PRODUCTS AND PLETHYSMS OF CHARACTERS WITH ORTHOGONAL, SYMPLECTIC AND SYMMETRIC GROUPS

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1. Introduction. Murnaghan **(9)** has proposed the following method of analyzing the Kronecker product of two symmetric group representations.

If $(\lambda) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_i)$ is a partition of p, the representation of the symmetric group on n symbols corresponding to the partition $(n - p, \lambda_1, \ldots, \lambda_i)$ is denoted by $[\lambda]$ and is said to be of depth p.

If $[\lambda]$ is of depth p and $[\mu]$ of depth q, then the terms in the Kronecker product $[\lambda] \times [\mu]$ of depth $p + q$ are terms which correspond to the terms in the product of S-functions $\{\lambda\}\ \{\mu\}$. Murnaghan gives a similar formula for the terms of depth $p + q - 1$, $p + q - 2$, $p + q - 3$ and $p - q$. He uses these formulae to work out some of the terms in particular cases and uses various artifices to complete the analysis. But he gives no proof of the formulae and it is by no means clear what is the general result for terms of depth $p + q - r$.

In this paper there will be obtained the equivalent of Murnaghan's formulae, proof of the results, and extension to the general result, so that a complete method of analysing the Kronecker product of symmetric group representations will be obtained, or equivalently, of expanding the inner product of two S-functions **(5)**.

The method extends to give an analysis of the invariant matrices of symmetric group representations, and thus yields the most powerful method so far obtained of calculating the inner plethysm of S-functions **(6)**.

In addition the method can be used with even greater simplicity to calculate products and plethysms of characters with orthogonal and symplectic groups. These cases, being simpler, will be dealt with first.

2. Products of orthogonal and symplectic group characters. In this section and in 3, when (λ) is a partition of n, $\{\!\!\{\lambda\}\!\!\}$ will denote the character of the orthogonal group which is associated with this partition **(4**, p. 233**)**. The number of variables will generally be assumed to be large, at least twice the number of parts in any partition. For a smaller number of variables the correct result can always be inferred by using the modification rules **(8**, p. 282**)**.

It is required to find a formula which expresses a product $\{\!\!\{\lambda\}\!\!\}\ \{\!\!\{\mu\}\!\!\}$ in terms of orthogonal group characters.

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If (λ) is a partition of n, let

$$A = A_{i_1 i_2 \dots i_n}$$

be a tensor of type $\{\!\!\{\lambda\}\!\!\}$, which implies that under the full linear group it is of type $\{\lambda\}$, but that it is further reduced so that all contractions with the metric tensor g^{ij} are zero.

Similarly, if (μ) is a partition of m, let

$$B = B_{j_1 j_2 \dots j_m}$$

be a tensor of rank m and type $\{\!\!\{\mu\}\!\!\}$. Consider the product

$$AB = A_{i_1 \dots i_n} B_{j_1 \dots j_m}.$$

Under the full linear group this is of type corresponding to the product $\{\lambda\}\ \{\mu\}$, but some of the contractions with g^{ij} are zero while others are not. The suffixes i, j cannot be contracted with a pair of suffixes of A, nor with a pair from B, for these would lead to a zero result since the contractions have already been removed. But contraction is still possible if one suffix is contracted with a suffix of A and the other with a suffix of B. The contraction gives a non-zero result in the general case since this contraction has not previously been removed.

Let the product AB be contracted with a concomitant of degree r of g^{ij}, the r first suffixes being contracted with A and the r second suffixes with B. Let the r first suffixes be subject to symmetrizing operators corresponding to the S-function $\{\xi\}$ of weight r, and the r second suffixes corresponding to the S-function $\{\eta\}$ of weight r. The symmetrizing operator on the g^{ij}'s therefore corresponds to $\{\xi\} \cdot \{\eta\}$. Since the g^{ij}'s are equal the only possible symmetrizing relation between them is the symmetric one corresponding to $\{r\}$. Hence $\{\xi\} \cdot \{\eta\}$ must contain $\{r\}$, which is only possible if $\{\xi\} = \{\eta\}$.

The contraction of A with a contravariant tensor of type $\{\xi\}$ is of type $\sum \Gamma_{\xi\zeta\lambda}\{\zeta\}$ where $\Gamma_{\xi\zeta\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\xi\}\ \{\zeta\}$. The contraction of B is of type $\sum \Gamma_{\xi\nu\mu}\{\nu\}$.

Hence the contraction of the product $A\ B$ is of type

$$\sum \Gamma_{\xi\zeta\lambda}\, \Gamma_{\xi\nu\mu}\{\zeta\}\{\nu\}.$$

The principal part of this contracted tensor is in general distinct from zero. Hence for each S-function which appears there is a corresponding orthogonal group character in the expansion of $\{\!\!\{\lambda\}\!\!\}\ \{\!\!\{\mu\}\!\!\}$. This is true for every suitable S-function $\{\xi\}$.

THEOREM I. *If*

$$\sum \Gamma_{\xi\zeta\lambda}\, \Gamma_{\xi\nu\mu}\{\zeta\}\{\nu\} = \sum K_{\lambda\mu\rho}\{\rho\}$$

the summation on the left being with respect to all possible S-functions including $\{\xi\} = \{0\}$, *then*

$$\{\!\!\{\lambda\}\!\!\}\{\!\!\{\mu\}\!\!\} = \sum K_{\lambda\mu\rho}\{\!\!\{\rho\}\!\!\}.$$

As an example consider the product $[2][1^2]$. Corresponding to $\{\xi\} = \{0\}$, the product is

$$\{2\}\,\{1^2\} = \{31\} + \{21^2\}.$$

For $\{\xi\} = \{1\}$, the product is

$$\{1\}\,\{1\} = \{2\} + \{1^2\}$$

and no other value of $\{\xi\}$ is possible. Thus

$$[2][1^2] = [31] + [21^2] + [2] + [1^2]$$

a result easily checked by other means.

The method is equally applicable to the symplectic groups. The skew-symmetric fundamental form is available for contractions in exactly the same way as the symmetric metric. In view of the different significance of orthogonal group and symplectic group characters it seems somewhat surprising that the multiplication laws for the two sets of characters are identical.

THEOREM II. *Under the same conditions as in Theorem I*

$$\langle\lambda\rangle\langle\mu\rangle = \sum K_{\lambda\mu\rho}\langle\rho\rangle.$$

Here $\langle\lambda\rangle$ denotes a symplectic group character.

