BULL. AUSTRAL. MATH. SOC. 16A27
VOL. 31 (1985), 355-363. (20C07)

NORMAL AND SUBNORMAL SUBGROUPS IN THE GROUP OF UNITS OF GROUP RINGS

JAIRO ZACARIAS GONCALVES

Let KG be the group ring of the group G over the infinite field K, and let $U(KG)$ be its group of units. If G is torsion, we obtain necessary and sufficient conditions for a finite subgroup H of G to be either normal or subnormal in $U(KG)$. Actually, if H is subnormal in $U(KG)$, we can handle not only the case H finite, but the precise assumptions depend on the characteristic of K.

1. Introduction

Let RG be the group ring of the group G over an integral domain R, and let $U(RG)$ be its group of units. When $R = K$, an infinite field of characteristic $p > 0$, and G is a torsion group we show that finite normal and subnormal subgroups of $U(KG)$ are central or "almost" central. This has a strong resemblance to the case in which G is finite, and is in the same line as Pearson [6], and Pearson and Taylor [7].

If $R = Z$, the ring of rational integers, we conclude that G is subnormal in $U(ZG)$ if, and only if, G is an abelian or a Hamiltonian 2-group; as a corollary we obtain [8], Theorem 1.

Our technique, which already appears in [3], is inspired by Herstein

Received 2 November 1984. This work was completed while the author was visiting the University of Alberta, and was supported by CNPq.

https://doi.org/10.1017/S000497270000931X Published online by Cambridge University Press

[5]. We are indebted to Arnaldo Mandel for many useful conversations.

2. Preliminary results

If H is a subgroup of G we write $H \lhd G$ to indicate that H is normal in G, and $H \lhd\lhd G$ to indicate that H is subnormal in G.

LEMMA 2.1. *Let Π be a nonempty subset of the set of rational primes, and let N be a Π-subgroup of G, $N \lhd\lhd G$. Then there exists a Π-subgroup M, $M \lhd G$, such that $N \subseteq M$.*

Proof. See [5], Lemma 1.

Let $KG[X]$ be the polynomial ring in the commutative indeterminate X with coefficients in KG.

LEMMA 2.2 (van der Monde determinant argument). *Let $f(X)$ be an element of $KG[X]$. If $f(X)$ assumes the same value for infinitely many elements of K, then f is constant.*

Proof. The claim is obviously equivalent to the statement that, if $f(X)$ has an infinite number of zeros in K, then $f(X)$ is zero. Hence, let $f(X) = a_0 + a_1X + \ldots + a_nX^n$ and let $\lambda_0, \lambda_1, \ldots, \lambda_n$ be a set of $n+1$ distinct zeros of $f(X)$ in K. Then, using matrix notation

$$[a_0, a_1, \ldots, a_n]\begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_0 & \lambda_1 & \ldots & \lambda_n \\ \lambda_0^2 & \lambda_1^2 & \ldots & \lambda_n^2 \\ \vdots & & & \\ \lambda_0^n & \lambda_1^n & \ldots & \lambda_n^n \end{bmatrix} = [0, 0, \ldots, 0] .$$

The matrix is invertible, whence $[a_0, a_1, \ldots, a_n] = [0, 0, \ldots, 0]$.

PROPOSITION 2.3. *Let m and n be positive integers, $n > 1$, let $f(X) = a_0 + a_1X + \ldots + a_{mn}X^{mn}$ be an element of $KG[X]$, and suppose that the rational function $g(X) = f(X)/(1-X^n)^m$ assumes the same value for infinitely many elements of K. Then $a_1 = 0$.*

Proof. Let B be an infinite subset of K, and let c be an

https://doi.org/10.1017/S000497270000931X Published online by Cambridge University Press

element of KG such that

$$g(\lambda) = c \text{ for every } \lambda \in B .$$

Then $f(\lambda) = c(1-\lambda^n)^m$ for every $\lambda \in B$, and since the right hand side has no term of first degree in λ, by Lemma 2.2, $a_1 = 0$.

Given an element $u \in KG$, char $K = p > 0$, we define an inner derivation in KG in the usual way: if $w \in KG$, then $w^{(0)} = w$, $w' = wu - uw$ and $w^{(i+1)} = (w^{(i)})'$, for every $i \geq 1$. Now, if r is a positive integer, we recall the formula

$$w^{(p^r)} = wu^{p^r} - u^{p^r}w . \tag{I}$$

3. Normal subgroups of the group of units

We denote by $\zeta U(KG)$ the center of $U(KG)$.

