E^\times$ if and only if $\nu\nu' = -1$*

*Proof.* Part (i) is straightforward. Part (ii) follows from part (i), thanks to Proposition 10.2. Part (iii) follows from Proposition 2.8 together with the identity (1) in the remark following this proposition. □

PROPOSITION 10.4. *Assume that $F_0$ is either Archimedean, or has odd residue characteristic. Let $\pi$ be a discrete series representation of $\mathrm{GL}_2(F_0)$ which is supercuspidal if $F_0$ is non-Archimedean. Then, if $X^{++}$ is non-empty, so is $X^{--}$, and conversely. In the Archimedean case, $X^{++}$ and $X^{--}$ are both empty sets, and $X^{+-}$ and $X^{-+}$ are sets with one element.*

*Proof.* The proof of the proposition is rather trivial in the Archimedean case, so we assume in the rest of the proof that $F_0$ is non-Archimedean. Since $F_0$ has odd residue characteristic, the number of characters of $F_0^\times$ of order dividing 2 is four, and further $\pi$ has a self-twist by a non-trivial character $\mu$ of order 2. Note that $\alpha \mapsto \alpha\mu$ takes $X^{\nu,\nu'}$ to $X^{\nu\mu(-1),\nu'\mu(-1)}$.

If $\mu(-1) = 1$, the sets $X^{\nu,\nu'}$ are stabilized by multiplication by $\mu$, and hence their cardinalities are even integers. Given that $X^{++}, X^{+-}, X^{--}, X^{-+}$ are disjoint sets of total cardinality 4 with the cardinalities of $X^{+-}$ and $X^{-+}$ equal, we easily deduce that it is not possible for $X^{++}$ to be non-empty but $X^{--}$ to be empty, and conversely; here we have also utilized Proposition 10.2.

If $\mu(-1) = -1$, then $\alpha \mapsto \alpha\mu$ gives a bijection between $X^{++}$ and $X^{--}$, and once again the proposition follows. □



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*Remark.* We are unable to prove Proposition 10.4 in residue characteristic 2 one way or the other which seems like an interesting *exercice dyadiques*.

We will need the following local result in order to prove Lemma 10.1.

Lemma 10.5. *Let $E_0$ be a separable quadratic algebra over a local field $F_0$, with $\omega_0$ as the corresponding character of $F_0^\times$ with $\omega_0^2 = 1$. Let $\widetilde{\Pi}_0$ be an irreducible discrete series representation of $\mathrm{PGL}_2(F_0)$. If $E_0$ is a quadratic field extension of $F_0$, and if $F_0$ is non-Archimedean, assume further that it is of odd residue characteristic, and that $\widetilde{\Pi}_0$ is a supercuspidal representation. Let $\beta_0$ be a quadratic character of $F_0^\times$. Then there exists a quadratic character $\gamma_0$ of $F_0^\times$ with,*

$$\frac{\epsilon(\widetilde{\Pi}_0 \otimes \gamma_0)}{\gamma_0(-1)} = -\frac{\epsilon(\widetilde{\Pi}_0 \otimes \beta_0)}{\beta_0(-1)},$$

*and,*

$$\epsilon(\widetilde{\Pi}_0 \otimes \gamma_0)\epsilon(\widetilde{\Pi}_0 \otimes \omega_0\gamma_0) = \epsilon(\widetilde{\Pi}_0 \otimes \beta_0)\epsilon(\widetilde{\Pi}_0 \otimes \omega_0\beta_0).$$

*Proof.* It is convenient to use the sets $X^{\nu,\nu'}$ to prove the lemma. By Proposition 10.3(i) and Proposition 10.4, we know that $X^{+-} = \emptyset \iff X^{-+} = \emptyset$ and $X^{++} = \emptyset \iff X^{--} = \emptyset$, under our assumptions.

Now if $\beta_0$ belongs to $X^{+-}$, then choose $\gamma_0$ from the non-empty set $X^{-+}$, and conversely. If $\beta_0$ belongs to $X^{--}$, then choose $\gamma_0$ from the non-empty set $X^{++}$, and conversely, in order to conclude the lemma. □

*Proof of Lemma 10.1.* Since the representation $\widetilde{\Pi}_v$ of $D_v^\times$ is $\beta_v \circ \mathbb{N}m$-distinguished for the subgroup $E_v^\times$, by the Saito–Tunnell theorem, we have,

$$\epsilon(\widetilde{\Pi}_v \otimes \beta_v)\epsilon(\widetilde{\Pi}_v \otimes \omega_v\beta_v) = \omega_v(-1)\omega_{D_v}(-1),$$

where $\omega_{D_v}(-1) = -1$ if $D_v$ is ramified, and $\omega_{D_v}(-1) = 1$ if $D_v \cong M_2(F_v)$. It follows that,

$$\epsilon(\widetilde{\Pi} \otimes \beta)\epsilon(\widetilde{\Pi} \otimes \beta\omega) = \prod_v \omega_{D_v}(-1) = 1,$$

the last equality following from the fact that the number of ramified primes of $D$ is even.

Therefore, if $\epsilon(\widetilde{\Pi} \otimes \beta) = 1$, then so is $\epsilon(\widetilde{\Pi} \otimes \beta\omega)$, and $\eta = \beta$ has all the desired properties to apply Conjecture 1.3.

If $\epsilon(\widetilde{\Pi} \otimes \beta) = -1$, we will use the fact that $\widetilde{\Pi}$ has a square integrable component at $v_0$ to modify $\beta$ to construct $\eta$ such that $\epsilon(\widetilde{\Pi} \otimes \eta) = 1 = \epsilon(\widetilde{\Pi} \otimes \eta\omega)$, with

$$\epsilon(\widetilde{\Pi}_v \otimes \eta_v)\epsilon(\widetilde{\Pi}_v \otimes \omega_v\eta_v) = \epsilon(\widetilde{\Pi}_v \otimes \beta_v)\epsilon(\widetilde{\Pi}_v \otimes \omega_v\beta_v)$$

for all $v$.