3. Plethysm with orthogonal and symplectic group characters. One of the pleasing features of the method is that it extends directly to plethysm. Formerly the general method of evaluating say $[\lambda] \otimes \{\mu\}$ was to express $[\lambda]$ in terms of S-functions, evaluate the plethysm and convert back into orthogonal group characters (**4**, p. 94). This made the labour of calculation rather tedious, and it was to avoid this tedious calculation that work was done showing that the orthogonal groups in certain numbers of variables were simply isomorphic with certain other groups (**3**). But this only simplified the problem in certain cases, notably in 3 and 4 variables.

The method described here gives a general method which will evaluate for the orthogonal group $[\lambda] \otimes \{\mu\}$, or for the symplectic group $\langle\lambda\rangle \otimes \{\mu\}$, in any number of variables.

The method is best described by means of an example. Consider $[21] \otimes \{2\}$. The expansion of $[21][21]$ may be obtained from Theorem I, and this is equal to

$$[21][21] = [21] \otimes \{2\} + [21] \otimes \{1^2\}.$$

To calculate $[21][21]$ the following terms are obtained

$$\begin{aligned}
\{21\}\,\{21\} &= \{42\} + \{41^2\} + \{3^2\} + 2\{321\} + \{2^3\} + \{31^3\} + \{2^21^2\},\\
\{2\}\;\;\{2\} &= \{4\} + \{31\} + \{2^2\},\\
\{1^2\}\,\{1^2\} &= \{2^2\} + \{21^2\} + \{1^4\},\\
\{2\}\,\{1^2\} &= \{31\} + \{21^2\},\\
\{1^2\}\;\;\{2\} &= \{31\} + \{21^2\},\\
\{1\}\;\;\{1\} &= \{2\} + \{1^2\}, \{1\{\;\{1\} = \{2\} + \{1^2\},\\
\{0\}\;\;\{0\} &= \{0\}.
\end{aligned}$$

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Considering the product of two equal tensors of rank 3, $A_{ijk}\,A_{pqr}$ it is clear that, of the terms corresponding to the product $\{21\}\,\{21\}$, the symmetric part corresponds to $\{21\} \otimes \{2\}$ and the skew-symmetric part to $\{21\} \otimes \{1^2\}$.

Of the first contraction $g^{ip}A_{ijk}\,A_{pqr}$ the terms corresponding to $\{2\}\,\{2\}$ are changed into themselves by the interchange of A_{ijk} and A_{pqr}. The symmetric part will thus correspond to $\{2\} \otimes \{2\}$ and the skew symmetric part to $\{2\} \otimes \{1^2\}$. Similar results are obtained for the terms corresponding to $\{1^2\}\,\{1^2\}$.

There are certain terms which correspond to $\{2\}\,\{1^2\}$. Interchanging A_{ijk} and A_{pqr} these are changed into different terms corresponding to $\{1^2\}\,\{2\}$. Clearly there is no reduction here corresponding to plethysm, but just one of the two products $\{2\}\,\{1^2\}$ and $\{1^2\}\,\{2\}$ is retained either for $\langle 21\rangle \otimes \{2\}$ or for $\langle 21\rangle \otimes \{1^2\}$.

Treating all the terms in this way the expansion of $\langle 21\rangle \otimes \{2\}$ corresponds to the expansion

$$\begin{aligned}\{21\} \otimes \{2\} + \{2\} \otimes \{2\} + \{1^2\} \otimes \{2\} + \{2\}\,\{1^2\} + 2\{1\} \otimes \{2\} + \{0\} \otimes \{2\} \\ = \{42\} + \{2^3\} + \{321\} + \{31^3\} + \{4\} + 2\{2^2\} + \{1^4\} + \{31\} \\ + \{21^2\} + 2\{2\} + \{0\}.\end{aligned}$$

Hence

$$\begin{aligned}\langle 21\rangle \otimes \{2\} = \langle 42\rangle + \langle 2^3\rangle + \langle 321\rangle + \langle 31^3\rangle + \langle 4\rangle + 2\langle 2^2\rangle + \langle 1^4\rangle + \langle 31\rangle \\ + \langle 21^2\rangle + 2\langle 2\rangle + \langle 0\rangle.\end{aligned}$$

THEOREM III. *If* (μ) *is a partition of* 2, *then*

$$\langle\lambda\rangle \otimes \{\mu\} = \sum H_{\lambda\mu\nu}\langle\nu\rangle$$

where

$$\sum H_{\lambda\mu\nu}\{\nu\} = \sum\, (\Gamma_{\xi\eta\lambda}\{\eta\}) \otimes \{\mu\} + \sum\, \Gamma_{\xi\eta\lambda}\,\Gamma_{\xi\zeta\lambda}\{\eta\}\{\zeta\}, \qquad (\eta) \neq (\zeta).$$

summed for all suitable S-functions $\{\xi\}$, $\{\eta\}$, $\{\zeta\}$, *the last term not being repeated for the interchange of* $\{\eta\}$ *and* $\{\zeta\}$.

The only aspect of the Theorem which is not obvious from the above example is the position of the coefficient $\Gamma_{\xi\eta\lambda}$ in the first summation when this coefficient exceeds 1. Such a case occurs for $\langle 321\rangle \otimes \{2\}$ when $\{\xi\} = \{21\}$, $\{\eta\} = \{21\}$. In this case $\Gamma_{\xi\eta\lambda} = 2$. Referring to contractions of the product of two tensors, there will exist for each tensor two corresponding contractions of type $\{21\}$. If the same contraction is taken for each tensor, these will be interchangeable to give a term corresponding to $\{21\} \otimes \{2\}$ in each case. If, however, different contractions are taken for the two tensors these will not be interchangeable, and there will be a term $\{21\}\,\{21\}$.

Taken together these terms correspond to

$$2(\{21\} \otimes \{2\}) + \{21\}\,\{21\} = (2\{21\}) \otimes \{2\}.$$

The generalization for any value of $\Gamma_{\xi\eta\lambda}$ presents no difficulty.

For the symplectic group the result is slightly different. When the two tensors are interchangeable allowance must be made for the skew-symmetry of the fundamental form. The difference occurs only when (ξ) is a partition of an odd number, say a partition of $m = 2k + 1$.