THEOREM 3.1. *Let K be an infinite field, let G be a group generated by torsion elements and let H be a subgroup of $U(KG)$. Suppose moreover that, either H is finite or H is abelian and $H \subseteq G$. Then $H \lhd U(KG)$ if, and only if, $H \subseteq \zeta U(KG)$.*

Proof. Only necessity requires a proof.

(i) H is finite. Let $h \in H$, and let $g \in G$ be a generator of G of order n. Let λ be an element of K such that $\lambda^n \neq 1$. Then

$$(1-\lambda g)^{-1} = \frac{1+\lambda g+\ldots+\lambda^{n-1}g^{n-1}}{1-\lambda^n} ,$$

and let us consider for such λ the element

$$h_\lambda = h(1-\lambda g)h^{-1}(1-\lambda g)^{-1} .$$

Developing the above expression as a polynomial in λ we obtain

$$h_\lambda = \frac{1+\lambda\left(g-hgh^{-1}\right)+\lambda^2 q_2(g,h)+\ldots+\lambda^n q_n(g,h)}{1-\lambda^n}$$

where $q_2(g, h), \ldots, q_n(g, h) \in KG$.

https://doi.org/10.1017/S000497270000931X Published online by Cambridge University Press

Now, since $H \lhd U(KG)$, the rational function

$$\phi(X) = \frac{1+\left(g-hgh^{-1}\right)X+q_2X^2+\ldots+q_nX^n}{1-X^n}$$

assume values in H, for an infinite number of λ in K. Thus, since H is finite, the Pigeon-Hole Principle implies that $\phi(X)$ assumes the same value for infinitely many λ in K. By Proposition 2.3,

$$g - hgh^{-1} = 0$$

and

$$gh = hg .$$

Since every $h \in H$ commutes with every torsion generator $g \in G$ the conclusion follows.

(ii) H is abelian and $H \subseteq G$. Arguing as in (i), we observe that for infinitely many λ belonging to K we have that $\left(1-\lambda^n\right)\phi(\lambda) \in KH$. Hence, solving the system of equations as in Lemma 2.2, we conclude that

$$g - hgh^{-1} \in KH ,$$

or

$$gh - hg \in KH .$$

So, if $g \in G\backslash H$ is a torsion generator of G, we obtain $gh = hg$, as was to be proved.

4. Subnormal subgroups of the groups of units

We cannot handle the subnormality question as easily, and so we will study separately the cases $p = 0$ and $p > 0$.

Let R be an integral domain. We denote by $V(RG)$ the group of normalized units of RG, that is, the set of elements of $U(RG)$ with augmentation one. The proposition below is an easy generalization of [9], Theorem II 5.1.

PROPOSITION 4.1. *Let G be a group, let R be an integral domain of characteristic 0 such that no rational prime is a unit of R, and let H be a torsion subnormal subgroup of $V(RG)$. Then $H \subseteq G$ and H is an* *abelian or a Hamiltonian 2-group, with every subgroup normal in* G .

Proof. By Lemma 2.1 there exists a torsion normal subgroup N of $V(RG)$ such that $H \subseteq N$. By [9], Theorem II 5.1, the conclusion follows.

THEOREM 4.2. *Let* G *be a torsion group. Then* $G \lhd \lhd U(ZG)$ *if, and only if* G *is an abelian or a Hamiltonian 2-group.*

Proof. Necessity. Since $G \subseteq V(ZG)$ we have that $G \lhd \lhd V(ZG)$, and applying Proposition 4.1 we arrive at the desired conclusion.

Sufficiency. Apply [9], Corollary II 2.5.

The corollary below implies in particular, [8], Theorem 1.

COROLLARY 4.3. *Let* G *be a torsion group. Then* $U(ZG)$ *is nilpotent if, and only if,* G *is an abelian or a Hamiltonian 2-group.*

Proof. Necessity. Since $U(ZG)$ is nilpotent every subgroup of $U(ZG)$ is subnormal. So $G \lhd \lhd U(ZG)$, and by Theorem 4.2, G is an abelian or a Hamiltonian 2-group.

Sufficiency. If G is abelian there is nothing to prove. If G is a Hamiltonian 2-group apply [9], Corollary II 2.5.

THEOREM 4.4. *Let* K *be a field of characteristic* 0 , *let* G *be a torsion group and let* H *be a subgroup of* G . *Then* $H \lhd \lhd U(KG)$ *if, and only if* $H \subseteq \zeta G$.

Proof. Only necessity requires a proof.

Since $H \lhd \lhd U(KG)$ it follows that $H \lhd \lhd U(ZG)$ so, by Proposition 4.1, H is either an abelian or a Hamiltonian 2-group.

