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OUTER AUTOMORPHISM GROUPS OF CERTAIN TREE PRODUCTS OF ABELIAN GROUPS

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Abstract

We prove that certain tree products of finitely generated Abelian groups have Property E. Using this fact, we show that the outer automorphism groups of those tree products of Abelian groups and Brauner's groups are residually finite.

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1. Introduction

It is well known that automorphism groups of finitely generated residually finite ($\mathcal{RF}$) groups are $\mathcal{RF}$ (Baumslag [2]). In general, outer automorphism groups (Out G) of finitely generated $\mathcal{RF}$ groups need not be $\mathcal{RF}$ (Wise [12]). It seems interesting to determine which finitely generated $\mathcal{RF}$ groups have residually finite outer automorphism groups. In fact, Gilman [3] showed that Out F_r is residually finite alternating and residually finite symmetric where F_r is a free group of rank $r \geq 3$. In [4], Grossman showed that the outer automorphism group, Out $\pi_1(M)$, is $\mathcal{RF}$ where M is a closed orientable surface of genus k. It follows that mapping class groups of orientable surfaces of genus k are $\mathcal{RF}$. In [1], Allenby *et al.* showed that Out G is $\mathcal{RF}$ if G is the generalized free product of two free groups amalgamating a maximal cyclic subgroup. From this it follows that mapping class groups of all closed surfaces (orientable or non-orientable) are $\mathcal{RF}$. Johannson [6] showed that mapping class groups of simple 3-manifolds are finite, from which he derived that outer automorphism groups of fundamental groups of simple 3-manifolds are finite.

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We note that fundamental groups of closed surfaces of genus k are conjugacy separable 1-relator groups and their outer automorphism groups are $\mathcal{RF}$ [1]. It would be interesting to know whether outer automorphism groups of conjugacy separable 1-relator groups are $\mathcal{RF}$.

In this paper, we develop a group property, Property E, which extends Grossman's Property A [4]. Exploiting this property, we show that the outer automorphism groups of certain tree products of Abelian groups and Brauner's groups are $\mathcal{RF}$.

In Section 2, we give the basic background materials that are needed for this paper. In Section 3, we determine a criterion for a generalized free product of a group with Property E and an Abelian group to have Property E. Applying this we prove that outer automorphism groups of certain tree products (stem products) of Abelian groups are $\mathcal{RF}$. In Section 4, applying these results to certain knot and linkage groups known as Brauner's groups, we prove that outer automorphism groups of these groups are $\mathcal{RF}$.

2. Preliminary results

Throughout this paper we use standard notation and terminology.

A group G is *residually finite* ($\mathcal{RF}$) if, for each non-trivial element $x \in G$, there exists a finite homomorphic image $\overline{G}$ of G such that the image of x in $\overline{G}$ is not trivial.

A group G is *conjugacy separable* if, for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\overline{G}$ of G such that the images of x and y in $\overline{G}$ are not conjugate in $\overline{G}$.

If A and B are groups, then $A *_H B$ denotes the generalized free product of A and B amalgamating the subgroup H. In particular, if $g \in A *_H B$, then we use $\|g\|$ to denote the generalized free product length of g.

We use $\operatorname{Inn} g$ to denote the inner automorphism of G induced by $g \in G$. Out G denotes the outer automorphism group, $\operatorname{Aut}(G)/\operatorname{Inn}(G)$, of G. $C_G(g)$ denotes the centralizer of g in G and $Z(G)$ denotes the center of G. We use $\{x\}^G$ to denote the conjugacy class of x in G.

If H is a subgroup of G, we use $x \sim_H y$ to denote that x is conjugate to y by an element in H.

DEFINITION 2.1. By a *conjugating endomorphism/automorphism* of a group G we mean an endomorphism/automorphism α which is such that, for each $g \in G$, there exists $k_g \in G$, depending on g, so that $\alpha(g) = k_g^{-1} g k_g$.