Let $\gamma$ be a quadratic Grössencharacter of $\mathbb{A}_F^\times/F^\times$ with $\gamma_{v_0} = \gamma_0$ as in Lemma 10.5, and which at the other places $v$ of $F$ where either $D$ or $\widetilde{\Pi}$ is ramified is $\beta_v$ (and no constraints outside the ramified primes of $\widetilde{\Pi}$). By a well-known calculation about the epsilon factor of principal series representations of $\mathrm{PGL}_2(F_v)$, it follows that

$$\frac{\epsilon(\widetilde{\Pi}_v \otimes \chi)}{\chi(-1)} = \epsilon(\widetilde{\Pi}_v),$$

for $\Pi_v$ a principal series representation of $\mathrm{PGL}_2(F_v)$, and $\chi$ any character of $F_v^\times$ with $\chi^2 = 1$. Therefore we have,

$$\epsilon(\widetilde{\Pi} \otimes \gamma) = \prod_v \frac{\epsilon(\widetilde{\Pi}_v \otimes \gamma_v)}{\gamma_v(-1)} = -\frac{\epsilon(\widetilde{\Pi}_0 \otimes \beta_0)}{\beta_0(-1)} \prod_{v \neq v_0} \frac{\epsilon(\widetilde{\Pi}_v \otimes \beta_v)}{\beta_v(-1)} = -\epsilon(\widetilde{\Pi} \otimes \beta),$$

proving Lemma 10.1. □

*Proof of Theorem 1.4.* Appealing to Conjecture 1.3, we obtain a Grössencharacter, say $\eta'$, of $\mathbb{A}_F^\times$ with $(\eta')^2 = 1$, such that:

(i) $L(\frac{1}{2}, \widetilde{\Pi} \otimes \eta') \neq 0 \neq L(\frac{1}{2}, \widetilde{\Pi} \otimes \omega\eta')$;

(ii) $\eta'$ agrees with $\eta$ at all the places $S$ of $F$ containing the infinite places of $F$, and the places of $F$ where $\widetilde{\Pi}$ or $D$ is ramified.

Given this, equation (3) of Lemma 10.1 continues to be satisfied with $\eta'$ instead of $\eta$ at all places $v$ of $F$, since outside of $S$ there is no condition, as can be easily checked. Thus, $\widetilde{\Pi} \otimes \eta'$ is distinguished with respect to $E_v^\times$ at all the places $v$. Since $L(\frac{1}{2}, \widetilde{\Pi} \otimes \eta') \neq 0 \neq L(\frac{1}{2}, \widetilde{\Pi} \otimes \omega\eta')$, the non-vanishing of the toric period on $\widetilde{\Pi} \otimes \eta'$ follows by the work of Waldspurger. This is enough to conclude that there is a member in the $L$-packet of $\Pi$ on which the $\mathbb{A}_E^1$-period integral is non-vanishing by Proposition 3.4. This finishes the proof of Theorem 1.4. □

*Remark.* One could ask whether it is possible to remove the further technical restrictions at the discrete series place in Theorem 1.4. This is possible if $\Pi$ has two discrete series places and further if $D$ is allowed to vary; i.e., given a cuspidal representation $\Pi$ of $\mathrm{SL}_1(D)(\mathbb{A}_F)$, with at least two discrete series components, that is locally distinguished with respect to $\mathbb{A}_E^1$, there exists a quaternion algebra $D'$ over $F$ containing $E$ such that the $L$-packet of $\Pi'$ is globally $\mathbb{A}_E^1$-distinguished, where the $L$-packet of $\Pi'$ is the Jacquet–Langlands correspondent of the $L$-packet of $\Pi$.

The previous arguments work as well for the split toric period of a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_F)$. In fact, in this case, since $\omega_{E/F} = 1$, there are not two $L$-values to control, but a single one, whose non-vanishing is the main theorem of the paper of Friedberg and Hoffstein [FH95] which we recall presently, so we do not need to resort to Conjecture 1.3 in the split toric case, and we obtain an unconditional theorem.

Theorem 10.6 (Friedberg–Hoffstein). *Let $\pi$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. If $\pi$ is self-dual, let $\eta$ be a quadratic character such that*

$$\epsilon(\pi \otimes \eta) = 1.$$

*If $\pi$ is not self-dual, let $\eta$ be any quadratic character. Then there are infinitely many quadratic characters $\eta'$ of $\mathbb{A}_F^\times/F^\times$, which agree with $\eta$ at any finitely many prescribed places of $F$, and such that*

$$L(\tfrac{1}{2}, \pi \otimes \eta') \neq 0.$$

Using Theorem 10.6, we obtain the following theorem regarding the period integral on the split torus. Actually, in this case the local-global principle holds true for individual automorphic representations, since automorphic representations (in an $L$-packet) are all $F^\times$ conjugates of each other and therefore if the period integral is nonzero on one automorphic representation, it is nonzero on any other automorphic member of the $L$-packet. Furthermore, for the split torus, there are no local conditions either (of course, the central character must be trivial). Thus we have the following unconditional theorem.

THEOREM 10.7. *Let $\Pi = \otimes_v \Pi_v$ be a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_F)$ with trivial central character contained in an automorphic representation $\widetilde{\Pi} = \otimes_v \widetilde{\Pi}_v$ of $\mathrm{GL}_2(\mathbb{A}_F)$, also with trivial central character. Suppose either that the global epsilon factor, $\epsilon(\widetilde{\Pi}) = 1$, or that $\Pi$ has at least one square integrable component at a place $v_0$ of $F$. Then, $\Pi$ is distinguished with respect to $\mathbb{A}_F^\times$ sitting inside $\mathrm{SL}_2(\mathbb{A}_F)$ as the diagonal subgroup $\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$, $x \in \mathbb{A}_F^\times$.*

*Remark.* We have already mentioned a result of Waldspurger, that for an automorphic representation $\widetilde{\Pi} = \otimes_v \widetilde{\Pi}_v$ of $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central character and with at least one square integrable component at a place $v_0$ of $F$, there is a character $\alpha$ of $\mathbb{A}_F^\times / F^\times$ with $\alpha^2 = 1$ such that the global epsilon factor, $\epsilon(\widetilde{\Pi} \otimes \alpha) = 1$. It may be instructive to recall the proof.

*Proof.* The proof of this remark is a direct consequence of Proposition 10.2 according to which

$$\frac{\epsilon(\pi \otimes \mu)}{\mu(-1)}$$

takes both the values $\pm 1$ if $\pi$ is a discrete series representation of $\mathrm{PGL}_2(F_{v_0})$.

