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Introduction and Motivation: Are Birds Smarter Than Nerds?

Everyone has seen "flocking," by which I mean the collective, coherent motion of large numbers of organisms [8, 9, 10, 11]. Flocks of birds and schools of fish, and herds of wildebeest, are all familiar sights (although the latter possibly only in nature documentaries). Perhaps nowadays it is most commonly seen in the simulations used for digital cinematic special effects [8, 9, 10, 11]; these have led to the only Oscar ever given for a physics project!

In the past couple of decades, many synthetic systems of self-propelled particles have been fabricated [12, 13] that also exhibit flocking. In addition to providing important experimental realizations of this phenomenon, these experiments make clear that flocking does *not* depend on intelligent decision making by the flockers, but, rather, can arise spontaneously from simple short-ranged interactions.

I will hereafter refer to all such collective motions – flocks, swarms, herds, collections of synthetic self-propelled objects, etc. – as "flocking"; for convenience, I will also refer to the "flockers" as "birds," or, alternatively, "boids."

Note that flocking can occur over an enormous range of length scales: from kilometers (herds of wildebeest) to microns (e.g., the microorganism *Dictyostelium discoideum* [14, 15, 16, 17]).

Remarkably, despite the familiarity and widespread nature of the phenomenon, it is only in the past three decades that many of the universal features of flocks have been identified and understood. It is my goal in this book to explain how we've come to understand one particular type of "flocking": namely, "polar ordered dry active fluids," which I'll define soon. In the process, I hope to introduce those of you unfamiliar with it to the "hydrodynamic" approach, which is a powerful technique that can be applied to any large-scale collective phenomenon.

1.1 An Example of Flocking: the Vicsek Model

To my knowledge, the first physicist to think about flocking – certainly the physicist who kicked off the modern field of active matter – was Thomas Vicsek [18, 19].

https://doi.org/10.1017/9781108993623.002 Published online by Cambridge University Press

He was, as far as I know, the first to recognize that flocks fall into the broad category of nonequilibrium dynamical systems with many degrees of freedom that has, over the past few decades, been studied using powerful techniques originally developed for equilibrium condensed matter and statistical physics (e.g., scaling, the renormalization group, etc.). In particular, Vicsek noted an analogy between flocking and ferromagnetism: The velocity vector of the individual birds is like the magnetic spin of an iron atom in a ferromagnet. The usual "moving phase" of a flock, in which all the birds, on average, are moving in the same direction, is then the analog of the "ferromagnetic" phase of iron, in which all the spins, on average, point in the same direction. Another way to say this is that the development of a nonzero mean center of mass velocity $\langle \mathbf{v} \rangle$ for the flock as a whole therefore requires spontaneous breaking of a continuous symmetry (namely, rotational), precisely as the development of a nonzero magnetization $\mathbf{M} \equiv \langle \mathbf{S} \rangle$ of the spin in a ferromagnet breaks the continuous[1] spin rotational symmetry of the Heisenberg magnet.

Because $\langle \mathbf{v} \rangle$ is only nonzero in the ordered state, it is an "order parameter" for flocking [20].

To make this analogy complete obviously requires that the birds, like the spin in a ferromagnet, live in a rotation invariant environment; that is, that the spins have nothing external that tells them in which direction to point, and the birds have nothing external that tells them which way to fly.

To study this phenomenon (the spontaneous breaking of rotation invariance by collective motion – which is what I will mean henceforth by the term "flocking"), Vicsek formulated his deservedly famous algorithm. I will now describe this algorithm in detail.

The model incorporates the following general features.

(1) A large number (a "flock") of point particles ("boids"[2]) each move over time through a space of dimension d $(= 2, 3, \ldots)$, *attempting* at all times to "follow" (i.e., move in the same direction as) their neighbors.
(2) The interactions are purely short ranged: Each "boid" responds only to its neighbors, defined as those "boids" within some fixed, finite distance R_0, which is assumed to be independent of L, the linear size of the "flock."
(3) The "following" is not perfect: The "boids" make errors at all times, which are modeled as a stochastic noise. This noise is assumed to have only short-ranged spatio-temporal correlations.