Let the fundamental skew-symmetric tensor be r^{ij}. A single contraction between the two tensors, or a contraction m times, will introduce a skew-symmetric relation between the two tensors. This will have the effect of changing $\{\eta\} \otimes \{\mu\}$ into $\{\eta\} \otimes \{\tilde{\mu}\}$ where $(\tilde{\mu})$ is the partition conjugate to (μ).

THEOREM IV. *If* (μ) *is a partition of* 2

$$\langle\lambda\rangle \otimes \{\mu\} = \sum J_{\lambda\mu\nu}\langle\nu\rangle$$

where

$$\sum J_{\lambda\mu\nu}\{\nu\} = \sum (\Gamma_{\xi\eta\lambda}\{\eta\}) \otimes (\{\mu\}.\{\epsilon\}) + \sum \Gamma_{\xi\eta\lambda}\,\Gamma_{\xi\zeta\lambda}\{\eta\}\{\zeta\}, \qquad (\eta) \neq (\zeta),$$

in which $(\epsilon) = (2)$ *if* $\{\xi\}$ *is of even weight, but* $(\epsilon) = (1^2)$ *if* (ξ) *is of odd weight.*

As an example the expansion of $\langle 21\rangle \otimes \{2\}$ will be obtained. The corresponding expansion is

$$\{21\}\otimes\{2\} + \{2\}\otimes\{1^2\} + \{1^2\}\otimes\{1^2\} + \{2\}\{1^2\} + 2\{1\}\otimes\{2\} + \{0\}\otimes\{1^2\}$$
$$= \{42\} + \{2^3\} + \{321\} + \{31^3\} + 2\{31\} + 2\{21^2\} + 2\{2\}.$$

Thus

$$\langle 21\rangle \otimes \{2\} = \langle 42\rangle + \langle 2^3\rangle + \langle 321\rangle + \langle 31^3\rangle + 2\langle 31\rangle + 2\langle 21^2\rangle + 2\langle 2\rangle.$$

To obtain the expansion of $\langle\!-\lambda-\!\rangle \otimes \{\mu\}$ where (μ) is a partition of 3, a procedure is adopted which will first be illustrated with an example. To evaluate $\langle\!-3-\!\rangle \otimes \{3\}$ consider the product of 3 equal tensors of rank 3, each of type $\langle\!-3-\!\rangle$, say

$$A_{ijk}\,A_{pqr}\,A_{stu}.$$

Leaving out contractions with respect to the fundamental tensor g^{ij}, the type is

$$\{3\} \otimes \{3\} = \{9\} + \{72\} + \{63\} + \{52^2\} + \{4^21\}.$$

Allowing one contraction with g^{ij} between the first and second tensors, these two tensors remain symmetric and the type is

$$(\{2\} \otimes \{2\})\,\{3\} = \{7\} + \{61\} + 2\{52\} + \{43\} + \{421\} + \{32^2\}.$$

One contraction between first and second, one between first and third tensors allows the symmetric interchange of the second and third tensors to give

$$\{1\}\,(\{2\} \otimes \{2\}) = \{5\} + \{41\} + \{32\} + \{2^21\}.$$

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Three contractions, one between each pair, allow the three tensors to be permuted symmetrically, and give

$$\{1\} \otimes \{3\} = \{3\}.$$

Two contractions between first and second tensors give

$$(\{1\} \otimes \{2\})\, \{3\} = \{5\} + \{41\} + \{32\}.$$

Two between first and second, one between first and third give

$$\{1\}\, \{2\} = \{3\} + \{21\}.$$

Two between first and second, one between first and third, one between second and third gives

$$\{1\}.$$

Finally three contractions between the first two tensors gives

$$\{3\}.$$

Hence

$$\begin{aligned} \langle 3\rangle \otimes \{3\} = {} & \langle 9\rangle + \langle 72\rangle + \langle 63\rangle + \langle 52^2\rangle + \langle 4^21\rangle + \langle 7\rangle \\ & + \langle 61\rangle + 2\langle 52\rangle + \langle 43\rangle + \langle 421\rangle + \langle 32^2\rangle + 2\langle 5\rangle \\ & + 2\langle 41\rangle + 2\langle 32\rangle + \langle 2^21\rangle + 3\langle 3\rangle + \langle 21\rangle + \langle 1\rangle. \end{aligned}$$

Consider now the general case $\langle\lambda\rangle \otimes \{\mu\}$ with (μ) a partition of 3. It is required to obtain the contractions of the product of three tensors, each of type $\langle\lambda\rangle$. Let the contractions with g^{ij} between the first and second correspond to the S-function $\{\gamma\}$, between the first and third to $\{\beta\}$ and between the second and third to $\{\alpha\}$. The two sets of contractions of the first correspond to $\{\beta\}$ and $\{\gamma\}$, so that the contracted tensor is of type $\sum \Gamma_{\xi\beta\gamma\lambda}\{\xi\}$, where $\Gamma_{\xi\beta\gamma\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\xi\}\, \{\beta\}\, \{\gamma\}$. The type of the contracted product is thus

$$\sum \Gamma_{\xi\beta\gamma\lambda}\, \Gamma_{\eta\alpha\gamma\lambda}\, \Gamma_{\zeta\alpha\beta\lambda}\{\xi\}\{\eta\}\{\zeta\}.$$

Allowing permutations of the three tensors each such term is repeated 6 times, except in certain cases of equality. But only $f^{(\mu)}$ of the 6 terms are retained for $\langle\lambda\rangle \otimes \{\mu\}$, where $f^{(\mu)}$ is the degree of the representation corresponding to (μ) of the symmetric group on 3 symbols.