If $\epsilon(\widetilde{\Pi}) = 1$, we can take $\alpha = 1$. Else, $\epsilon(\widetilde{\Pi}) = -1$. In this case, use Proposition 10.2 at the place $v_0$ of $F$ where $\pi = \pi_0$ is given to be a discrete series representation, to change the value of $\epsilon(\pi \otimes \mu)/\mu(-1)$, so that $\epsilon(\pi \otimes \mu)/\mu(-1) = -\epsilon(\pi)$. Now let $\alpha$ be a quadratic character of $\mathbb{A}_F^\times / F^\times$ which is $\mu$ at the place $v_0$ of $F$, which is equal to 1 at the finite set of places of $F$ where $\Pi$ is not a principal series representation. Noting that for $\pi$ a principal series,

$$\frac{\epsilon(\pi \otimes \mu)}{\mu(-1)} = \epsilon(\pi),$$

we find that

$$\epsilon(\widetilde{\Pi} \otimes \alpha) = \frac{\epsilon(\widetilde{\Pi} \otimes \alpha)}{\alpha(-1)} = -\epsilon(\widetilde{\Pi}) = 1,$$

completing the proof of the remark. □

*Remark.* Similarly, Proposition 10.2 implies that if we are given two automorphic representations $\Pi_1$ and $\Pi_2$ of $\mathrm{PGL}_2(\mathbb{A}_F)$, such that there are two distinct places $v_1$ and $v_2$ of $F$ such that $\Pi_1$ at $v_1$ is a discrete series but $\Pi_2$ is a principal series at $v_1$, and similarly, $\Pi_2$ at $v_2$ is a discrete series but $\Pi_1$ is a principal series at $v_2$, then there are quadratic characters $\alpha$ of $\mathbb{A}_F^\times / F^\times$ such that,

$$\epsilon(\Pi_1 \otimes \alpha) = \epsilon(\Pi_2 \otimes \alpha) = 1.$$

However, if $\Pi_1$ and $\Pi_2$ have discrete series components at the same set of primes (or on an empty set of primes), then this method does not work. In our work, we have to deal with Conjecture 1.3 on simultaneous non-vanishing of twists of $L$-values for $\Pi_1$ and $\Pi_2$ which are themselves twists of each other, where it may not be possible to use discrete series components to achieve

$$\epsilon(\Pi_1 \otimes \alpha) = \epsilon(\Pi_2 \otimes \alpha) = 1,$$

as can be seen through simple examples. In Lemma 10.1, we have managed to make two global epsilon factors 1 after twisting, under the hypothesis of local distinction at all the places of $F$.



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## 11. Local-global principle for toric periods

Let $E$ be a quadratic extension of a number field $F$. Fix an embedding of $E^\times$ in $\mathrm{GL}_2(F)$, and hence an embedding of $E^1$ into $\mathrm{SL}_2(F)$. Let $\Pi = \otimes\Pi_v$ be an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_F)$. The group $\mathbb{A}_F^\times$ sitting inside $\mathrm{GL}_2(\mathbb{A}_F)$ as

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

operates on $\mathrm{SL}_2(\mathbb{A}_F)$ via conjugation action, and therefore on the set of isomorphism classes of representations of $\mathrm{SL}_2(\mathbb{A}_F)$. The orbit of $\Pi$ under the action of $\mathbb{A}_F^\times$ is precisely the global $L$-packet of representations of $\mathrm{SL}_2(\mathbb{A}_F)$ containing $\Pi$. Let $G_\Pi \subset \mathbb{A}_F^\times$, $G_\Pi = \prod G_v$, be the stabilizer of the representation $\Pi = \otimes\Pi_v$, where $G_v$ is the stabilizer inside $F_v^\times$ of the representation $\Pi_v$.

The action of $F^\times$ on $\mathrm{SL}_2(\mathbb{A}_F)$ is transitive on the set of automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ contained in the global $L$-packet determined by $\Pi$. Clearly if $\Pi$ has a nonzero period integral on the given embedding of $E^1(\mathbb{A}_F)$ in $\mathrm{SL}_2(\mathbb{A}_F)$, then so will all its conjugates under $\mathbb{N}(E^\times)$. If we can prove that these are the only automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ which have local periods with respect to $E^1(F_v)$ for all places $v$ of $F$, we will have proved the local-global principle for toric periods. However, the proof in the toric case will not be so simple, and will depend on using another related group $\mathrm{GL}_2^+(\mathbb{A}_F)$, defined as follows:

$$\mathrm{GL}_2^+(\mathbb{A}_F) = \{g \in \mathrm{GL}_2(\mathbb{A}_F) \mid \det g \in \mathbb{N}(\mathbb{A}_E^\times)\}.$$

We will prove that the part of the $L$-packet of $\mathrm{SL}_2(\mathbb{A}_F)$ determined by the restriction of an irreducible automorphic representation $\Pi^+$ of $\mathrm{GL}_2^+(\mathbb{A}_F)$ has the local-global property if $\Pi^+$ is globally distinguished by a quadratic character of $\mathbb{A}_F^\times$.

THEOREM 11.1. *Suppose $\Pi^+$ is an irreducible cuspidal representation of $\mathrm{GL}_2^+(\mathbb{A}_F)$ which is globally $\mathbb{A}_E^\times$ distinguished by a quadratic character $\omega$ of $\mathbb{A}_F^\times/F^\times$; i.e., by the character $\omega \circ \mathbb{N}$ of $\mathbb{A}_E^\times/E^\times$. Then any automorphic representation of $\mathrm{SL}_2(\mathbb{A}_F)$ contained in the restriction of $\Pi^+$ has nonzero period integral on $\mathbb{A}_E^1/E^1$.*

*Proof.* Define groups analogous to the ones defined in § 7:

$$\begin{aligned} H_0 &= \mathbb{A}_F^\times, \\ H_1 &= \mathbb{N}(\mathbb{A}_E^\times)G_\Pi, \\ H_2 &= F^\times G_\Pi, \\ H_3 &= \mathbb{N}(E^\times)G_\Pi. \end{aligned}$$

Let $\Pi$ be an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_F)$ contained in $\Pi^+ = \otimes_v \Pi_v^+$ which is globally distinguished by $\mathbb{A}_E^1$; its existence follows from Proposition 3.3. It is clear that automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ of the form $H_3 \cdot \Pi$ are globally distinguished by $\mathbb{A}_E^1$, whereas representations of the form $H_1 \cdot \Pi$ are all the irreducible components of $\Pi^+$ restricted to $\mathrm{SL}_2(\mathbb{A}_F)$, and among these, representations of the form $H_2 \cdot \Pi$ are automorphic. Thus the following result proves the theorem. □