DEFINITION 2.2 (Grossman [4]). A group G has *Property A* if for each conjugating automorphism α of G, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1} g k$ for all $g \in G$, that is $\alpha = \operatorname{Inn} k$.

We extend Grossman's Property A to include endomorphisms.

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DEFINITION 2.3. A group G has *Property E* if, for each conjugating endomorphism α of G, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1}gk$ for all $g \in G$, that is $\alpha = \operatorname{Inn} k$.

Clearly, every group having Property E has Property A. We will make use of the following results.

THEOREM 2.4 (Grossman [4]). *Let B be a finitely generated, conjugacy separable group with Property A. Then* $\operatorname{Out} B$ *is* $\mathcal{RF}$.

THEOREM 2.5 (Magnus *et al.* [8, Theorem 4.6]). *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced, and that $x \sim_G y$.*

(1) *If $\|x\| = 0$, then $\|y\| \leq 1$ and, if $y \in A$, then there is a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_{A(B)} h_r = x$.*
(2) *If $\|x\| = 1$, then $\|y\| = 1$ and either $x, y \in A$ and $x \sim_A y$, or $x, y \in B$ and $x \sim_B y$.*
(3) *If $\|x\| \geq 2$, then $\|x\| = \|y\|$ and $y \sim_H x^*$ where x^* is a cyclic permutation of x.*

The following lemma is a slight modification of Lemma 3.14 in [7].

LEMMA 2.6. *Let G be a tree product of any groups A_i $(1 \leq i \leq n)$ amalgamating edge groups, where edge groups are contained in the centers of vertex groups. If $x \sim_G y$ for $x \in Z(A_i)$ and $y \in A_j$, then $x = y$.*

3. Main results

In this section we prove that certain generalized free products of groups with Property E have Property E. Applying these results we show that outer automorphism groups of certain tree products of Abelian groups are $\mathcal{RF}$.

THEOREM 3.1. *Let $G = A *_H B$ where B is Abelian. Suppose that the following hold:*

(C1) $\bigcap_{a \in A}(C_A(a)H) = Z(A)H$;
(C2) *there exists $a \in A$ such that $\{a\}^A \cap H = \emptyset$;*
(C3) *A has Property E.*

Then $G = A *_H B$ has Property E.

PROOF. If $B = H$ then $G = A$ has Property E by (C3). Hence we assume that $B \neq H$.

Let α be a conjugating endomorphism of G and $\alpha(g) = k_g^{-1} g k_g$ for $g \in G$. Without loss of generality, we can assume that $\alpha(a) = a$, where a satisfies (C2). Thus $a \notin H$.

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STEP 1. We show that, for each $b \in B\backslash H$, we can choose k_b in A.

Let $b \in B\backslash H$ and $\alpha(b) = k_b^{-1} b k_b$ for some $k_b \in G$. Let $k_b = u_1 u_2 \cdots u_r$ be an alternating product of the shortest length in G. Since B is Abelian, we may assume that $u_1 \in A$. Then

$$k_{ba}^{-1}(ba)k_{ba} = \alpha(ba) = k_b^{-1} b k_b \cdot a = u_r^{-1} \cdots u_2^{-1} \cdot u_1^{-1} b u_1 \cdot u_2 \cdots u_{r-1} \cdot u_r \cdot a.$$

Thus

$$ba \sim_G u_{r-1}^{-1} \cdots u_2^{-1} \cdot u_1^{-1} b u_1 \cdot u_2 \cdots u_{r-1} \cdot u_r a u_r^{-1}. \tag{3.1}$$

If $r \geq 2$ and $u_r \in B\backslash H$, then the right-hand side of (3.1) is cyclically reduced of length $2(r+1)$. By Theorem 2.5 this is impossible, since the left-hand side of (3.1) is cyclically reduced of length 2. If $r \geq 2$ and $u_r \in A$ then, by (C2), $u_r a u_r^{-1} \notin H$. In this case, the right-hand side of (3.1) is cyclically reduced of length $2r$. Since the left-hand side of (3.1) is of length 2, this is also impossible by Theorem 2.5. Therefore $r \leq 1$. Hence we can choose $k_b = u_1 \in A$.