Consider now the cases of equality. Such a case arises when $\{\alpha\} = \{\beta\}$, $\{\xi\} = \{\eta\}$. The corresponding term for $\langle\lambda\rangle\langle\lambda\rangle\langle\lambda\rangle$ is

$$\Gamma^2_{\xi\alpha\gamma\lambda}\, \Gamma_{\zeta\alpha\alpha\lambda}\, \{\xi\}\, \{\xi\}\, \{\zeta\}.$$

The interchange of the first two tensors leaves this unaltered. It has the effect of interchanging the two α's in the coefficient $\Gamma_{\zeta\alpha\alpha\lambda}$ and also the two $\{\xi\}$'s in the product $\{\xi\}\, \{\xi\}$. In the case of $\langle\lambda\rangle \otimes \{3\}$ either both interchanges must be symmetric or both skew-symmetric. Let $\Gamma_{\zeta\alpha'\lambda}$ be the coefficient of $\{\lambda\}$ in

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$$\{\zeta\}\ (\{\alpha\} \otimes \{2\}),$$

and $\Gamma_{\zeta\alpha''\lambda}$ the coefficient $\{\lambda\}$ in

$$\{\zeta\}\ (\{\alpha\} \otimes \{1^2\}).$$

The corresponding term for $\langle\lambda\rangle \otimes \{3\}$ is

$$\sum \Gamma_{\zeta\alpha'\lambda}\{\zeta\}[(\Gamma_{\xi\alpha\gamma\lambda}\{\xi\}) \otimes \{2\}] + \sum \Gamma_{\zeta\alpha''\lambda}\{\zeta\}[(\Gamma_{\xi\alpha\gamma\lambda}\{\xi\}) \otimes \{1^2\}].$$

The term for $\langle\lambda\rangle \otimes \{1^3\}$ is obtained from this by interchanging α' and α''.

Since $\{21\}$ appears in both the products $\{2\}\ \{1\}$ and $\{1^2\}\ \{1\}$, the corresponding term for $\langle\lambda\rangle \otimes \{21\}$ is the sum of the two, or

$$\sum \Gamma_{\zeta\alpha\alpha\lambda}\,\Gamma^2_{\xi\alpha\gamma\lambda}\{\xi\}\{\xi\}\{\zeta\}.$$

The cases when $\{\alpha\} = \{\gamma\}, \{\xi\} = \{\zeta\}$, or when $\{\beta\} = \{\gamma\}, \{\eta\} = \{\zeta\}$ become equivalent to the above case by a rearrangement of the three tensors, and these cases need not be considered.

There remains only the case

$$\{\alpha\} = \{\beta\} = \{\gamma\}, \qquad \{\xi\} = \{\eta\} = \{\zeta\}.$$

The numerical coefficient becomes

$$\Gamma^3_{\xi\alpha\alpha\lambda} = (\Gamma_{\xi\alpha'\lambda} + \Gamma_{\xi\alpha''\lambda})^3 = \Gamma^3_{\xi\alpha'\lambda} + \Gamma^3_{\xi\alpha''\lambda} + 3\Gamma^2_{\xi\alpha'\lambda}\,\Gamma_{\xi\alpha''\lambda} + 3\Gamma^2_{\xi\alpha''\lambda}\,\Gamma_{\xi\alpha'\lambda}.$$

The terms which correspond to

$$\Gamma^2_{\xi\alpha'\lambda}\,\Gamma_{\xi\alpha''\lambda}$$

and to

$$\Gamma^2_{\xi\alpha''\lambda}\,\Gamma_{\xi\alpha'\lambda}$$

are treated in the same way as the case considered above when $\{\alpha\} = \{\beta\}$, $\{\xi\} = \{\eta\}$.

The term

$$\Gamma^3_{\xi\alpha'\lambda}$$

for $\langle\lambda\rangle \otimes \{\mu\}$ corresponds to

$$\sum (\Gamma_{\xi\alpha'\lambda}\{\xi\}) \otimes \{\mu\}.$$

The term

$$\Gamma^3_{\xi\alpha''\lambda}$$

implies a skew-symmetry for every interchange among the 3 tensors. This has the effect of converting $\{\mu\}$ into $\{\tilde{\mu}\}$, $(\tilde{\mu})$ being the conjugate partition to (μ).

THEOREM V. *If* (μ) *is a partition of* 3,

$$\langle\lambda\rangle \otimes \{\mu\} = \sum H_{\lambda\mu\nu}\langle\nu\rangle$$

where

$$\begin{aligned}\sum \mathrm{H}_{\lambda\mu\nu}\{\nu\} &= f^{(\mu)} \sum \Gamma_{\xi\beta\gamma\lambda}\,\Gamma_{\eta\alpha\gamma\lambda}\,\Gamma_{\zeta\alpha\beta\lambda}\{\xi\}\{\eta\}\{\zeta\} \\ &+ \sum (\Gamma_{\xi\alpha\gamma\lambda}\{\xi\}) \otimes \{\mu'\}\,\Gamma_{\zeta\alpha'\lambda}\{\zeta\} + \sum (\Gamma_{\xi\alpha\gamma\lambda}\{\xi\}) \otimes \{\tilde{\mu}\}\Gamma_{\zeta\alpha''\lambda}\{\zeta\} \\ &+ \sum (\Gamma_{\xi\alpha'\lambda}\{\xi\}) \otimes \{\mu\} + \sum (\Gamma_{\xi\alpha''\lambda}\{\xi\}) \otimes \{\tilde{\mu}\} \\ &+ \sum (\Gamma_{\xi\alpha'\lambda}\{\xi\}) \otimes \{\tilde{\mu}'\}\Gamma_{\xi\alpha''\lambda}\{\xi\} + \sum (\Gamma_{\xi\alpha''\lambda}\{\xi\}) \otimes \{\mu'\}\,\Gamma_{\xi\alpha'\lambda}\{\xi\}.\end{aligned}$$

In this expression, $f^{(\mu)}$ is the degree of the representation corresponding to $\chi^{(\mu)}$; $(\mu') = (2)$ if $(\mu) = (3)$, $(\mu') = (1^2)$ if $(\mu) = (1^3)$, $(\mu') = (2) + (1^2)$ if $(\mu) = (21)$; $\Gamma_{\xi\beta\gamma\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\xi\}\ \{\beta\}\ \{\gamma\}$, $\Gamma_{\xi\alpha'\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\xi\}\ (\{\alpha\} \otimes \{2\})$, $\Gamma_{\xi\alpha''\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\xi\}\ (\{\alpha\} \otimes \{1^2\}$. Terms are omitted in any summation when cases of equality lead to corresponding terms in a later summation.

The case of the symplectic group is very similar. The only differences arise by making allowance for the skew-symmetry of the fundamental form.

