A NEW CHARACTERIZATION OF DEDEKIND DOMAINS

by D. D. ANDERSON and E. W. JOHNSON

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Throughout this paper all rings are assumed commutative with identity. Among integral domains, Dedekind domains are characterized by the property that every ideal is a product of prime ideals. For a history and proof of this result the reader is referred to Cohen [**2**, pp. 31–32]. More generally, Mori [**5**] has shown that a ring has the property that every ideal is a product of prime ideals if and only if it is a finite direct product of Dedekind domains and special principal ideal rings (SPIRS). Rings with this property are called general Z.P.I.-rings.

Since in a Dedekind domain every nonzero ideal is a finite intersection of powers of maximal ideals, it follows that in a general Z.P.I.-ring every ideal is a finite intersection of powers of prime ideals. Butts and Gilmer [**1**] proved the converse.

The purpose of this paper is to generalize and unify the results of Mori and Butts and Gilmer. Hence, if R is a ring, we denote by $\mathscr{L}(R)$ the set of ideals of R, by $\mathscr{P}(R)$ the set of prime ideals of R and by $\hat{\mathscr{P}}(R)$ the closure of $\mathscr{P}(R)$ under products and finite intersections. For convenience we set $R = P^0$ for any prime ideal P, so that $R \in \hat{\mathscr{P}}(R)$. We show that R is a general Z.P.I.-ring if and only if $\mathscr{L}(R) = \hat{\mathscr{P}}(R)$.

We note three elementary but useful facts. If $A \in \hat{\mathscr{P}}(R)$, $A \neq R$, then there are only finitely many prime ideals minimal over A. If $A \in \hat{\mathscr{P}}(R)$ and S is a multiplicatively closed subset of R, then $A_S \in \hat{\mathscr{P}}(R_S)$. And if A and B are ideals with $A \subseteq B$ and $B \in \hat{\mathscr{P}}(R)$, then $B/A \in \hat{\mathscr{P}}(R/A)$. In particular, the property $\mathscr{L}(R) = \hat{\mathscr{P}}(R)$ carries over to R_S and to R/A.

LEMMA 1. *Let R be an integral domain satisfying $\mathscr{L}(R) = \hat{\mathscr{P}}(R)$. Let P be a prime ideal of R minimal over a nonzero principal ideal. Then R_P is a* DVR *and hence* rank $P = 1$.

Proof. Let P be minimal over (a). Then R_P satisfies $\mathscr{L}(R_P) = \hat{\mathscr{P}}(R_P)$ and P_P is minimal over $(a)_P$. Thus we can assume that (R, M) is a quasilocal domain satisfying $\mathscr{L}(R) = \hat{\mathscr{P}}(R)$ and that M is minimal over (a). We must show that R is a discrete valuation ring (DVR). Since M is the only prime ideal containing (a), (a) must be a power of M. Hence M is invertible. Since R is quasilocal, M is principal, say $M = (p)$. If $\bigcap_{n=1}^{\infty} (p^n) = 0$, then (p) is the only nonzero prime ideal of R, so R is a DVR. Hence we may assume that $0 \neq b \in \bigcap_{n=1}^{\infty} (p^n)$. Since each prime $Q \subsetneqq (p)$ satisfies $Q = pQ$ and since for A and B with $pA = A$ and $pB = B$, we have $p(AB) = AB$ and $p(A \cap B) = pA \cap pB = A \cap B$, it follows from $(b) \in \hat{\mathscr{P}}(R)$ that $(b) = p(b)$. Thus by Nakayama's Lemma, $(b) = 0$. Thus R must be a DVR.

LEMMA 2. *Let (R, M) be a quasilocal ring satisfying $\mathscr{L}(R) = \hat{\mathscr{P}}(R)$. Then either $M = M^2$ or M is principal.*

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Proof. Suppose that $M \neq M^2$. Let $x \in M - M^2$. Let $y \in M$, then since $(x, y^2) \in \hat{\mathscr{P}}$ and $(x, y^2) \not\subseteq M^2$, (x, y^2) must be an intersection of prime ideals. Hence $(x, y) \subseteq \sqrt{(x, y^2)}$, so $(x, y) = (x, y^2)$. By Nakayama's Lemma, $(x, y) = (x)$. Since y was arbitrary, $M = (x)$.