STEP 2. There exists a fixed $w \in A$ such that $\alpha(y) = w^{-1} y w$ for all $y \in B$.

Let $b \in B\backslash H$ be fixed and let $y \in B\backslash H$ be arbitrary. By Step 1, we can assume that $k_b = w \in A$ and $k_y \in A$. Then $k_{by}^{-1}(by)k_{by} = \alpha(by) = \alpha(b)\alpha(y) = w^{-1} b w \cdot k_y^{-1} y k_y$. Hence,

$$by \sim_G b \cdot w k_y^{-1} \cdot y \cdot k_y w^{-1}. \tag{3.2}$$

Since $b, y \in B\backslash H$, if $k_y w^{-1} \in A\backslash H$ then the right-hand side of (3.2) is cyclically reduced of length 4. Thus (3.2) does not hold. This implies that $k_y w^{-1} \in H$. Let $k_y = h_y w$, where $h_y \in H$ depends on y. Since B is Abelian, $\alpha(y) = k_y^{-1} y k_y = w^{-1} h_y^{-1} y h_y w = w^{-1} y w$. This shows that $\alpha(y) = w^{-1} y w$ for all $y \in B\backslash H$. Now $\alpha(h) = \alpha(hy \cdot y^{-1}) = \alpha(hy)\alpha(y^{-1}) = w^{-1}(hy)w \cdot w^{-1} y^{-1} w = w^{-1} h w$ for all $h \in H$. Hence $\alpha(y) = w^{-1} y w$ for all $y \in B$.

By (3.1), we have $ba \sim_G w^{-1} b w a \sim_G b w a w^{-1}$. It follows from Theorem 2.5 that $ba \sim_H b w a w^{-1}$. Let $ba = h^{-1} b w a w^{-1} h$ for some $h \in H$. Since B is Abelian, $a = h^{-1} w a w^{-1} h$. Let $u = w^{-1} h$. Then $u \in A$ and $[u, a] = 1$.

Let $\overline{\alpha} = \mathrm{Inn}\, u \circ \alpha$. Then $\overline{\alpha}$ is also a conjugating endomorphism of G and $\overline{\alpha}(y) = u^{-1}(\alpha(y))u = u^{-1}(w^{-1} y w)u = h^{-1} y h = y$ for all $y \in B$. Moreover, $\overline{\alpha}(a) = u^{-1}(\alpha(a))u = u^{-1} a u = a$. We shall show that $\overline{\alpha}$ is an inner automorphism of G. For convenience, we again write $\overline{\alpha}(g) = k_g^{-1} g k_g$ for $g \in G$.

STEP 3. We show that $k_x \in A$ for all $x \in A\backslash H$.

Let $x \in A\backslash H$ and $k_x = u_1 u_2 \cdots u_r$ be an alternating product of the shortest length in G such that $\overline{\alpha}(x) = k_x^{-1} x k_x$. Then $k_{xa}^{-1}(xa)k_{xa} = \overline{\alpha}(xa) = k_x^{-1} x k_x \cdot a = u_r^{-1} \cdots u_1^{-1} x u_1 \cdots u_r \cdot a$. Hence

$$xa \sim_G u_{r-1}^{-1} \cdots u_2^{-1} \cdot u_1^{-1} x u_1 \cdot u_2 \cdots u_{r-1} \cdot u_r a u_r^{-1}. \tag{3.3}$$