THEOREM 11.2. *The group $(H_1 \cap H_2)/H_3$ is trivial.*

*Proof.* We will prove that $(H_1 \cap H_2)/H_3$ is trivial by proving that its character group is trivial. Noting that $(H_1 \cap H_2)/H_3$ is nothing but the kernel of the map,

$$H_1/H_3 \to H_0/H_2,$$



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the character group of $(H_1 \cap H_2)/H_3$ is the cokernel of the natural map

$$X(H_0/H_2) \rightarrow X(H_1/H_3).$$

Therefore to prove the theorem, it suffices to prove the surjectivity of the natural map

$$X(H_0/H_2) \rightarrow X(H_1/H_3).$$

Equivalently, we need to prove that a character of $(\mathbb{N}(\mathbb{A}_E^\times)G_\Pi)/[\mathbb{N}(E^\times)G_\Pi]$, can be extended to a Grössencharacter of $\mathbb{A}_F^\times/F^\times$ which is a self-twist of $\tilde{\Pi}$.

Since $G_\Pi$ and hence $\mathbb{N}(E^\times)G_\Pi$ is an open subgroup of $\mathbb{A}_F^\times$, $\mathbb{A}_F^\times/[\mathbb{N}(E^\times)G_\Pi]$ is a discrete group, hence a character $\chi$ of $[\mathbb{N}(\mathbb{A}_E^\times)G_\Pi]/[\mathbb{N}(E^\times)G_\Pi]$ can be thought of as a character of $\mathbb{A}_F^\times/[\mathbb{N}(E^\times)G_\Pi]$, so that $\widetilde{\Pi} \cong \widetilde{\Pi} \otimes \chi$. Our aim is to eventually get one which is a Grössencharacter.

Let $\mathrm{BC}(\widetilde{\Pi})$ denote the base change lift of the representation $\widetilde{\Pi}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ to $\mathrm{GL}_2(\mathbb{A}_E)$. By local considerations, it is clear that

$$\mathrm{BC}(\widetilde{\Pi}) \cong \mathrm{BC}(\widetilde{\Pi} \otimes \chi) \cong \mathrm{BC}(\widetilde{\Pi}) \otimes \chi \circ \mathbb{N}. \tag{4}$$

Note that although we do not know that $\chi$ is a Grössencharacter on $\mathbb{A}_F^\times/F^\times$, but since it is trivial on $\mathbb{N}(E^\times)$, the character $\chi \circ \mathbb{N}$ of $\mathbb{A}_E^\times$ is a Grössencharacter on $\mathbb{A}_E^\times/E^\times$. Further, the Grössencharacter $\chi \circ \mathbb{N}$ on $\mathbb{A}_E^\times/E^\times$ is naturally Galois-invariant. Therefore, the Grössencharacter $\chi \circ \mathbb{N}$ on $\mathbb{A}_E^\times/E^\times$ can be descended to a Grössencharacter, say $\mu$ on $\mathbb{A}_F^\times/F^\times$, i.e.,

$$\chi \circ \mathbb{N} = \mu \circ \mathbb{N}.$$

So (4) can be rewritten as

$$\mathrm{BC}(\widetilde{\Pi}) \cong \mathrm{BC}(\widetilde{\Pi} \otimes \chi) \cong \mathrm{BC}(\widetilde{\Pi}) \otimes \chi \circ \mathbb{N} \cong \mathrm{BC}(\widetilde{\Pi}) \otimes \mu \circ \mathbb{N} \cong \mathrm{BC}(\widetilde{\Pi} \otimes \mu). \tag{5}$$

This gives

$$\mathrm{BC}(\widetilde{\Pi}) \cong \mathrm{BC}(\widetilde{\Pi} \otimes \mu).$$

Just like the previous case dealing with the Asai lift (cf. Theorem 6.5), appealing now to the (this time, much better known) theorem about fibers of the base change map, we find that either:

(i) $\widetilde{\Pi} \cong \widetilde{\Pi} \otimes \mu$; or

(ii) $\widetilde{\Pi} \otimes \omega_{E/F} \cong \widetilde{\Pi} \otimes \mu$.

In case (i), the character $\mu$ is trivial on $G_\Pi$ (by the very definition of $G_\Pi$), and since $\chi \circ \mathbb{N} = \mu \circ \mathbb{N}$, we find that $\chi$ and $\mu$ are the same on the subgroup $\mathbb{N}(\mathbb{A}_E^\times)$ of $\mathbb{A}_F^\times$, therefore the character $\mu$ on $\mathbb{A}_F^\times/F^\times$ is the desired extension of the character $\chi$ initially defined on $(\mathbb{N}(\mathbb{A}_E^\times)G_\Pi)/[\mathbb{N}(E^\times)G_\Pi]$.

In case (ii), the character $\mu\omega_{E/F}$ is trivial on $G_\Pi$, and since $\chi \circ \mathbb{N} = \mu \circ \mathbb{N}$, we find that $\chi$ and $\mu\omega_{E/F}$ are the same on the subgroup $\mathbb{N}(\mathbb{A}_E^\times)$ of $\mathbb{A}_F^\times$, therefore the character $\mu\omega_{E/F}$ on $\mathbb{A}_F^\times/F^\times$ is the desired extension of the character $\chi$ initially defined on $(\mathbb{N}(\mathbb{A}_E^\times)G_\Pi)/[\mathbb{N}(E^\times)G_\Pi]$. □

It may be useful to isolate a fact of independent interest from the above proof which was actually the crux of the argument for the proof of Theorem 11.1.

THEOREM 11.3. *Suppose $\Pi^+$ is an irreducible cuspidal representation of $\mathrm{GL}_2^+(\mathbb{A}_F)$. Then $E^\times \subset \mathrm{GL}_2^+(\mathbb{A}_F)$ acts transitively on the set of automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ contained in the restriction of $\Pi^+$.*

Theorem 11.1 holds true in the analogous division algebra case, and the proof is the same after we have noted that the group $H_0 = \mathbb{A}_F^\times$, which is used as a subgroup of $\mathrm{GL}_2(\mathbb{A}_F)$, can also be treated as a quotient group via the determinant map, and then it once again operates on $SL_2(\mathbb{A}_F)$ via conjugation, well-defined up to inner-automorphisms, so also on its representations; this then allows one to define $H_0$ for $D^\times(\mathbb{A}_F)$ as the image of the reduced norm mapping, and $H_1, H_2, H_3$ as subgroups of this norm mapping. The appeal to base change from $GL_2(\mathbb{A}_F)$ to $GL_2(\mathbb{A}_E)$ in the previous argument can now be done using Jacquet–Langlands correspondence from automorphic representations of $D^\times(A_F)$ to automorphic representations of $GL_2(\mathbb{A}_F)$ and then to $GL_2(\mathbb{A}_E)$. We state this as the following theorem.