THEOREM VI. *If* (μ) *is a partition of* 3,

$$\langle\lambda\rangle \otimes \{\mu\} = \sum J_{\lambda\mu\nu}\langle\nu\rangle$$

where the definition of $\sum J_{\lambda\mu\nu}\{\nu\}$ *differs from that of* $\sum \mathrm{H}_{\lambda\mu\nu}\{\nu\}$ *in Theorem V only by the interchange of* $\{\mu'\}$ *and* $\{\tilde{\mu}'\}$ *in the second and third summations when* (γ) *is a partition of an odd number, and by the interchange of* $\{\mu'\}$ *and* $\{\tilde{\mu}'\}$, $\{\mu\}$ *and* $\{\tilde{\mu}\}$ *in the last* 4 *summations when* (α) *is a partition of an odd number.*

The method extends readily to the cases when (μ) is a partition of 4, 5, 6, etc., no essentially new concept being required. But the details become more and more complicated. The statement of a Theorem even for $n = 4$ must involve so many special cases that it does not seem worth while to enunciate.

4. Symmetric group representations. The method can be applied to representations of the symmetric group, the results being equivalent to evaluating the inner product **(5)** and the inner plethysm **(6)** of S-functions.

Henceforward in this paper $[\lambda] \equiv [\lambda_1, \ldots, \lambda_i]$, where (λ) is a partition of p, will denote the S-function $\{n - p, \lambda_1, \ldots, \lambda_i\}$.

The symmetric group of permutations on n symbols is the group of n-rowed permutation matrices, and is thus a sub-group of the full linear group on n variables $x_1, x_2, \ldots, x_n$. It is in fact the restricted group which leaves invariant a set of forms of respective degrees $1, 2, \ldots, n$, namely, the forms

$$S_1 = \sum x_i,\ S_2 = \sum x_i^2, \ldots,\ S_n = \sum x_i^n.$$

Clearly a permutation of the x_i's will leave invariant these symmetric functions of the x_i's, and conversely if the values of $S_1, S_2, \ldots, S_n$ are assigned the x_i's will be the roots of a certain equation of degree n and the only possible transformations will be the permutations of the roots.

The tensor coefficients of these forms will be denoted by R_i, R_{ij}, R_{ijk}, etc. Although the forms are algebraically independent, the tensors are connected in that every one can be expressed as a concomitant tensor of the quadratic and cubic tensors R_{ij} and R_{ijk}. Since the quadratic tensor R_{ij} is available for raising and lowering suffixes, which it does without modification, there is no distinction between upper and lower suffixes. Then clearly

$$R_{ij}\,R_{ijk} = R_k$$
$$R_{ijk}\,R_{ipq} = R_{jk\,pq}$$

with similar results for tensors of rank 5, 6, etc.

Since transformations which leave certain tensors invariant also leave invariant every concomitant tensor, the following Theorem results.

THEOREM VII. *The symmetric group on n symbols is the subgroup of the full linear group in n-variables which leaves a quadratic form and a cubic form invariant.*

The linear concomitant can be used to reduce the number of variables from n to $n - 1$. The full linear group in $n - 1$ variables may therefore be taken if it is assumed that the linear concomitant is identically zero.

The characters of the symmetric group can be obtained from those of the full linear group in a similar manner to that used for the orthogonal group, namely by considering a tensor corresponding to any partition (λ) of any integer n, and removing all possible contractions with the fundamental forms (**2**, p. 392). The remainder when all contractions are removed is an irreducible character, provided that $n - p \geqslant \lambda_1$, and it is not difficult to see that it is in fact the character of the symmetric group corresponding to the partition $(n - p, \lambda_1, \ldots, \lambda_i)$. It is convenient to represent by $[\lambda]$ not this character, but the corresponding S-function

$$[\lambda] = \{n - p, \lambda_1, \ldots, \lambda_i\}.$$

The inner product and inner plethysm of these S-functions correspond exactly to products and plethysms of symmetric group characters (**5**; **6**).

THEOREM VIII. *If*

$$A_{i_1 i_2 \ldots i_r}$$

is an irreducible tensor under the symmetric group corresponding to the S-function $[\lambda]$, *where* (λ) *is a partition of* r, *then for every non-zero term in the tensor all the suffixes will be different.*

To prove this it is sufficient to show that if two suffixes are equal there exists a non-zero contraction. Suppose that $i_1 = i_2$ and consider the contraction

$$B_{j i_3 i_4 \ldots i_r} = R_{i_1 i_2 j}\,A_{i_1 i_2 \ldots i_r}.$$

Then the term corresponding to $j = i_1 = i_2$ is non-zero and the contracted tensor does not vanish, contrary to hypothesis. The Theorem follows readily.

Consider now the inner product of two S-functions $[\lambda].[\mu]$, where (λ) is a partition of p and (μ) a partition of q. Corresponding to $[\lambda]$, $[\mu]$ respectively are two tensors

$$A = A_{i_1 i_2 \ldots i_p}, \qquad B = B_{j_1 j_2 \ldots j_q}.$$

These two tensors are of type $\{\lambda\}$, $\{\mu\}$ respectively over the full linear group, but have had removed from them all contractions with R_{ij}, R_{ijk}.

The product is of type under the full linear group corresponding to the product

$$\{\lambda\}\{\mu\} = \sum \Gamma_{\lambda\mu\nu}\{\nu\}.$$

The subtensor corresponding to $\{\nu\}$ will have had some, but not all, of its contractions removed. The principal part of this tensor, of type $[\nu]$, will not be zero since no contraction at all is involved. It follows that the inner product $[\lambda].[\mu]$ includes $\sum \Gamma_{\lambda\mu\nu}[\nu]$. To determine what other terms are involved it is necessary to determine what non-zero contractions can be formed with R_{ij}, R_{ijk}, etc.

Since the suffixes in A are all distinct and the suffixes of R_{ij}, R_{ijk} or R_{ijkp} are all equal, it is clear that for a non-zero result only one contraction can occur between these two tensors, and only one contraction between the fundamental tensor and B. Thus exactly two of the suffixes of the fundamental tensor can be contracted away. If the fundamental tensor is quartic, two suffixes remain and there is no reduction in the rank. The contractions we are seeking, however, are of lower rank. There are thus two possibilities, contraction with R_{ij}, just as for the orthogonal group, and contraction with R_{ijk} leaving one uncontracted suffix in the place of two.

If the tensor R_{ij} is used r times the suffixes removed from A will correspond to a partition (α) of r, and since the tensors R_{ij} are symmetrically disposed, the r suffixes removed from B will correspond to the same partition (α) of r, just as with the orthogonal group.