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(i) Suppose that $u_1 \in A$ and $r \geq 2$. In this case we may assume that $u_1^{-1}xu_1 \notin H$ for, if $u_1^{-1}xu_1 = h \in H$ then $u_2^{-1}u_1^{-1}xu_1u_2 = u_2^{-1}hu_2 = h = u_1^{-1}xu_1$. This reduces the length of k_x. Hence we may assume that $u_1^{-1}xu_1 \notin H$. If $u_r \in B$ then the right-hand side of (3.3) is cyclically reduced of length $2r$. If $u_r \in A$ then, from (C2), $u_rau_r^{-1} \notin H$, hence the right-hand side of (3.3) is cyclically reduced of length $2(r-1)$. Since the left-hand side of (3.3) is of length ≤ 1, by Theorem 2.5, both cases are impossible.

(ii) Suppose that $u_1 \in B$. If $r \geq 2$ and $u_r \in B\backslash H$, then the right-hand side of (3.3) is cyclically reduced of length $2(r+1)$. If $u_r \in A$ then $u_rau_r^{-1} \notin H$ by (C2). Hence if $r \geq 2$ and $u_r \in A\backslash H$ then the right-hand side of (3.3) is cyclically reduced of length $2r$. Hence, by Theorem 2.5, both cases are impossible. Now, if $r = 1$ and $k_x = u_1 \in B\backslash H$, then the length of the right-hand side of (3.3) is 4. Clearly it is also impossible.

This shows that $k_x = u_1 \in A$ for all $x \in A\backslash H$.

Since $H \subset B$, $\overline{\alpha}(h) = h$ for all $h \in H$. Hence $\overline{\alpha}$ restricted to A is a conjugating endomorphism of A. Since A has Property E by (C3), $\overline{\alpha}$ restricted to A is an inner automorphism of A. Thus there exists a fixed $g \in A$ such that $\overline{\alpha}(x) = g^{-1}xg$ for all $x \in A$. Since $h = \overline{\alpha}(h) = g^{-1}hg$, $g \in \bigcap_{d\in H} C_A(d) \subset \bigcap_{d\in H}(C_A(d)H)$. For $x \in A\backslash H$, $xb \sim_G \overline{\alpha}(xb) = g^{-1}xg \cdot b$, where $b \in B\backslash H$. By Theorem 2.5, $xb \sim_H g^{-1}xg \cdot b$. This implies that $x = h_1^{-1}g^{-1}xgh_2$ and $b = h_2^{-1}bh_1$ for some $h_1, h_2 \in H$. Since B is Abelian, $h_1 = h_2$. Thus $x = h_1^{-1}g^{-1}xgh_1$. This implies that $gh_1 \in C_A(x)$. Hence $g \in C_A(x)H$ for all $x \in A\backslash H$. Thus $g \in \bigcap_{x\in A}(C_A(x)H)$. By (C1), this implies that $g = zh_3$ for some $h_3 \in H$ and $z \in Z(A)$. This means that $\overline{\alpha}(x) = h_3^{-1}xh_3$ for all $x \in A$. Since B is Abelian, $\overline{\alpha}(y) = y = h_3^{-1}yh_3$ for all $y \in B$. Hence $\overline{\alpha} = \mathrm{Inn}\, h_3$. This shows that G has Property E. □

Since Abelian groups have Property E, we have the following corollary.

COROLLARY 3.2. *Let A, B be finitely generated Abelian groups. Then $A *_H B$ has Property E.*

PROOF. If $A = H$ (similarly $B = H$) then $A *_H B = B$ has Property E. Suppose that $A \neq H \neq B$. Since A is Abelian, A satisfies (C1), (C2) and (C3). Hence $A *_H B$ has Property E by Theorem 3.1. □