Theorem 11.4. *Let $D$ be a quaternion algebra over a number field $F$, and $E$ a quadratic extension of $F$ contained in $E$. Let $D^+(\mathbb{A}_F)$ be the subgroup of $D^\times(\mathbb{A}_F)$ consisting of those elements with reduced norm in $\mathbb{N}(\mathbb{A}_E^\times)$. Suppose $\Pi^+$ is an irreducible cuspidal representation of $D^+(\mathbb{A}_F)$ which is globally $\mathbb{A}_E^\times$-distinguished by a quadratic character $\omega$ of $\mathbb{A}_F^\times/F^\times$; i.e., by the character $\omega\circ\mathbb{N}$ of $\mathbb{A}_E^\times/E^\times$. Then any isotypical piece of automorphic representations of $SL_1(D)(\mathbb{A}_F)$ contained in the restriction of $\Pi^+$ has a nonzero period integral on $\mathbb{A}_E^1/E^1$.*

The strategy in the present paper to come to grips with those automorphic representations of $SL_2(\mathbb{A}_F)$ in a given global $L$-packet which have nonzero period integral for a given embedding of $E^1(\mathbb{A}_F)$ inside $SL_2(\mathbb{A}_F)$ is to prove that such global packets which have no local obstructions for non-vanishing are conjugate to each other by an element of $\mathbb{N}E^\times \subset F^\times$ instead of just being conjugate by $F^\times$, which is the case as they belong to the same $L$-packet. The following lemma suggests that this strategy will not succeed in the presence of certain principal series components, which one may call *supersingular primes*, being analogues of supersingular primes for elliptic curves.

Lemma 11.5. *Let $K$ be a quadratic unramified extension of a local field $k$ of odd residue characteristic. Let $\mu$ be an unramified character of $k^\times$ of order 4 with $\mu^2=\omega_{K/k}$. Then the principal series representation $\pi=\mathrm{Ps}(\mu,\mu\omega_{K/k})$ of $GL_2(k)$ decomposes as a sum of two irreducible representations $\pi^+$ and $\pi^-$ when restricted to $GL_2^+(k)$ in which $\pi^+$ is the one which is spherical, i.e., contains a vector fixed under $GL_2(\mathcal{O}_k)$. Fix an embedding of $K^\times$ in $GL_2^+(k)$ such that $K^\times \subset k^\times\cdot GL_2(\mathcal{O}_k)$. Then the trivial representation of $K^\times$ appears in $\pi^+$, and the ramified character of order 2 of $K^\times/k^\times$ appears in $\pi^-$.*

*Proof.* Let $\varpi$ be a uniformizing element in $k$, and $\mathcal{O}_k, \mathcal{O}_K$ be respectively the maximal compact subrings of $k$ and $K$. Since $K^\times \subset k^\times\cdot GL_2(\mathcal{O}_k)$, $\pi^+$ has trivial central character, and $\pi^+$ has a vector fixed under $GL_2(\mathcal{O}_k)$, the trivial representation of $K^\times$ appears in $\pi^+$. The representation $\pi^-$ is obtained from $\pi^+$ by conjugating by the matrix,

$$\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix},$$

hence it is clear that $\pi^-$ has a subrepresentation on which

$$\Gamma_0(\varpi)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2(\mathcal{O}_k)\;\middle|\;\varpi|c\right\}$$

acts trivially. This means that $\pi^-$ must contain the Steinberg representation of $GL_2(\mathbb{F}_q)$ where $\mathbb{F}_q$ is the residue field of $k$ as the Steinberg is the only non-trivial irreducible representation of $PGL_2(\mathbb{F}_q)$ with a fixed vector under the group of upper triangular matrices. Since the Steinberg representation contains all non-trivial characters of $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$, the conclusion about the ramified character of order 2 of $K^\times/k^\times$ appearing in $\pi^-$ follows. □

It should be noted that for an automorphic representation $\widetilde{\Pi}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ of trivial central character, at a place $v_0$ of odd residue characteristic where $\widetilde{\Pi}_0$ is unramified, the representation $\widetilde{\Pi}_0$ decomposes when restricted to $\mathrm{GL}_2^+(F_0)$ if and only if $\widetilde{\Pi}_0$ is as in the previous lemma, i.e., the principal series representation $\pi = \mathrm{Ps}(\mu, \mu\omega_{K/k})$ with $\mu^2 = \omega_{K/k}$. These are what are called supersingular primes in the classical language, and are interpreted by vanishing of the Fourier-coefficient: $a_v = 0$. It is expected that for non-CM modular forms of weight $\geqslant 4$, there are only finitely many supersingular primes (for arbitrary $F$); for example, a famous conjecture of Lehmer asserts that there are no supersingular primes for the Ramanujan Delta function. Thus the following theorem is not without content, although its applicability at the moment is only theoretical; besides, its proof also depends on Conjecture 1.3 about simultaneous non-vanishing of $L$-values.

