SOLUTION TO A MATROID PROBLEM POSED BY D. J. A. WELSH

D. T. Bean

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The pair (S, M) is a <u>matroid</u> if S is a finite set and M a collection of subsets of S such that (1) every subset of a set of M is in M, and (2) all maximal sets in M have a common cardinality. The <u>span</u> of a set $A \subset S$ is $\Gamma(A)$ where $y \in \Gamma(A)$ if and only if $y \in A$ or there is $A' \subset A$, $A' \in M$ and $\{y\} \cup A' \notin M$. A maximal set in M is called a <u>base</u>. For each base B in M define $B: S \to M$ by: $B(x)$ is $\{x\}$ if $x \in B$; otherwise $B(x)$ is the unique minimal subset of B such that $B(x) \cup \{x\} \notin M$.

<u>Remark</u>. Using the natural correspondence between bases and circuits the set $B(x) \cup \{x\}$ is just the atom $J(D; x)$ of Tutte [1] where D is the dendroid $E - B$ when B is a base and $x \in S - B$.

For $A \subset S$ and $x \in S$, x is B-orthogonal to A, written $x \in O_B(A)$, if $B(x) \cap B(a) = \phi$ for all $a \in A$.

A problem posed by Welsh [2] is the following: If $A \in M$, B is a base and $x \in O_B(A)$ is $x \in O_B(\Gamma(A))$? The answer is yes and is in fact true for any $A \subset S$. To prove this two lemmas are required.

LEMMA 1. *Let* (S, M) *be a matroid and* $b \in B$, B *a base in* M. *Then* $B' = (B - \{b\}) \cup \{a\}$ *is a base if and only if* $b \in B(a)$.

<u>Proof</u>. Necessity. If $a \in B$ then $B' = B$ and $\{b\} = B(a)$. If $a \notin B$ then $B'(b) \subset B'$ and $B'(b) \cup \{b\} \notin M$. Thus $a \in B'(b)$ and $B(a) = (B'(b) \cup \{b\}) - \{a\}$ by uniqueness.

<u>Sufficiency</u>. If $a \in B$ then $b = a$ and so $B' = B$. If $a \notin B$ and B' is dependent then there is $A \subset B'$, A minimal dependent. Since B is independent, $a \in A \subset (B - \{b\}) \cup \{a\}$ so that $A - \{a\} = B(a)$ and $b \notin B(a)$, a contradiction. B' is independent and $|B| = |B'|$. Hence B' is a base. ($|X|$ denotes the cardinality of X.)

LEMMA 2. *Let* (S, M) *be a matroid*, B *a base*, $a \in S - B$, $b \in B(a)$, *and* $B' = (B - \{b\}) \cup \{a\}$. *Then for* $p \in S$,

(i) $B'(p) = B(p)$ *if* $b \notin B(p)$,

(ii) $B'(b) = (B(a) \cup \{a\}) - \{b\}$,

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(iii) $B'(a) = \{a\}$, and

(iv) $B(a) + B(p) \subset B'(p) \subset B(a) \cup B(p) \cup \{a\}$ if $b \in B(p)$, $b \neq p$.

($X + Y$ denotes the symmetric difference of X and Y.)

Proof. B' is a base by Lemma 1.

(i) If $b \notin B(p)$ then $B(p) \subset B'$. Hence $B'(p) = B(p)$.

(ii) The proof of Lemma 1 shows that $B'(b) = (B(a) \cup \{a\}) - \{b\}$.

(iii) By definition.

(iv) Right hand side. The minimal dependent sets form a "circuit" matroid (Edmonds [3, Prop. 2]).

Since $b \in B(p) \cup \{p\}$ and $b \in B(a) \cup \{a\}$, there is a minimal dependent set C with $p \in C \subset B(a) \cup B(p) \cup \{a, p\} - \{b\} \subset B' \cup \{p\}$. Hence $B'(p) = C - \{p\} \subset B(a) \cup B(p) \cup \{a\}$. Also $a \in B'(p)$ by Lemma 1 since $B = (B' - \{a\}) \cup \{b\}$.