Suppose that the tensor R_{ijk} is used s times, the suffixes removed from A will correspond to a partition (β) of s, the suffixes removed from B to a partition (γ) of s. In order that the s tensors R_{ijk} may be symmetrically disposed the remaining s uncontracted suffixes of the R_{ijk} must correspond to $\{\beta\}.\{\gamma\}$.

The types of the contracted tensors A and B respectively are

$$\sum \Gamma_{\alpha\beta\xi\lambda}\{\xi\} \quad \text{and} \quad \sum \Gamma_{\alpha\gamma\eta\mu}\{\eta\}.$$

The type of the contracted product is thus

$$\sum \Gamma_{\alpha\beta\xi\lambda}\,\Gamma_{\alpha\gamma\eta\mu}\{\xi\}\{\eta\}(\{\beta\}.\{\gamma\}).$$

THEOREM IX. *The inner product of two S-functions* $[\lambda]$, $[\mu]$, *each of weight* n, *is given by*

$$[\lambda].[\mu] = \sum P_{\lambda\mu\nu}[\nu]$$

where

$$\sum P_{\lambda\mu\nu}\{\nu\} = \sum \Gamma_{\alpha\beta\xi\lambda}\,\Gamma_{\alpha\gamma\eta\mu}\{\xi\}\{\eta\}(\{\beta\}.\{\gamma\}).$$

The coefficient $\Gamma_{\alpha\beta\xi\lambda}$ is defined as the coefficient of $\{\lambda\}$ in the product $\{\alpha\}\,\{\beta\}\,\{\xi\}$. The summation is with respect to all suitable partitions (α), (β), (γ) of which (β) and (γ) are partitions of the same integer. Those cases are included for which $(\alpha) = (0)$, and/or $(\beta) = (\gamma) = (0)$.

As an example consider the inner product $[21]\cdot[21]$. First,

$$\{21\}\,\{21\} = \{42\} + \{41^2\} + \{3^2\} + 2\{321\} + \{2^3\} + \{31^3\} + \{2^21^2\}.$$

Take next the cases for which $(\beta) = (\gamma) = (0)$ with respectively $(\alpha) = (1)$, $(\alpha) = (2)$, $(\alpha) = (1^2)$, $(\alpha) = (21)$. These give

$$(\{2\} + \{1^2\})\,(\{2\} + \{1^2\}) + \{1\}\,\{1\} + \{1\}\,\{1\} + \{0\}$$
$$= \{4\} + 3\{31\} + 2\{2^2\} + 3\{21^2\} + \{1^4\} + 2\{2\} + 2\{1^2\} + \{0\}.$$

Next with $(\beta) = (\gamma) = (1)$ and (α) respectively (0), (1), (2), (1^2) the following terms result

$$(\{2\} + \{1^2\})\,(\{2\} + \{1^2\})\,\{1\} + 4\{1\}\,\{1\}\,\{1\} + \{1\} + \{1\}$$
$$= \{5\} + 4\{41\} + 5\{32\} + 6\{31^2\} + 5\{2^21^2\} + 4\{21^3\} + \{1^5\}$$
$$+ 4\{3\} + 8\{21\} + 4\{1^3\} + 2\{1\}.$$

With $(\beta) = (\gamma) = (2)$, and $(\alpha) = (0)$, $(\alpha) = (1)$, the terms are

$$\{1\}\,\{1\}\,\{2\} + \{0\}\,\{0\}\,\{2\} = \{4\} + 2\{31\} + \{2^2\} + \{21^2\} + \{2\}.$$

Also $(\beta) = (\gamma) = (1^2)$ gives exactly the same result.

But $(\beta) = (2)$, $(\gamma) = (1^2)$ gives

$$\{1\}\,\{1\}\,\{1^2\} + \{0\}\,\{0\}\,\{1^2\} = \{31\} + \{2^2\} + 2\{21^2\} + \{1^4\} + \{1^2\},$$

with precisely the same result for $(\beta) = (1^2)$, $(\gamma) = (2)$.

Lastly when $(\beta) = (\gamma) = (21)$ the result is

$$\{21\}\cdot\{21\} = \{3\} + \{21\} + \{1^3\}.$$

Hence

$$[21]\cdot[21] = [42] + [41^2] + [3^2] + 2[321] + [2^3] + [31^3]$$
$$+ [2^21^2] + [5] + 4[41] + 5[32] + 6[31^2] + 5[2^21] + 4[21^3] + [1^5]$$
$$+ 3[4] + 9[31] + 6[2^2] + 9[21^2] + 3[1^4] + 5[3] + 9[21] + 5[1^3] + 4[2]$$
$$+ 4[1^2] + 2[1] + [0].$$

The result conforms with that given by Murnaghan **(7)**.

5. Inner plethysm of S-functions. The method extends immediately to the evaluation of inner plethysms. To evaluate $[\lambda] \odot \{\mu\}$ where (μ) is a partition of 2, consider the inner product $[\lambda]\cdot[\lambda]$ as given by Theorem IX. The coefficients $P_{\lambda\lambda\mu}$ are obtained from the expression

$$\sum \Gamma_{\alpha\beta\xi\lambda}\,\Gamma_{\alpha\gamma\eta\lambda}\{\xi\}\{\eta\}\,(\{\beta\}\,.\,\{\gamma\}).$$

For each term in this expansion there is an equal term obtained by inter-

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changing $\{\beta\}$ and $\{\gamma\}$, $\{\xi\}$ and $\{\eta\}$. Provided that these two equal terms are distinct, one only of the pair will appear for $[\lambda] \odot \{2\}$ and one for $[\lambda] \odot \{1^2\}$.

The situation is different if $\{\xi\} = \{\eta\}$, $\{\beta\} = \{\gamma\}$, for then the interchange of the two factors $[\lambda]$ in the inner product changes this term into itself. It is therefore possible to separate the symmetric and the skew-symmetric components of the product. This term in $[\lambda].[\lambda]$ is then

$$\Gamma^2_{\alpha\beta\xi\lambda}\{\xi\}\{\xi\}(\{\beta\}.\{\beta\}) = \Gamma^2_{\alpha\beta\xi\lambda}(\{\xi\}\otimes\{2\}+\{\xi\}\otimes\{1^2\})(\{\beta\}\odot\{2\}+\{\beta\}\odot\{1^2\}.$$

Of the four terms obtained by expanding the right hand side the choice of $\{\xi\} \otimes \{1^2\}$ rather than $\{\xi\} \otimes \{2\}$ indicates a change of sign for the interchange. Similarly the choice of $\{\beta\} \odot \{1^2\}$ rather than $\{\beta\} \odot \{2\}$ also indicate a change of sign.