Theorem 11.6. *Assume Conjecture 1.3. Let $\widetilde{\Pi}$ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central character, and with a discrete series local component at an odd place, say $v_0$, and only finitely many supersingular primes. Then any automorphic representation $\Pi^+$ of $\mathrm{GL}_2^+(\mathbb{A}_F)$ contained in $\widetilde{\Pi}$ which is locally distinguished by $E^1(\mathbb{A}_F)$ is globally $\lambda$-distinguished for a character $\lambda$ of $\mathbb{A}_E^\times/E^\times\mathbb{A}_F^\times$ of order 2, and hence by Theorem 11.1, the local-global principle holds for automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ contained in $\widetilde{\Pi}$ for the subgroup $\mathbb{A}_E^1$.*

*Proof.* Let $S$ be a finite set of places containing all ramified places of $\widetilde{\Pi}$, places of residue characteristic 2, as well as infinite places, and all the supersingular primes which we are assuming is a finite set. By the remarks above, the representation $\widetilde{\Pi}_v$ remains irreducible when restricted to $\mathrm{GL}_2^+(F_v)$ for places $v$ outside $S$. Since $\Pi^+$ is locally distinguished by $E^1(\mathbb{A}_F)$, it is $\lambda_v$-distinguished for some characters $\lambda_v$ of $F_v^\times$ of order $\leqslant 2$. Globalize these characters $\lambda_v$ for $v$ in $S$ to a quadratic character $\lambda$ of $\mathbb{A}_F^\times/F^\times$ for which we then know that $\Pi^+$ is locally $\lambda$-distinguished at all places of $F$ because of an easy observation that an irreducible principal series representation of $\mathrm{PGL}_2(F_v)$ is $\lambda_v$-distinguished for any quadratic character $\lambda_v$ of $E_v^\times/F_v^\times$. By Lemma 10.1, there are quadratic characters $\eta$ of $\mathbb{A}_F^\times/F^\times$ matching with $\lambda$ at places in $S\backslash\{v_0\}$ such that the following global epsilon factors are 1:

$$\epsilon(\widetilde{\Pi} \otimes \eta) = \epsilon(\widetilde{\Pi} \otimes \eta\omega_{E/F}) = 1.$$

We are then in the context of Conjecture 1.3 which gives a character $\mu$ of $\mathbb{A}_F^\times/F^\times$ of order 2 matching with $\eta$ at all places of $S$ such that

$$L(\tfrac{1}{2}, \widetilde{\Pi} \otimes \mu) \neq 0,$$
$$L(\tfrac{1}{2}, \widetilde{\Pi} \otimes \mu\omega_{E/F}) \neq 0.$$

Since Lemma 10.1 also guarantees

$$\epsilon(\widetilde{\Pi}_v \otimes \eta_v)\epsilon(\widetilde{\Pi}_v \otimes \omega_v\eta_v) = \epsilon(\widetilde{\Pi}_v \otimes \lambda_v)\epsilon(\widetilde{\Pi}_v \otimes \omega_v\lambda_v)$$

at each place $v$ of $F$, by the Saito–Tunnell theorem, we see that $\tilde{\Pi}_{v_0}$ is both $\eta_{v_0}$-distinguished and $\lambda_{v_0}$-distinguished even if $\eta_{v_0} \neq \lambda_{v_0}$. By the theorem of Waldspurger, the above non-vanishing of $L$-values then implies that $\widetilde{\Pi}$ is globally $\mu$-distinguished. By local multiplicity one, this is enough to conclude that $\Pi^+$ is globally $\mu$-distinguished, provided we know that $\Pi_{v_0}^+$ is $\mu_{v_0}$-distinguished. We prove that $\Pi_{v_0}^+$ is $\mu_{v_0}$-distinguished by proving that both the characters of $E_v^\times/F_v^\times E_v^{\times 2}$ appear on the same component of the restriction of $\tilde{\Pi}_v$ to $\mathrm{GL}_2^+(F_v)$ (so the possible difference between $\mu$ and $\lambda$ at $v_0$ has no consequence for the question of distinction), and this follows from the following lemma. □



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Lemma 11.7. *Let $E/F$ be a quadratic extension of $p$-adic fields with $p$ odd. Let $\pi^+$ be a supercuspidal representation of $\mathrm{GL}_2^+(F)$ of trivial central character which is distinguished with respect to $E^1$. Additionally, let $\tilde{\pi}$ be a supercuspidal representation of $\mathrm{GL}_2(F)$ which is distinguished with respect to $E^\times$ such that $\pi^+$ occurs in the restriction of $\tilde{\pi}$ to $\mathrm{GL}_2^+(F)$. Then the multiplicity of the trivial representation of $E^1$ in $\pi^+$ is 2.*

*Proof.* Note that since $p$ is odd, we have $E^\times/F^\times E^{\times 2} = \mathbb{Z}/2$, and we claim that both the characters of $E^\times/F^\times E^{\times 2}$ do appear in $\tilde{\pi}$. By Proposition 10.3(iii), $\tilde{\pi}$ is $\alpha \circ \mathbb{N}m$-distinguished with respect to $E^\times$ for a quadratic character $\alpha$ of $F^\times$ if and only if $\alpha \in X^{\nu,\nu'}$ with $\nu\nu' = 1$. Now by our assumption $\tilde{\pi}$ is distinguished with respect to $E^\times$ and therefore we conclude that $1 \in X^{++}$. By Proposition 10.4, $X^{--} \neq \emptyset$, and we can choose a character $\gamma \in X^{--}$. It follows that $\gamma \circ \mathbb{N}m$ is the non-trivial character of $E^\times/F^\times E^{\times 2}$ and that $\tilde{\pi}$ is $\gamma \circ \mathbb{N}m$-distinguished with respect to $E^\times$. This proves the claim. Thus if $\tilde{\pi}$ does not split into a direct sum of two representations on $\mathrm{GL}_2^+(F)$, the assertion of the lemma is obvious.

So we assume that $\tilde{\pi}$ restricts to $\pi^+ \oplus \pi^-$ on $\mathrm{GL}_2^+(F)$. We need to show that $\pi^+$ is $\mu$-distinguished as well, where $\mu$ is the non-trivial character of $E^\times/F^\times E^{\times 2}$. In this case, $\tilde{\pi}$ corresponds to a monomial representation of $W_F$ of the form $\mathrm{Ind}_{W_E}^{W_F}\chi$ for a character $\chi : E^\times \to \mathbb{C}^\times$ with $\chi|_{F^\times} = \omega_{E/F}$. By the extension of the Saito–Tunnell theorem to $\mathrm{GL}_2^+(F)$ due to the second author [Pra94, Theorem 1.2], what we need to show is that

$$\epsilon(\chi\mu, \psi) = \epsilon(\chi, \psi),$$

where we take $\psi$ to be a non-trivial character of $E/F$. Note that we can take $\psi$ to be unramified, i.e., trivial on $\mathcal{O}_E$ but not on $\varpi_E^{-1}\mathcal{O}_E$.

We will prove the above equality by making use of a theorem of Fröhlich–Queyrut [FQ73, Theorem 3], according to which $\epsilon(\tau, \psi) = 1$ if $\tau|_{F^\times} = 1$, as well as the behaviour of degree one epsilon factors under unramified character twists. In the following, $f(\chi)$ denotes the conductor of the multiplicative character $\chi$.