Left hand side. (α) For $z \in B(p) - B(a)$, $B'' = (B - \{z\}) \cup \{p\}$ is a base by Lemma 1 and since $z \notin B(a)$, $B(a) = B''(a)$ by Lemma 2, (i). $a \in B'(p) \cup \{p\}$ is a circuit, and if $z \notin B'(p)$ then $B'(p) \cup \{p\} \subset B'' \cup \{a\}$, giving $b \in B(a) = B''(a) = B'(p) \cup \{p\} - \{a\}$, a contradiction. ($\beta$) For $z \in B(a) - B(p)$, $B'' = (B - \{z\}) \cup \{a\}$ is a base by Lemma 1 and since $z \notin B(p)$, $b \in B''(p) = B(p)$ by Lemma 2, (i). If $z \notin B'(p)$ then $B'(p) \subset (B(p) \cup B(a) \cup \{a\}) - \{z\} \subset B''$. Since $B'(p) \cup \{p\}$ is a circuit in $B'' \cup \{p\}$ we have $b \in B'(p) = B''(p) = B'(p)$, a contradiction. Thus $B(p) + B(a) \subset B'(p)$.

THEOREM. Let (S, M) be a matroid, B a base, $A \subset S$ and $x \in O_B(A)$. Then $x \in O_B(\Gamma(A))$.

Proof. If $B(x) = \phi$ there is nothing to prove. If $y \in \Gamma(A)$ with $B(x) \cap B(y) \neq \phi$ then $y \notin A$ and there is a circuit C, $y \in C = A' \cup \{y\}$ with $A' \subset A$, $A' \in M$. Then $x \in O_B(A')$, $x \notin O_B(\Gamma(A'))$ so that it is sufficient to prove the theorem for sets $A \in M$.

We now have: if the theorem is not true there is a triple (x, A, B) where $x \in S$, B is a base, $A \in M$, $0 \leq |A - B|$, $x \in O_B(A)$ and $x \notin O_B(\Gamma(A))$. Suppose (x, A, B) such a triple. If $A \subset B$ then $\bigcup_{a \in A} B(a) = A$ and for $y \in \Gamma(A) - A$ there is a circuit C with $y \in C = A' \cup \{y\}$, $A' \subset A$. Hence $B(y) = A' \subset A$ and $B(x) \cap B(y) = \phi$, a contradiction. Thus for such triple, $0 < |A - B|$ and we may choose one with $|A - B|$ least.

Take $a_0 \in A - B$. Since $a_0 \in M$ there is $b \in B(a_0)$ and by hypothesis

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$b \notin B(x)$. By Lemma 1, $B' = (B - \{b\}) \cup \{a_0\}$ is a base and by Lemma 2 (i) $B(x) = B'(x)$. Now $a_0 \notin B'(x)$, for otherwise by Lemma 2 (iv) (with B' and B interchanged) $B'(a_0) + B'(x) = B(x) = B'(x)$ giving $B'(a_0) = \{a_0\} = \phi$, a contradiction. For the remaining $a \in A$ again using Lemma 2 (iv), $B'(a) \subset B(a) \cup B(a_0) \cup \{a\}$ so that $B'(x) \cap B'(a) = \phi$ for all $a \in A$. Thus $x \in O_{B'}(A)$ and by the minimality of $|A - B|$, $x \in O_{B'}(\Gamma(A))$.

Now take $y \in \Gamma(A)$. If $b \notin B(y)$ then by Lemma 2 (i), $\phi = B'(x) \cap B'(y) = B(x) \cap B(y)$. If $b \in B(y)$, $B'(y) \supset B(a_0) + B(y)$ by Lemma 2 (iv) and $\phi = B'(x) \cap B'(y) = B(x) \cap (B(a_0) + B(y)) = B(x) \cap B(y)$. In all cases, $B(x) \cap B(y) = \phi$ and so $x \in O_B(\Gamma(A))$, a contradiction to the existence of a triple (x, A, B). Thus the theorem is proved.

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York University
Toronto