[1] Of course, in a real crystalline ferrogmagnet, crystal symmetry breaking fields make the rotational symmetry of the spins discrete, rather than continuous, since there are only a discrete set of orientations for the spin preferred by the lattice. In flocks, there are no such symmetry breaking fields, so the rotational symmetry *is* continuous, as it is in the idealized $0(n)$ Heisenberg model of a ferromagnet. Everything I say hereafter about ferromagnetic systems implicitly refers to this fully rotationally invariant $0(n)$ model.

[2] I will frequently use the term "boid" (a short form for "birdoid"), coined by C. Reynolds [8].

(4) The underlying model has complete rotational symmetry: The flock is equally likely, a priori, to move in any direction.

Any model that incorporates these general features should belong to the same "universality class," in the sense that term is used in critical phenomena and condensed matter physics. That is, all such systems should be described by the same simple, universal scaling laws at large distances and times. To see this "universality," of course, we need a large flock: The universality becomes exact in the "thermodynamic" limit; i.e., as $N \rightarrow \infty$, where N is the number of "boids" in the flock. The specific model proposed and simulated numerically by Vicsek is the following.

In Vicsek's discrete time model, a number of birds labeled by i move in a two-dimensional plane with positions $\{\mathbf{r}_i(t)\}$, with time t being a discrete (integer) variable. At each integer time, all of the birds simultaneously choose the direction they will move on the next time step (taken to be of duration $\Delta t = 1$) by averaging the directions of motion of all of those birds within a circle of radius R_0 (in the most convenient units of length $R_0 = 1$) on the previous time step (i.e., updating is simultaneous). The distance R_0 is assumed to be $\ll L$, the size of the flock. The direction the bird actually moves on the next time step differs from the previously described direction by a random angle $\eta_i(t)$, with zero mean and standard deviation Δ. The distribution of $\eta_i(t)$ is identical for all birds, time independent, *and* uncorrelated between different birds and different time steps. Each bird then, on the next time step, moves in the direction so chosen a distance $v_0 \Delta t$, where the speed v_0 is the same for all birds.

To summarize, the rule for bird motion in $d = 2$ is

$$\theta_i(t+1) = \left\langle \theta_j(t) \right\rangle_n + f_i(t), \tag{1.1.1}$$

$$\mathbf{r}_i\,(t+1) = \mathbf{r}_i(t) + v_0\,(\cos\theta(t+1), \sin\theta(t+1))\,, \tag{1.1.2}$$

$$\langle f_i(t) \rangle = 0, \tag{1.1.3}$$

$$\left\langle f_i(t) f_j(t') \right\rangle = 2D\delta_{ij}\delta_{tt'}, \tag{1.1.4}$$

where the symbol $\langle\rangle_n$ denotes an average over "neighbors," which are defined as the set of birds j satisfying

$$\left| \mathbf{r}_j(t) - \mathbf{r}_i(t) \right| < R_0\,. \tag{1.1.5}$$

Here $\langle\rangle$ without the subscript n denote averages over the random distribution of the noises $f_i(t)$, and $\theta_i(t)$ is the angle of the direction of motion of the ith bird (relative to some fixed reference axis) on the time step that ends at t.

https://doi.org/10.1017/9781108993623.002 Published online by Cambridge University Press

The quantity $\langle\theta_j(t)\rangle_n$ is defined via

$$\langle\theta_j(t)\rangle_n \equiv \arctan\left(\frac{\langle\sin(\theta_j(t))\rangle_n}{\langle\cos(\theta_j(t))\rangle_n}\right), \tag{1.1.6}$$

where

$$\langle\sin(\theta_j(t))\rangle_n = \frac{\sum\limits_{j\in n}\sin(\theta_j(t))}{N_n},$$
$$\langle\cos(\theta_j(t))\rangle_n = \frac{\sum\limits_{j\in n}\cos(\theta_j(t))}{N_n} \tag{1.1.7}$$

with $\sum_{j\in n}$ denoting a sum over all neighbors (that is, all birds) satisfying (1.1.5), and N_n the number of birds satisfying (1.1.5).