In this paper, we are mainly interested in certain tree products, so called *stem products*, of finitely generated Abelian groups. Hence we consider tree products of any groups A_i $(1 \leq i \leq n)$ amalgamating central edge subgroups [7], that is, the tree product of the form

$$G_n = A_1 *_{H_1} A_2 *_{H_2} \cdots *_{H_{n-1}} A_n, \quad H_i \subset Z(A_i) \cap Z(A_{i+1}), \tag{3.4}$$

where $A_i \cap A_{i+1} = H_i$, $A_i \neq H_i$ and $H_i \neq A_{i+1}$. If $n \geq 2$ then

$$G_n = G_{n-1} *_{H_{n-1}} A_n, \tag{3.5}$$

and $Z(G_n) = H_1 \cap \cdots \cap H_{n-1} = H_1 \cap H_{n-1}$ (see [8, p. 211]). For convenience, we let $H_0 = 1$ and $H_n = 1$.

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LEMMA 3.3. *Let G_n be as in (3.4). If $x \in A_i \backslash (H_{i-1} \cup H_i)$ then $C_{G_n}(x) \subset A_i$.*

PROOF. Let $y \in C_{G_n}(x)$. We shall show that $y \in A_i$ by induction on n.

For $n = 2$, $G_2 = A_1 *_{H_1} A_2$ and $x \in A_1 \backslash H_1$. Since $H_1 \subset Z(G_2)$, x is not in any conjugate of H_1 in G_2. Then, by [8, Theorem 4.5, p. 209], either (i) y is in a conjugate of H_1 in G_2 or (ii) $y \in A_1$. Since $Z(G_2) = H_1$, (i) also implies that $y \in H_1 \subset A_1$. Hence $C_{G_2}(x) \subset A_1$. Thus the lemma is true for $n = 2$.

Assume that the lemma is true for G_{n-1}, that is $C_{G_{n-1}}(a) \subset A_j$ if $a \in A_j \backslash (H_{j-1} \cup H_j)$ for $1 \le j \le n-1$. Let $G_n = G_{n-1} *_{H_{n-1}} A_n$ and let $x \in A_i \backslash (H_{i-1} \cup H_i)$. We shall show that $C_{G_n}(x) \subset A_i$.

CASE 1 ($x \in A_i$ for some $1 \le i \le n-1$). Clearly $x \in G_{n-1} \backslash H_{n-1}$. Let $y \in C_{G_n}(x)$. Since $x \in A_i \backslash H_{n-1}$ for $1 \le i \le n-1$ and $H_{n-1} \subset Z(A_n)$, by Lemma 2.6, x is not in any conjugate of H_{n-1} in $G_n = G_{n-1} *_{H_{n-1}} A_n$. Thus, by [8, Theorem 4.5, p. 209], either (i) y is in a conjugate of H_{n-1} in G_n or (ii) $y \in G_{n-1}$. We shall show that (i) also implies that $y \in G_{n-1}$. For this, let $y = g^{-1}hg$, where $g \in G_n$ and $h \in H_{n-1}$. Let $g = u_1 u_2 \cdots u_r$ be an alternating product of the shortest length in $G_n = G_{n-1} *_{H_{n-1}} A_n$ such that $y = g^{-1}hg$. Since $h \in H_{n-1} \subset Z(A_n)$, we may assume that $u_1 \in G_{n-1}$ and $u_1^{-1}hu_1 \ne h$. Then, by Lemma 2.6, $u_1^{-1}hu_1 \notin H_{n-1}$. We shall show that $g \in G_{n-1}$, that is $r \le 1$. Suppose that $r \ge 2$. Then $y = u_r^{-1} \cdots u_2^{-1}(u_1^{-1}hu_1)u_2 \cdots u_r$ is a reduced element of length $2r - 1$. Since $xy = yx$,

$$xu_r^{-1} \cdots u_2^{-1}(u_1^{-1}hu_1)u_2 \cdots u_r = u_r^{-1} \cdots u_2^{-1}(u_1^{-1}hu_1)u_2 \cdots u_r x. \tag{3.6}$$