Since $\chi|_{F^\times} = \omega_{E/F}$, we have

$$\epsilon(\chi\widetilde{\omega}, \psi) = 1$$

by the theorem of Fröhlich–Queyrut, where $\widetilde{\omega}$ denotes an extension of $\omega_{E/F}$ to $E^\times$. Suppose $E/F$ is unramified. Then, it follows that

$$\epsilon(\chi, \psi) = (-1)^{f(\chi)},$$

as we can choose $\widetilde{\omega}$ to be unramified. Similarly, we obtain

$$\epsilon(\chi\mu, \psi) = (-1)^{f(\chi\mu)}.$$

Thus if $f(\chi) > 1$, then $f(\chi\mu) = f(\chi)$ and the equality of the epsilon factors follows. The case $f(\chi) = 0$ does not arise since this would mean that $\chi = \chi^\sigma$ which is not possible since $\tilde{\pi}$ is supercuspidal. If $f(\chi) = 1$, we claim that once again $f(\chi\mu) = f(\chi)$, as the only other option is $f(\chi\mu) = 0$, and this also implies that $\chi = \chi^\sigma$ since $\mu = \mu^\sigma$ by the uniqueness of the quadratic character of $E^\times/F^\times E^{\times 2}$.

Now suppose $E/F$ is ramified. This forces $\mu$ to be unramified. Therefore,

$$\epsilon(\chi\mu, \psi) = (-1)^{f(\chi)}\epsilon(\chi, \psi),$$

and thus we only need to note that $f(\chi)$ is even by our assumptions. Indeed, $f(\chi)$ is either even or 1 since $E/F$ is ramified and $\chi|_{F^\times} = \omega_{E/F}$, and $f(\chi) = 1$ is ruled out when $q \equiv 1 \bmod 4$ since in this case, $\chi$ is forced to be Galois-invariant, hence $\tilde{\pi}$ cannot be supercuspidal. Also, $q$ cannot



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be 3 mod 4, since in that case $\tilde{\pi}$ is neither distinguished nor $\mu$-distinguished as can be seen by an application of the Saito–Tunnell theorem. □

Note that the above proof goes through and proves an analogous lemma in the division algebra case except that at the very last step, when $E/F$ is ramified and $q \equiv 3 \bmod 4$, $\epsilon(\chi\mu, \psi) = -\epsilon(\chi, \psi)$ if $f(\chi) = 1$. However, for purposes of the local-global principle this is no problem: if a character $\chi$ of $F^\times$ thought of as a character of $E^\times$ through the norm mapping appears in a representation $\pi^+$ of $D^+$, then clearly so does $\chi\omega_{E/F}$ (being the same character of $E^\times$). For $q \equiv 3 \bmod 4$,

$$\frac{\epsilon(\widetilde{\pi} \otimes \chi)}{\chi(-1)} = -\frac{\epsilon(\widetilde{\pi} \otimes \chi\omega_{E/F})}{\chi(-1)\omega_{E/F}(-1)}.$$

This gives the required change of sign argument used earlier to prove the local-global principle for $E^1(\mathbb{A}_F)$ contained in $\mathrm{SL}_1(D)(\mathbb{A}_F)$, assuming finitely many supersingular primes, Conjecture 1.3 and one odd prime where the representation is discrete series; we omit the details.

*Remark.* To prove Theorem 11.6 without the finiteness condition on supersingular primes, we will need a finer version of Conjecture 1.3 which has allowed us the existence of the quadratic character $\eta$ at the end of this theorem. The refinement would seek to construct $\eta$ with prescribed behaviour inside $S$, which is unramified at those places outside $S$ where $\Pi$ is supersingular. This is because, as we noted earlier, the behaviour of $\eta$ outside of $S$ and the supersingular primes does not matter for distinction questions as the representation $\Pi_v$ of $\mathrm{GL}_2(F_v)$ remains irreducible when restricted to $\mathrm{GL}_2^+(F_v)$. At least in the non-CM case, since the supersingular set is rather 'thin', one hopes that this strengthening may be possible.

## 12. A final remark

The two cases of the local-global principle studied in the paper relied on the Asai lift and the base change map. One part of the argument had to do with the fibers of these functorial maps. The other part consisted in proving that for $E/F$ a quadratic extension of number fields, certain characters of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_F^\times$ are Grössencharacters are themselves Grössencharacters, if we know certain properties of these characters under base change or Asai lift as the case may be. It seems worthwhile to isolate these as questions. Before we do this, it must be added that at the moment, automorphy of the tensor product $\Pi \boxtimes \Pi'$, or of the Asai lift, is known only in certain cases, so either the questions below could be asked for only those cases, or we should be willing to grant these in general.

*Question.* Suppose $E$ is a number field, and $\Pi = \otimes\Pi_v$ is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_E)$, and $\Pi' = \otimes\Pi'_v$ is an automorphic representation of $\mathrm{GL}_m(\mathbb{A}_E)$.

(i) Suppose that $\Pi \boxtimes \Pi'$ is automorphic. Then is there an automorphic representation $\Pi''$ of $\mathrm{GL}_n(\mathbb{A}_E)$ with $\Pi'' \boxtimes \Pi' \cong \Pi \boxtimes \Pi'$? What are the various automorphic representations $\Pi''$ of $\mathrm{GL}_n(\mathbb{A}_E)$ with this property? (This part of the question generalizes the notion of self-twists of automorphic representations.)

(ii) Suppose that $\mathrm{BC}(\Pi \boxtimes \Pi')$ is automorphic. Then is there an automorphic representation $\Pi''$ of $\mathrm{GL}_n(\mathbb{A}_E)$ with $\mathrm{BC}(\Pi'' \boxtimes \Pi') \cong \mathrm{BC}(\Pi \boxtimes \Pi')$?

(iii) Suppose that $\mathrm{As}(\Pi)$ is automorphic. Then is there an automorphic representation $\Pi''$ of $\mathrm{GL}_n(\mathbb{A}_E)$ with $\mathrm{As}(\Pi'') \cong \mathrm{As}(\Pi)$? What are the various automorphic representations $\Pi''$ of $\mathrm{GL}_n(\mathbb{A}_E)$ with this property?