This definition is equivalent to saying that the direction each bird moves between time t and time $t+1$ would, in the absence of noise, be the direction of the average of the velocity *vectors* $\mathbf{v}_j(t)$ of its neighbors at time t.

The reason for this convoluted definition of the average $\langle\theta_j(t)\rangle_n$ is that using the more obvious definition

$$\langle\theta_j(t)\rangle_{n\text{wrong}} = \frac{\sum\limits_{j\in n}\theta_j(t)}{N_n} \tag{1.1.8}$$

has pathologies associated with the fact that θ_j, like all angles, is defined only modulo 2π. To have an unambiguous definition of θ_j, therefore, one must introduce a "cut"; that is, define θ_j to always lie within some range of width 2π.

For example, one could choose to define θ_j to always lie in the interval $0 < \theta_j < 2\pi$, with $\theta_j = 0$ defined to point to the east. But then consider a situation in which a bird had two neighbors, one heading one degree due south of east, the other heading one degree due north of east. With our convention, we'd define $\theta_1 = 1^\circ$, and $\theta_2 = 359^\circ$. Thus, on our next step, if we used the rule (1.1.8), our bird would head off at 180°; i.e., almost exactly *opposite* the direction of its neighbors. This is clearly *not* following your neighbors!

You might think you could fix this problem by putting the "cut" due west; that is, by defining θ_j to always lie in the interval $-\pi < \theta_j < \pi$. However, you can easily convince yourself that, while this fixes the problem just described when your neighbors are heading almost due east, it gives you the same problem if they're heading almost due west. Indeed, in general, one will always have problems with the rule (1.1.8) if your neighbors are moving close to, but on opposite sides of, the direction in which you choose to put the cut.

You can easily convince yourself that the rule (1.1.7) has no such problems.

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The flock evolves through the iteration of this rule. Note that the "neighbors" of a given bird may change on each time step, since birds do not, in general, move in exactly the same direction as their neighbors.

I have been rather precise and detailed in explaining this algorithm. However, we actually believe that most of the details of this algorithm do not matter for the scaling properties of the flock. Only a few features (all of which the Vicsek algorithm possesses) *do* matter for those scaling laws. These features are: activity, conservation laws, symmetries, short-ranged interactions, and noisiness. We now elaborate on these.

(1) Activity: A large number (a "flock") of point particles ("boids") each move over time through a space of dimension d $(= 2, 3, \ldots)$, *attempting* at all times to "follow" (i.e., move in the same direction as) their neighbors. This motion is due to some form of self-propulsion; in Vicsek's algorithm, the rule is that the speed of each creature is constant. Departures from this rule are not important, provided that the boids prefer to be in a state of motion, rather than at rest. This is what is meant by the word "active" in "polar ordered dry active fluids."

    This self-propulsion requires an energy source; it also requires that the system be out of equilibrium. Dead birds don't flock!

(2) Conservation laws: The underlying model does *not* conserve momentum; the total momentum of the flock can change. Indeed, it does so every time a creature turns. We imagine this violation of momentum conservation can happen because the creatures move either over a fixed surface, in two dimensions, or through some fixed matrix (e.g., a gel) in three dimensions, with which they interact frictionally. This surface or matrix therefore acts like a momentum "source" or "sink."

    This lack of momentum conservation is what is meant by the term "dry" in "polar ordered dry active fluids." Note that many active systems – e.g., many active nematics – are "wet," by which we mean momentum is conserved. Note, incidentally, that real birds (and not only water birds!) are "wet" in this sense, since the sum of their momentum and the momentum of the air through which they fly is conserved. This changes the dynamics considerably. The problem of wet flocks can still be treated by a hydrodynamic approach [3, 4, 5, 6], but the hydrodynamic model is different because of momentum conservation. I will not discuss that case further here.