THEOREM 3. *Let R be an integral domain. Then R satisfies* $\mathscr{L}(R) = \hat{\mathscr{P}}(R)$ *if and only if R is a Dedekind domain.*

Proof. Since every ideal in a Dedekind domain is a product of prime ideals, a Dedekind domain satisfies $\mathscr{L}(R) = \hat{\mathscr{P}}(R)$. Conversely, suppose that $\mathscr{L}(R) = \hat{\mathscr{P}}(R)$ and that R is not a field. It suffices to show that every rank one prime ideal of R is maximal. For then by Lemma 1, R_M will be a DVR for each maximal ideal M of R. Also, for $0 \neq b \in R$, $(b) \in \hat{\mathscr{P}}(R)$ and hence $\sqrt{(b)}$ is a finite intersection of prime ideals, so (b) is contained in only finitely many maximal ideals of R. Then by [**3**, Theorem 37.2], R will be a Dedekind domain. Let P be a rank one prime ideal of R. Assume that P is not maximal, say $P \subsetneq M$, a maximal ideal. Let $0 \neq a \in P$. Let the other rank one prime ideals containing (a) be $P_1, \ldots, P_n$. Choose $b \in M - (P \cup P_1 \cup \ldots \cup P_n)$. Hence $(a, b) \subseteq M$, but (a, b) is contained in no rank one prime ideal. Shrink M to a prime ideal Q minimal over (a, b). Pass to R_Q. Then $\mathscr{L}(R_Q) = \hat{\mathscr{P}}(R_Q)$ and Q_Q is minimal over $(a, b)_Q$, so $(a, b)_Q = Q_Q^s$ for some $s \geq 1$. Hence by Nakayama's Lemma, we must have $Q_Q \neq Q_Q^2$. By Lemma 2, Q_Q is principal. But then by Lemma 1, R_Q is a DVR, so rank $Q = 1$. This contradiction shows that P must be maximal.

It may be of interest to note that to this point we have used only that for all $x, y \in R$ (an integral domain), $(x) \in \hat{\mathscr{P}}(R)$, $(y) \in \hat{\mathscr{P}}(R)$ and $(x) + (y) \in \hat{\mathscr{P}}(R)$.

THEOREM 4. *A ring R has the property that every proper ideal is in* $\hat{\mathscr{P}}(R)$ *if and only if R is a general* Z.P.I.-*ring.*

Proof. If R is a general Z.P.I.-ring, then every proper ideal is a product of prime ideals and hence is in $\hat{\mathscr{P}}(R)$. Conversely, suppose that every proper ideal of R is in $\hat{\mathscr{P}}(R)$. If P is a prime ideal of R, then $\mathscr{L}(R/P) = \hat{\mathscr{P}}(R/P)$ and hence by Theorem 3 R/P is a Dedekind domain. It follows that $\dim R \leq 1$. Let M be a maximal ideal of R. We show that R_M is either a SPIR or a DVR. Then since (0) is a product of prime ideals of R and since for each maximal ideal M of R, R_M is a SPIR or a DVR, it easily follows that R is a finite direct product of SPIRs and Dedekind domains. (See, for example, the proof of [**3**, Theorem 46.11].) Thus we are reduced to proving that a quasilocal ring (R, M) with $\dim R \leq 1$ satisfying $\mathscr{L}(R) = \hat{\mathscr{P}}(R)$ is either a SPIR or a DVR. If $\dim R = 0$, then every principal ideal of R is a power of M and hence R is a SPIR. Suppose that $\dim R = 1$. Since R has only finitely many minimal prime ideals, M is minimal over a principal ideal (a). Thus (a) is a power of M, so $M \neq M^2$. Hence by Lemma 2, M is principal. Hence $P = \bigcap_{n=1}^{\infty} M^n = MP$ is the unique minimal prime ideal of R. Since every principal ideal of R is a power of M or of P it easily follows that P is principal. Hence $P = 0$. So R is a DVR.

Along the lines of the note following Theorem 3, we note here that Theorem 4 remains

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valid under the assumption that every element of $\mathscr{L}(R)$ is the sum (not necessarily finite) of elements of $\hat{\mathscr{P}}(R)$ and that $A, B \in \hat{\mathscr{P}}(R)$ implies $A + B \in \hat{\mathscr{P}}(R)$.

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UNIVERSITY OF IOWA
IOWA CITY
IOWA 52242, USA