THEOREM X. *If* (μ) *is a partition of* 2,

$$[\lambda] \odot \{\mu\} = \sum Q_{\lambda\mu\nu}[\nu]$$

where

$$\sum Q_{\lambda\mu\nu}\{\nu\} = \sum \Gamma_{\alpha\beta\xi\lambda}\,\Gamma_{\alpha\gamma\eta\lambda}\{\xi\}\{\eta\}(\{\beta\}.\{\gamma\}) + \sum [(\Gamma_{\alpha\beta\xi\lambda}\{\xi\}) \otimes \{2\}(\{\beta\} \odot \{\mu\}) + (\Gamma_{\alpha\beta\xi\lambda}\{\xi\} \otimes \{1^2\}(\{\beta\} \odot \{\tilde{\mu}\})].$$

In the first summation the term is not repeated for the interchange of $\{\xi\}$ and $\{\eta\}$, $\{\beta\}$ and $\{\gamma\}$, and those terms are omitted for which $\{\xi\} = \{\eta\}$, $\{\beta\} = \{\gamma\}$.

As an example consider $[21] \odot \{2\}$. Of the terms considered above for $[21]\cdot[21]$, the term $\{21\}\{21\}$ is replaced by

$$\{21\} \otimes \{2\} = \{42\} + \{321\} + \{31^3\} + \{2^3\}.$$

The cases $(\beta) = (\gamma) = (0)$, $(\alpha) = (1), (2), (1^2), (21)$ give

$$\begin{aligned}\{2\}\otimes\{2\} + \{1^2\}\otimes\{2\} + \{2\}\{1^2\} + \{1\}\otimes\{2\} + \{1\}\otimes\{2\} + \{0\}\otimes\{2\} \\ = \{4\} + \{31\} + 2\{2^2\} + \{21^2\} + \{1^4\} + 2\{2\} + \{0\}.\end{aligned}$$

For $(\beta) = (\gamma) = (1)$, $(\alpha) = (0), (1), (2), (1^2)$, the terms are

$$\begin{aligned}(\{2\} \otimes \{2\} + \{1^2\} \otimes \{2\} + \{2\}\{1^2\})\{1\} + [(2\{1\}) \otimes \{2\}]\{1\} + \{1\} + \{1\} \\ = \{5\} + 2\{41\} + 3\{32\} + 2\{31^2\} + 3\{2^21\} + 2\{21^3\} + \{1^5\} + 3\{3\} \\ + 4\{21\} + \{1^3\} + 2\{1\}.\end{aligned}$$

For $(\beta) = (\gamma) = (2)$ with $(\alpha) = (0), (1)$,

$$(\{1\} \otimes \{2\})\{2\} + \{0\}\{2\} = \{4\} + \{31\} + \{2^2\} + \{2\},$$

with an equal result for $(\beta) = (\gamma) = (1^2)$.

For $(\beta) = (2)$, $(\gamma) = (1^2)$ the term is the same as in $[21].[21]$, namely,

$$\{1\}\{1\}\{1^2\} + \{0\}\{0\}\{1^2\} = \{31\} + \{2^2\} + 2\{21^2\} + \{1^4\} + \{1^2\},$$

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but this is taken once only. Lastly for $(\beta) = (\gamma) = (21)$ the terms are

$$\{21\} \odot \{2\} = \{3\} + \{21\}.$$

Summing this gives

$$\begin{aligned}[21] \odot \{2\} = [42] + [321] + [31^3] + [2^3] + [5] + 2[41] + 3[32] + 2[31^2] \\ + 3[2^21] + 2[21^3] + [1^5] + 3[4] + 4[31] + 5[2^2] + 3[21^2] + 2[1^4] + 4[3] \\ + 5[21] + [1^3] + 4[2] + [1^2] + 2[1] + [0].\end{aligned}$$

This result has been checked as follows. Taking $n = 10$, it gives the expansion of $\{721\} \odot \{2\}$. The total degree of the representations on the right is then found to be 12,880 which is correctly equal to $\frac{1}{2}(160 \times 161)$.

The extension to $[\lambda] \odot \{\mu\}$ where (μ) is a partition of any integer, is straightforward, but becomes complicated even in comparatively simple cases because of the multiplicity of the possible contractions with R_{ij}, R_{ijk}, R_{ijkp}, etc. It does not seem worth while to attempt to express a general theorem, but the method will be illustrated with respect to the comparatively simple case $[2] \odot \{3\}$.

Denote by C_{12} contractions from the first and second tensors with R_{ij}, by C_{12}' similar contractions with R_{ijk}. Denote by C_{123} contractions from all three tensors with R_{ijk}, and by C_{123}' similar contractions with R_{ijkp}.

The possibilities will be listed below.

$$\begin{aligned}
C_{12} &: \{2\}\{2\} &&= \{4\} + \{31\} + \{2^2\}.\\
C_{12}' &: \{2\}\{2\}\{1\} &&= \{5\} + 2\{41\} + 2\{32\} + \{31^2\} + \{2^21\}.\\
C_{12}^2 &: \{2\} &&= \{2\}.\\
C_{12}\, C_{12}' &: \{2\}\{1\} &&= \{3\} + \{21\}.\\
C_{12}^{12} &: \{2\}\{2\} &&= \{4\} + \{31\} + \{2^2\}.\\
C_{12}\, C_{13} &: \{2\} &&= \{2\}.\\
C_{12}\, C_{13}' &: \{1\}\{1\}\{1\} &&= \{3\} + 2\{21\} + \{1^3\}.\\
C_{12}'C_{13}' &: \{2\}\{2\} + \{1^2\}\{1^2\} &&= \{4\} + \{31\} + 2\{2\} + \{21^2\} + \{1^4\}.
\end{aligned}$$

Before proceeding, a word of explanation may be needed here. There are two tensors R_{ijk}, the first suffix of each being contracted with the first tensor of type [2]. The suffixes are necessarily symmetric. The second suffixes, contracted respectively with the second and third tensors, can be either symmetric or skew-symmetric. The uncontracted suffixes of the second and third tensors will likewise be symmetric or skew-symmetric in the respective cases. Further, if the second suffixes of the tensors R_{ijk} are skew-symmetric, then the third uncontracted suffixes must also be skew-symmetric.