    There *is* one conservation law in the Vicsek algorithm, however: The number of birds is conserved. That is, birds are not being born or dying "on the wing." You laugh, but there are many biological situations – bacteria swarms, and tissue development to name just two – in which this is not a good approximation: Bacteria or cells are being born and dying on the time scale of the motion.

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The hydrodynamics of this case is quite interesting [21, 22, 23], and will be discussed in Chapter 9.

(3) Symmetry: The underlying model has complete rotational symmetry; the flock is equally likely, a priori, to move in any direction. I will here consider models that do *not* have Galilean invariance: that is, they have a preferred Galilean frame. This frame is the one in which the background medium over or through which the boids move is stationary. In the Vicsek algorithm, this is the unique frame in which the *speeds* (i.e., the *magnitudes* of the velocity vectors, but *not* the velocity vectors themselves) are the same (and given by v_0).

(4) The interactions are purely short ranged: In Vicsek's model, each "boid" only responds to its neighbors. In Vicsek's model, these are defined as those "boids" within some fixed, finite distance R_0, which is assumed to be independent of L, the linear size of the "flock." Hence, in the limit of flock size going to infinity – i.e., the "thermodynamic limit" – the range of interaction is much smaller than the size of the flock. Variants on this rule – for example, interactions whose strength falls off exponentially with distance – can also be considered short ranged.

(5) The "following" is not perfect: The "boids" make errors at all times, which are modeled as a stochastic noise. This noise is assumed to have only short-ranged spatio-temporal correlations. Its role in this problem is very similar to the role of temperature in equilibrium systems: It tends to disorder the flock. As you'll see, one of the most interesting questions in this problem is whether the ordered state can survive this noise.

In addition to these symmetries of the equations of motion, which reflect the underlying symmetries of the physical situation under consideration, it is also necessary to treat correctly the symmetries of the *state* of the system under consideration. These may be different from those of the underlying system, precisely because the system may spontaneously break one or more of the underlying symmetries of the equations of motion. Indeed, this is precisely what happens in the ordered state of a ferromagnet: The underlying rotation invariance of the system as a whole is broken by the system in its steady state, in which a unique direction is picked out – namely, the direction of the spontaneous magnetization.

As should be apparent from our earlier discussion, this is also what happens in a spontaneously moving flock. Indeed, the symmetry that is broken – rotational – and the manner in which it is broken – namely, the development of a nonzero expectation value for some vector (the spin $\mathbf{S}$ in the ferromagnetic case; the velocity $\mathbf{v}$ in the flock) – are precisely the same in both cases.[3]

[3] The isotropic Heisenberg model of magnetism is invariant under uniform rotation of all the spins, without a corresponding rotation of the lattice on which they live. A flock, like an ordinary collection of interacting

https://doi.org/10.1017/9781108993623.002 Published online by Cambridge University Press

The fact that it is a unique *vector* that is singled out, rather than merely a unique *axis*, is the meaning of the word "polar" in "polar ordered dry active fluids."

Many different "phases,"[4] in this sense of the word, of a system with a given underlying symmetry are possible. Indeed, I have already described two such phases of flocks: the "ferromagnetic" or moving flock, and the "disordered," "paramagnetic," or stationary flock.

In equilibrium statistical mechanics, this is precisely how we classify different phases of matter: by the underlying symmetries that they break. Crystalline solids, for example, differ from fluids (liquid and gases) by breaking both translational and orientational symmetry. Less familiar to those outside the discipline of soft condensed matter physics are the host of mesophases known as liquid crystals, in some of which (e.g., nematics [24]) only orientational symmetry is broken, while in others, (e.g., smectic [24], which we'll revisit in Chapter 8) translational symmetry is only broken in *some* directions, not all.

It seems clear that, at least in principle, every phase known in condensed matter systems could also be found in flocks. In this book, I'm going to focus on just one phase: the "polar ordered dry active fluid phase," in which rotational symmetry is completely broken by the development of a nonzero average flock speed $\langle \mathbf{v} \rangle$, but all of the other symmetries of the dynamics (e.g., translation invariance) are preserved. The word "fluid" in "polar ordered active fluids" is what tells us that these are systems in which translational invariance is *not* broken.