*Remark.* We remark that Blasius [Bla94] has constructed examples of automorphic representations $\Pi_1$ and $\Pi_2$ on $\mathrm{GL}_n(\mathbb{A}_F)$ which are locally twists of each other at all places of $F$, but are not globally twists of each other by a Grössencharacter. This means that the answer to question (i) above is not always positive. This negative solution to question (i) is itself rather interesting; however, we are asking if there are ways of making it into a positive answer, by dictating either local or global conditions on $\Pi'$. A very specific suggestion would be to ask if question (i) has an affirmative answer if we assume that $\Pi'$ has a local component which is a Steinberg representation. We do not know if there are automorphic representations $\Pi_1$ and $\Pi_2$ on $\mathrm{GL}_n(\mathbb{A}_F)$ which are locally twists of each other at all (or, almost all) places of $F$, have a Steinberg local component say in $\Pi_1$, but are not globally twists of each other by a Grössencharacter.

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References

AKT04 U. K. Anandavardhanan, A. C. Kable and R. Tandon, *Distinguished representations and poles of twisted tensor L-functions*, Proc. Amer. Math. Soc. **132** (2004), 2875–2883; MR 2063106(2005g:11080).

AP03 U. K. Anandavardhanan and D. Prasad, *Distinguished representations for* SL(2), Math. Res. Lett. **10** (2003), 867–878; MR 2025061(2004j:22018).

AP06 U. K. Anandavardhanan and D. Prasad, *On the* SL(2) *period integral*, Amer. J. Math. **128** (2006), 1429–1453; MR 2275907(2008b:22014).

Bla94 D. Blasius, *On multiplicities for* SL($n$), Israel J. Math. **88** (1994), 237–251; MR 1303497(95i:11049).

Fli88 Y. Z. Flicker, *Twisted tensors and Euler products*, Bull. Soc. Math. France **116** (1988), 295–313; MR 984899(89m:11049).

Fli91 Y. Z. Flicker, *On distinguished representations*, J. Reine Angew. Math. **418** (1991), 139–172; MR 1111204(92i:22019).

FH94 Y. Z. Flicker and J. L. Hakim, *Quaternionic distinguished representations*, Amer. J. Math. **116** (1994), 683–736; MR 1277452(95i:22028).

FH95 S. Friedberg and J. Hoffstein, *Nonvanishing theorems for automorphic L-functions on* GL(2), Ann. of Math. (2) **142** (1995), 385–423; MR 1343325(96e:11072).

FQ73 A. Fröhlich and J. Queyrut, *On the functional equation of the Artin L-function for characters of real representations*, Invent. Math. **20** (1973), 125–138; MR 0321888(48#253).

GGP12 W. T. Gan, B. H. Gross and D. Prasad, *Symplectic local root numbers, central critical l-values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), 1–109.

GP92 B. H. Gross and D. Prasad, *On the decomposition of a representation of* $\mathrm{SO}_n$ *when restricted to* $\mathrm{SO}_{n-1}$, Canad. J. Math. **44** (1992), 974–1002; MR 1186476(93j:22031).

Hak91 J. Hakim, *Distinguished p-adic representations*, Duke Math. J. **62** (1991), 1–22; MR 1104321(92c:22037).

HLR86 G. Harder, R. P. Langlands and M. Rapoport, *Algebraische Zyklen auf Hilbert–Blumenthal–Flächen*, J. Reine Angew. Math. **366** (1986), 53–120; MR 833013(87k:11066).

Jac87 H. Jacquet, *On the nonvanishing of some L-functions*, Proc. Indian Acad. Sci. Math. Sci. **97** (1987), 117–155; MR 983610(90e:11079).

JL85 H. Jacquet and K. F. Lai, *A relative trace formula*, Compositio Math. **54** (1985), 243–310; MR 783512(86j:11059).

Kab04 A. C. Kable, *Asai L-functions and Jacquet's conjecture*, Amer. J. Math. **126** (2004), 789–820; MR 2075482(2005g:11083).

Kri03 M. Krishnamurthy, *The Asai transfer to* $\mathrm{GL}_4$ *via the Langlands–Shahidi method*, Int. Math. Res. Not. **41** (2003), 2221–2254; MR 2000968(2004i:11050).

Kri12 M. Krishnamurthy, *Determination of cusp forms on GL(2) by coefficients restricted to quadratic subfields*, J. Number Theory **132** (2012), 1359–1384; with an appendix by Dipendra Prasad and Dinakar Remakrishnan; MR 2899809.

Lar94 M. Larsen, *On the conjugacy of element-conjugate homomorphisms*, Israel J. Math. **88** (1994), 253–277; MR 1303498(95k:20073).

LL79 J.-P. Labesse and R. P. Langlands, *L-indistinguishability for* SL(2), Canad. J. Math. **31** (1979), 726–785; MR 540902(81b:22017).

MP00 V. K. Murty and D. Prasad, *Tate cycles on a product of two Hilbert modular surfaces*, J. Number Theory **80** (2000), 25–43; MR 1735646(2000m:14028).

Pra92 D. Prasad, *Invariant forms for representations of* $\mathrm{GL}_2$ *over a local field*, Amer. J. Math. **114** (1992), 1317–1363; MR 1198305(93m:22011).

Pra94 D. Prasad, *On an extension of a theorem of Tunnell*, Compositio Math. **94** (1994), 19–28; MR 1302309(95k:22023).

Pra00 D. Prasad, A relative local langlands conjecture.

Ram00 D. Ramakrishnan, *Modularity of the Rankin–Selberg L-series, and multiplicity one for* SL(2), Ann. of Math. (2) **152** (2000), 45–111; MR 1792292(2001g:11077).

Sai93 H. Saito, *On Tunnell's formula for characters of* GL(2), Compositio Math. **85** (1993), 99–108; MR 1199206(93m:22021).

SV00 Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties*.

Ser02 J.-P. Serre, *Galois cohomology*, Springer Monographs in Mathematics, English edition (Springer, Berlin, 2002), translated from the French by Patrick Ion and revised by the author; MR 1867431(2002i:12004.

Tun83 J. B. Tunnell, *Local* $\epsilon$*-factors and characters of* GL(2), Amer. J. Math. **105** (1983), 1277–1307; MR 721997(86a:22018).

Wal85 J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), 173–242; MR 783511(87g:11061b).

Wal91 J.-L. Waldspurger, *Correspondances de Shimura et quaternions*, Forum Math. **3** (1991), 219–307; MR 1103429(92g:11054).



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