If $u_r \in A_n \backslash H_{n-1}$ then both sides of (3.6) are cyclically reduced of length $2r$. Since $x \in G_{n-1} \backslash H_{n-1}$, this is impossible. Hence $u_r \in G_{n-1} \backslash H_{n-1}$. Since $r \ge 2$, this implies from (3.6) that $xu_r^{-1} = u_r^{-1}k$ for some $k \in H_{n-1}$. Then, by Lemma 2.6, $x = k$ which is also impossible. Hence $r \le 1$, that is, $g \in G_{n-1}$. Thus $y = g^{-1}hg \in G_{n-1}$ and $y \in C_{G_{n-1}}(x)$. Therefore, by induction, $y \in A_i$.

CASE 2 ($x \in A_n$). Let $y \in C_{G_n}(x)$. Since $x \in A_n \backslash H_{n-1}$, as in Case 1, either (i) y is in a conjugate of H_{n-1} or (ii) $y \in A_n$. We shall show that (i) implies that $y \in H_{n-1}$. As before, let $g = u_1 u_2 \cdots u_r$ be an alternating product of the shortest length in $G_n = G_{n-1} *_{H_{n-1}} A_n$ such that $y = g^{-1}hg$, where $h \in H_{n-1}$, $u_1 \in G_{n-1}$ and $u_1^{-1}hu_1 \ne h$. Suppose that $r \ge 1$. If $u_r \in G_{n-1} \backslash H_{n-1}$, then both sides of (3.6) are cyclically reduced of length $2r$. Since $x \in A_n \backslash H_{n-1}$, this is impossible. Hence $u_r \in A_n \backslash H_{n-1}$. Then (3.6) implies that $xu_{r-1}^{-1} \cdots u_1^{-1}hu_1 \cdots u_{r-1} = u_{r-1}^{-1} \cdots u_1^{-1}hu_1 \cdots u_{r-1}x$. Since $x \in A_n \backslash H_{n-1}$ and $u_{r-1} \in G_{n-1} \backslash H_{n-1}$, this is also impossible. Thus $r = 0$, that is, $g \in H_{n-1}$. This implies that $y = g^{-1}hg \in H_{n-1} \subset A_n$. □

LEMMA 3.4. *Let G_n be as in (3.4) and $H_n \le Z(A_n)$. Then $\bigcap_{a \in G_n}(C_{G_n}(a)H_n) = Z(G_n)H_n$.*

PROOF. Recall that $A_i \ne H_{i-1}$ and $A_i \ne H_i$ for each i and that $Z(G_n) = H_1 \cap \cdots \cap H_{n-1} = H_1 \cap H_{n-1}$ (see [8, p. 211]).

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Clearly there exist $x_1 \in A_1 \backslash H_1$ and $x_n \in A_n \backslash H_{n-1}$. We shall show that $C_{G_n}(x_1)H_n \cap C_{G_n}(x_n)H_n \subset Z(G_n)H_n$. Let $x \in C_{G_n}(x_1)H_n \cap C_{G_n}(x_n)H_n$. Then $x = a_1h_1 = a_nh_n$, where $a_i \in C_{G_n}(x_i)$ and $h_i \in H_n$. Then, by Lemma 3.3, $a_1 \in A_1$ and $a_n \in A_n$. Hence $a_1 = a_nh_nh_1^{-1} \in A_1 \cap A_n = H_1 \cap H_n = Z(G_n)$. It follows that $x = a_1h_1 \in Z(G_n)H_n$. Thus $C_{G_n}(x_1)H_n \cap C_{G_n}(x_n)H_n \subset Z(G_n)H_n$. Therefore $\bigcap_{a \in G_n}(C_{G_n}(a)H_n) = Z(G_n)H_n$ as required. □

THEOREM 3.5. *Let G_n be given by (3.4), where the A_i are finitely generated Abelian groups. Then G_n has Property E.*