The first, and to my mind still the biggest, surprise in the entire field of active matter is that a "polar ordered dry active fluid phase" is even possible in two dimensions. The reason Yuhai and I (and Vicsek) found this so surprising is the well-known "Mermin–Wagner–Hohenberg theorem" [25, 26, 27] of equilibrium statistical mechanics. This theorem states that in a thermal equilibrium model at nonzero temperature with short-ranged interactions, it is impossible to spontaneously break a continuous symmetry. This implies in particular that the equilibrium or "pointer" version of Vicsek's algorithm described earlier, in which the birds carry a vector $\mathbf{v}_i$ whose direction is updated according to Vicsek's algorithm, but in which the birds do not actually move, can never develop a true long-ranged ordered state in which all the $\mathbf{v}_i$ point, on average, in the same direction (more precisely, in which $\langle \mathbf{v} \rangle \equiv \frac{\Sigma_i \mathbf{v}_i}{N} \neq \vec{0}$) , since such a state breaks a continuous symmetry, namely rotation invariance.

Yet the *moving* flock evidently has no difficulty in doing so; as Vicsek's simulation shows, even two-dimensional flocks with rotationally invariant dynamics,

molecules, such as those which form liquid crystals, is invariant only under spatial rotations, which rotate both the position and the velocity vectors of the creatures of the flock.

[4] By "phases" in systems far from equilibrium, I simply mean nonequilibrium steady states of a given symmetry.

short-ranged interactions, and noise – i.e., seemingly all of the ingredients of the Mermin–Wagner–Hohenberg theorem – *do* move with a nonzero macroscopic velocity, which requires $\langle \mathbf{v} \rangle \neq \mathbf{0}$, which, in turn, breaks rotation invariance, in seeming violation of the theorem.

There are a pair of gedanken experiments that make the very paradoxical and surprising nature of this result more obvious. Both experiments start by putting a million people on a flat, featureless plane in the fog. The featurelessness of the plane, and the fog, ensure rotation invariance (since they leave the people with no external indication of a preferred direction), while the fog has the further role of ensuring that each person can see only a few of her nearest neighbors.

The first experiment now consists of asking everyone to try to point in the same direction. The result is that the people cannot all point in the same direction, no matter how good a job they do at aligning with their nearest neighbors (unless, of course, the alignment is perfect). If they make the slightest errors, those will accumulate over distance, so that, even though a given person may point in roughly the same direction as others not too far away from her, widely separated people will inevitably be pointing in wildly different directions.

The second gedanken experiment consists of slightly modifying the instructions given to these million folks: Now ask them to all *walk* in the same direction.

Amazingly, if this instruction is given to the same people, in the same fog, with the same errors, they *can* all walk in the same direction. Moving, apparently, is fundamentally different from pointing.

Why? There is a very simple explanation for this *apparent* "violation" of the Mermin–Wagner–Hohenberg theorem: One of the essential premises of the Mermin–Wagner–Hohenberg theorem does *not* apply to movers, namely, they are *not* systems in thermal equilibrium. The nonequilibrium aspect arises from the motion: You can't move forever in a medium with friction unless you're alive. And, if you're alive, you're not in thermal equilibrium (that's why we say "cold and dead").

Clearly, motion must be what stabilizes the order in $d = 2$: As described above, the motion is the *only* difference between the pointing and moving gedanken experiments just described.

But *how* does motion get around the Mermin–Wagner–Hohenberg theorem? And, more generally, how best to understand the large-scale, long-time dynamics of a very large, moving flock?

The answer to these questions can be found in the field of hydrodynamics. I will apply that body of knowledge to flocks in Chapter 4. But first, I'll explain why the Mermin–Wagner–Hohenberg theorem is true for pointers, which will give us some insight into why it's *not* true for walkers.