Suppose that H is a Hamiltonian 2-group. Then $H = K_8 \times E$, the direct product of the quaternion group of order 8 by an elementary abelian 2-group E . So $K_8 \lhd H$, and $K_8 \lhd \lhd U(KG)$ implies that $K_8 \lhd \lhd U(KK_8)$, in contradiction to [2], Theorem 2.4.

Therefore H is abelian and we claim that H is central. Suppose not. Then there exist $a \in H$ and $g \in G$ such that $(a, g) \neq 1$, and since $\langle a \rangle \lhd G$ it follows that $\tilde{G} = \langle a, g \rangle$, the subgroup generated by a and g , is finite. Again $\langle a \rangle \lhd H \lhd \lhd U(KG)$ and so $\langle a \rangle \lhd \lhd U(K\tilde{G})$, in contradiction to [2], Theorem 2.4.

Now we turn our attention to the case $p > 0$.

PROPOSITION 4.5. *Let* K *be an infinite field of characteristic* $p > 0$ *, let* G *be a group generated by torsion elements, and let* H *be a subnormal subgroup of* $U(KG)$ *such that either* H *is finite or* H *is nilpotent. Then there exists a positive integer* $l \geq 1$ *such that* $H^{p^l} \subseteq \zeta U(KG)$.

Proof. (i) H is finite. Let $H = N_r \lhd N_{r-1} \lhd \ldots \lhd N_1 \lhd N_0 = U(KG)$ be a subnormal series for H , let $h \in H$, let $g \in G$ be a generator of G of order n , and let $\lambda \in K$ be such that $\lambda^n \neq 1$. Then

$$(1-\lambda g)^{-1} = \frac{1+\lambda g+\ldots+\lambda^{n-1}g^{n-1}}{1-\lambda^n}$$

and we define recursively,

$$c_{\lambda 1} = (h, (1-\lambda g)), \; c_{\lambda 2} = (c_{\lambda 1}, h) \;, \ldots, \; c_{\lambda(i+1)} = (c_{\lambda i}, h)$$

for every positive integer i , and where $(x, y) = xyx^{-1}y^{-1}$.

As before

$$c_{\lambda 1} = \frac{1+\lambda(g'h^{-1})+\lambda^2 q_2(g,h)+\ldots+\lambda^n q_n(g,h)}{1-\lambda^n}$$

with $q_2, \ldots, q_n \in KG$. In general, an easy induction argument shows that

$$c_{\lambda m} = \frac{1+\lambda(g^{(m)}h^{-m})+\lambda^2 s_2(g,h)+\ldots+\lambda^{mn} s_{mn}(g,h)}{(1-\lambda^n)^m}$$

where $s_2, \ldots, s_{mn} \in KG$.

Now choose a positive integer l such that $m = p^l > r$. Then $c_{\lambda m} \in H$ for every $\lambda \in K$ such that $\lambda^n \neq 1$, and since H is finite the Pigeon-Hole Principle implies that the rational function

$$\phi(X) = \frac{1+X(g^{(m)}h^{-m})+X^2 s_2+\ldots+X^{mn} s_{mn}}{(1-X^n)^m}$$

https://doi.org/10.1017/S000497270000931X Published online by Cambridge University Press

assumes the same value for infinitely many $\lambda \in K$. By Proposition 2.3,

$$g^{(m)}h^{-m} = 0 ,$$

and by formula (I) we have

$$g^{(p^l)} = gh^{p^l} - h^{p^l}g = 0 ,$$

$$gh^{p^l} = h^{p^l}g ,$$

and the conclusion follows.

(ii) H is nilpotent. As in (i), we define inductively the elements $c_{\lambda i}$ for every positive integer i. Since H is nilpotent there exists a positive integer l such that, for $m = p^l$, we have $c_{\lambda m} = 1$ for every $\lambda \in K$ with $\lambda^n \neq 1$. Now repeat the argument of (i).

THEOREM 4.6. *Let K be an infinite field of characteristic $p > 0$, let G be a group generated by torsion elements, and let H be a subgroup of $U(KG)$ such that either H is finite or H is torsion nilpotent. Then $H \lhd\lhd U(KG)$ if, and only if:*

(a) $H = P \times Q$, the direct product of a p-group P by a p'-group Q;

(b) there exists a positive integer l such that

$$P^{p^l} \times Q \subseteq \zeta U(KG) \text{ and } P \lhd\lhd U(KG) .$$

Proof. Necessity. (i) H is finite. By Proposition 4.5 there exists a positive integer l such that

$$H^{p^l} \subseteq \zeta U(KG) \cap H \subseteq \zeta H .$$

Therefore $H/\zeta H$ is a finite p-group and hence H is nilpotent. Thus we can write $H = P \times Q$, the direct product of a finite p-group P by a finite p'-group Q. Moreover, since the order of every element of Q is prime to p we have that $Q \subseteq \zeta U(KG)$.