PROOF. We shall prove the theorem by induction on n. Since $G_1 = A_1$ is Abelian, clearly G_1 has Property E. By Corollary 3.2, G_2 has Property E. Suppose that $n \geq 3$ and G_{n-1} has Property E. Considering $G_n = G_{n-1} *_{H_{n-1}} A_n$, we need to show (C1) and (C2) in Theorem 3.1. By Lemma 3.4, (C1) holds. To prove (C2), let $a \in A_1 \backslash H_1$. Then $a \notin H_{n-1}$ and by Lemma 2.6, $\{a\}^{G_{n-1}} \cap H_{n-1} = \emptyset$. Hence (C2) holds. Thus, by Theorem 3.1, G_n has Property E. □

Since tree products of finitely generated Abelian groups are conjugacy separable [7], we have the following result from Theorem 2.4.

THEOREM 3.6. *Outer automorphism groups of tree products of finitely generated Abelian groups are $\mathcal{RF}$.*

If $G = A *_{h=b^m} \langle b \rangle$, where b is of infinite order, then we say that G is obtained by *adjoining an mth root* to an element of the group A (see [9, 10]). In the case of adjoining a root, we prove the following criterion which is simpler than Theorem 3.1.

THEOREM 3.7. *Suppose that A has Property E and satisfies:*

(C1) $\bigcap_{a \in A}(C_A(a)\langle h \rangle) = Z(A)\langle h \rangle$.

Then $G = A *_{h=b^m} \langle b \rangle$ has Property E.

PROOF. Let α be a conjugating endomorphism of G and $\alpha(g) = k_g^{-1}gk_g$ for $g \in G$. Without loss of generality, we can assume that $\alpha(b) = b$.

CLAIM. *We show that, for each $a \in A$, we can choose k_a in $\langle b \rangle$.*

PROOF. Let $a \in A \backslash \langle h \rangle$ and $k_a = u_1u_2 \cdots u_r$ be an alternating product of the shortest length in G such that $\alpha(a) = k_a^{-1}ak_a$. We shall first show that $r \leq 1$. Suppose that $r \geq 2$. Then, as in the proof of Theorem 3.1,

$$ab \sim_G \alpha(ab) \sim_G u_{r-1}^{-1} \cdots u_2^{-1} \cdot u_1^{-1}au_1 \cdot u_2 \cdots u_{r-1} \cdot u_rbu_r^{-1}. \tag{3.7}$$

(1) Suppose that $u_1 \in A$. Since $u_2 \in \langle b \rangle$, we may assume that $u_1^{-1}au_1 \notin \langle h \rangle$. In this case, if $u_r \in A$ then the right-hand side of (3.7) is cyclically reduced of length $2r$. If $u_r \in \langle b \rangle$ then the right-hand side of (3.7) is cyclically reduced of length $2(r-1)$. Since the left-hand side of (3.7) is cyclically reduced of length 2, (3.7) does not hold if $r \geq 3$. Thus we assume that $u_1^{-1}au_1 \notin \langle h \rangle$ and $r = 2$. Then (3.7) implies that

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$$ab \sim_G u_1^{-1} a u_1 u_2 b u_2^{-1} = u_1^{-1} a u_1 b. \tag{3.8}$$

Thus, by Theorem 2.5, $ab \sim_{\langle h \rangle} u_1^{-1} a u_1 b$. This means that $a = h^{-s} u_1^{-1} a u_1 h^{s_1}$ and $b = h^{-s_1} b h^s$ for some s, s_1. Since $h \in \langle b \rangle$, we have $s = s_1$ and $u_1^{-1} a u_1 = h^s a h^{-s}$. Then $\alpha(a) = u_2^{-1} u_1^{-1} a u_1 u_2 = u_2^{-1}(h^s a h^{-s}) u_2$. Thus we may choose $k_a = h^{-s} u_2 \in B$. This contradicts that $r \geq 2$ is the smallest length of k_a such that $\alpha(a) = k_a^{-1} a k_a$.