The remaining possibilities are as follows:

$$\begin{aligned}
C_{12}\, C_{13}\, C_{23} &: \{0\}.\\
C_{12}\, C_{13}\, C_{23}' &: \{1\}.\\
C_{12}\, C_{13}'C_{23}' &: \{2\}.\\
C_{12}\, C_{13}\, C_{23} &: \{3\}.\\
C_{123} &: \{3\}.
\end{aligned}$$

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$$
\begin{aligned}
C_{123}{}': \{3\}\{1\} &= \{4\} + \{31\}.\\
C_{123}\, C_{12} : \{1\}.&\\
C_{123}\, C_{12}{}': \{1\}\{1\} &= \{2\} + \{1^2\}.\\
C_{123}{}' C_{12} : \{1\}\{1\} &= \{2\} + \{1^2\}.\\
C_{123}{}' C_{12}{}': \{1\}\{1\}\{1\} &= \{3\} + 2\{21\} + \{1^3\}\\
C_{123}{}^2: \{0\}.&\\
C_{123}\, C_{123}{}': \{1\}.&\\
C_{123}{}'^2: \{2\}.&
\end{aligned}
$$

Summing

$$
\begin{aligned}
[2] \odot \{3\} = {} & [6] + [42] + [2^3] + [5] + 2[41] + 2[32] + [31^2] + [2^21]\\
& 4[4] + 4[31] + 4[2^2] + [21^2] + [1^4] + 5[3] + 5[21] + 2[1^3] + 6[2]\\
& 2[1^2] + 3[1] + 2[0].
\end{aligned}
$$

This result has been checked by obtaining the total degree of the representation for S-functions of weight 10, and gives correctly

$$7{,}770 = 35.36.37/6.$$

There is one case of special importance for which a general formula can be found. That is the case $[1] \odot \{\mu\}$. This is equivalent to expressing the general S-function $\{\mu\}$ as a sum of symmetric group characters, when the symmetric group is regarded as a sub-group of the full linear group.

To express the result it is convenient to employ an analogue of differential operators, following the method of Foulkes (**1**). Let

$$D(\{\lambda\})\{\mu\} = \sum \Gamma_{\lambda\nu\mu}\{\nu\}.$$

Such operators satisfy

$$
\begin{aligned}
D(\{\lambda\} + \{\mu\}) &= D(\{\lambda\}) + D(\{\mu\}),\\
D(\{\lambda\}\, \{\mu\}) &= D(\{\lambda\})\, D(\{\mu\}).
\end{aligned}
$$

Let (μ) be a partition of m. It is required to find $[1] \odot \{\mu\}$. Following the method used for the orthogonal group (**4**, p. 393), consider a general tensor of type $\{\mu\}$ and note all the tensor forms that can be obtained by contractions with the fundamental tensors.

Consider first the tensor R_{ij}, repeated r times. The total set of $2r$ contracted suffixes correspond to a concomitant of R_{ij}, and therefore to a term in the expansion of $\{2\} \otimes \{r\}$. The type of the tensor corresponding to $\{\mu\}$ is therefore reduced to $D(\{2\} \otimes \{r\})\, \{\mu\}$. Similarly for complete contractions with R_{ijk} the corresponding operator is $D(\{3\} \otimes \{r'\})$, and so on. There remain to consider contractions with fundamental tensors which leave one suffix of the fundamental tensor uncontracted.

Suppose that there are s tensors R_{ijk} for which the first two suffixes only are contracted with the tensor of type $\{\mu\}$. The type of the contracted suffixes is $\{2\} \otimes \{\lambda\}$ for some partition (λ) of s. In order that the s tensors R_{ijk} may be symmetrically disposed the last suffixes must correspond to the same partition (λ). The type of the operator is therefore

$$\{\lambda\}\ D(\{2\} \otimes \{\lambda\})\ \{\mu\}.$$

Similar results hold for the fundamental tensors of rank 4, 5, 6, etc.

THEOREM XI. *If*

$$\sum \{\lambda_2\}\{\lambda_3\} \ldots \{\lambda_i\}\ D(\{2\} \otimes \{\lambda_2\}) \ldots D(\{i\} \otimes \{\lambda_i\})\ D(\{2\} \otimes \{r_2\}) \ldots$$
$$\ldots D(\{j\} \otimes \{r_j\})\{\mu\} = \sum V_{\mu\nu}\{\nu\}$$

the summation on the left being with respect to any combination of partitions $(\lambda_2), \ldots, (\lambda_i)$ *and of integers* $r_2, \ldots, r_j$ *including* $(\lambda_k) = (0)$ *and* $r_k = 0$, *then*

$$[1] \odot \{\mu\} = \sum V_{\mu\nu}[\nu].$$

The following examples illustrate. Consider first $[1] \odot \{21\}$. Since

$$[1 + D(\{2\}) + \{1\}\ D\{2\}]\ \{21\} = \{21\} + \{1\} + \{2\} + \{1^2\},$$

therefore

$$[1] \odot \{21\} = [21] + [2] + [1^2] + [1].$$

As a check, for $n = 6$ this gives

$$\{51\} \odot \{21\} = \{321\} + \{42\} + \{41^2\} + \{51\}.$$

The degree of the representation is, correctly

$$\frac{1}{3} 4.5.6 = 16 + 9 + 10 + 5.$$

Next consider $[1] \odot \{31\}$. Since

$$[1 + D(\{2\}) + \{1\}\ D(\{2\}) + \{1^2\}\ D(\{31\}) + \{1\}\ D(\{2\})\ D(\{2\})$$
$$+ D(\{3\} + \{1\}\ D(\{3\})]\ \{31\} = \{31\} + \{2\} + \{1^2\} + \{3\} + 2\{21\}$$
$$+ \{1^3\} + \{1^2\} + \{1\} + \{1\} + \{2\} + \{1^2\},$$

therefore

$$[1] \odot \{31\} = [31] + [3] + 2[21] + [1^3] + 2[2] + 3[1^2] + 2[1].$$

This formula for $n = 7$ gives

$$\{61\} \odot \{31\} = \{3^21\} + \{43\} + 2\{421\} + \{41^3\} + 2\{52\} + 3\{51^2\} + 2\{61\}.$$

The degree of the representation gives, correctly

$$\frac{3}{24} 5.6.7.8 = 21 + 14 + 70 + 20 + 28 + 45 + 12.$$

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