Now, since $P \lhd H$, it follows that $P \lhd\lhd U(KG)$.

(ii) H is torsion nilpotent. Once more, since H is torsion nilpotent, we can write $H = P \times Q$, the direct product of a p-subgroup P by a p'-subgroup Q and repeat the reasoning above, invoking Proposition 4.5.

Sufficiency. Since $P \lhd \lhd U(KG)$ and $Q \subseteq \zeta U(KG)$ it follows that $H = P \times Q \lhd \lhd U(KG)$.

As a consequence of Theorem 4.6 above we can obtain the result of Pearson and Taylor [7] for infinite fields of nonzero characteristic.

COROLLARY 4.7. *Let K be an infinite field of nonzero characteristic p and let G be a finite group. Then a subgroup H of G is subnormal in $U(KG)$ if, and only if, $H = P \times Q$ where P is contained in $O_p(G)$, the maximum normal p-subgroup of G , and Q is a p'-group contained in ζG .*

Proof. Necessity. By Theorem 4.6, $H = P \times Q$, with P a subnormal p-subgroup of $U(KG)$ and $Q \subseteq \zeta U(KG)$ a p'-subgroup. But $P \subseteq G$, and since $P \lhd \lhd U(KG)$ implies $P \lhd \lhd G$, by Lemma 2.1, $P \subseteq O_p(G)$. The remaining part follows from the fact that $\zeta U(KG) \cap G = \zeta G$.

Sufficiency. See [7].

COROLLARY 4.8. *Let G be a finite group and let K be an infinite field. Then $G \lhd \lhd U(KG)$ if and only if $U(KG)$ is nilpotent.*

Proof. If $U(KG)$ is nilpotent certainly $G \lhd \lhd U(KG)$. So, let us assume that $G \lhd \lhd U(KG)$.

If $\operatorname{char} K = 0$, by Theorem 4.4, G is abelian, so $U(KG)$ is nilpotent.

If $\operatorname{char} K = p > 0$, by Theorem 4.6, $G = P \times Q$, the direct product of a p-subgroup P by a central p'-subgroup Q . Now by [1], $U(KG)$ is nilpotent.

COROLLARY 4.9. *Let K be an infinite field of characteristic $p > 0$ and let G be a group generated by torsion elements. Then a subnormal p'-subgroup H of $U(KG)$ is nilpotent if, and only if, $H \subseteq \zeta U(KG)$.*

COROLLARY 4.10. *Let K be an infinite field of characteristic $p > 0$, and let G be a torsion solvable group without p-elements. Then $G \lhd \lhd U(KG)$ if, and only if G is abelian.*

Proof. Suppose that $G \lhd \lhd U(KG)$. By [4], Lemma 1, G contains a nilpotent characteristic subgroup N of class at most two, such that $N \supseteq C_G(N)$, the centralizer of N in G .

Since N is a nilpotent p'-subgroup with $N \lhd \lhd U(KG)$, by Corollary 4.9 it follows that $N \subseteq \zeta G$. So, from $N \supseteq C_G(N)$ we conclude that $G = N$ and G is abelian.

References

[1] J.M. Bateman and D.B. Coleman, "Group algebras with nilpotent unit groups", *Proc. Amer. Math. Soc.* 19 (1968), 448-449.

[2] J.Z. Goncalves, "Free groups in subnormal subgroups and the residual nilpotence of the group of units of group rings", *Canad. Math. Bull.* 27 (1984), 365-370.

[3] J.Z. Goncalves and A. Mandel, "Are there free groups in division rings?", *Israel J. Math.* (to appear).

[4] B. Hartley, "Finite groups of automorphisms of locally soluble groups", *J. Algebra* 57 (1979), 242-257.

[5] I.N. Herstein, "Multiplicative commutators in division rings, II", *Rend. Circ. Mat. Palermo* (2) 29 (1980), 485-489.

[6] K.R. Pearson, "On the units of a modular group ring II", *Bull. Austral. Math. Soc.* 8 (1973), 435-442.

[7] K.R. Pearson and D.E. Taylor, "Groups subnormal in the units of their modular group ring", *Proc. London Math. Soc.* (3) 33 (1976), 313-328.

[8] C. Polcino Milies, "Integral group rings with nilpotent unit groups", *Canad. J. Math.* 28 (1976), 954-960.

[9] S.K. Sehgal, *Topics in group rings* (Marcel Dekker, New York, 1978).

Instituto de Mat e Estatistica,
Universidade de Sao Paulo,
Cx Postal 20570 Ag Iguatemi,
01000 Sao Paulo, Brasil.