(2) Suppose that $u_1 \in \langle b \rangle$. If $u_r \in A$ then the right-hand side of (3.7) is cyclically reduced of length $2(r+1)$. If $u_r \in \langle b \rangle$ then the right-hand side of (3.7) is cyclically reduced of length $2r$. Since $r \geq 2$ and the left-hand side of (3.7) is of length 2, both cases cannot occur.

Therefore $r \leq 1$. Thus (i) $k_a = u_1 \in A$ or (ii) $k_a = u_1 \in \langle b \rangle$. If $k_a = u_1 \in A$ then, by (3.7), $ab \sim_G u_1^{-1} a u_1 b$. Hence, as above, $a = h^{-s} u_1^{-1} a u_1 h^s$ for some s. Thus $\alpha(a) = u_1^{-1} a u_1 = h^s a h^{-s}$. This implies that we can choose $k_a = h^{-s} \in \langle b \rangle$. Hence, in both case (i) and (ii), $k_a \in \langle b \rangle$ for each $a \in A \backslash \langle h \rangle$. Since $\alpha(b) = b$, $\alpha(h^i) = h^i$ for all i.

Thus the claim holds. $\square$

Let $a \in A \backslash \langle h \rangle$ be fixed and $x \in A \backslash \langle h \rangle$ be arbitrary. By our claim, we can assume that $k_a = w \in \langle b \rangle$ and $k_x \in \langle b \rangle$. Then $ax \sim_G \alpha(ax) = w^{-1} a w \cdot k_x^{-1} x k_x \sim_G a w k_x^{-1} x k_x w^{-1}$. Since $a, x \in A \backslash \langle h \rangle$, $k_x w^{-1} \in \langle h \rangle$. Hence $k_x = h^{s_x} w$ where s_x depends on x.

Let $\overline{\alpha} = \operatorname{Inn} w^{-1} \circ \alpha$. Then $\overline{\alpha}$ is also a conjugating endomorphism of G and $\overline{\alpha}(a) = w(w^{-1} a w) w^{-1} = a$, $\overline{\alpha}(x) = w(k_x^{-1} x k_x) w^{-1} = h^{-s_x} x h^{s_x}$ for each $x \in A \backslash \langle h \rangle$. Moreover, $\overline{\alpha}(b) = w^{-1}(\alpha(b)) w = w^{-1} b w = b$. Since $h \in \langle b \rangle$, $\overline{\alpha}(h^i) = h^i$ for all i. Let $s_x = 0$ for $x \in \langle h \rangle$. Then $\overline{\alpha}(x) = h^{-s_x} x h^{s_x}$ for each $x \in A$. Thus $\overline{\alpha}$ restricted to A is a conjugating endomorphism of A. Since A has Property E, $\overline{\alpha}$ restricted to A is an inner automorphism of A. Let $g \in A$ such that $\overline{\alpha}(x) = g^{-1} x g$ for all $x \in A$. Thus $g^{-1} x g = h^{-s_x} x h^{s_x}$ for each $x \in A$. Hence $g h^{-s_x} \in C_A(x)$. This implies that $g \in C_A(x)\,\langle h \rangle$ for all $x \in A$. Since $\bigcap_{x \in A} (C_A(x)\,\langle h \rangle) = Z(A)\,\langle h \rangle$ by (C1), $g = z h^r$ for some r and $z \in Z(A)$. Thus $\overline{\alpha}(x) = h^{-r} x h^r$ for all $x \in A$ and $\overline{\alpha}(b) = b = h^{-r} b h^r$. Hence $\overline{\alpha} = \operatorname{Inn} h^r$ and α is an inner automorphism of G. Therefore, G has Property E. $\square$

4. On Brauner's groups

In [10], Stebe studied the cyclic subgroup separability of Brauner's groups. In this section we prove that the outer automorphism groups of these groups are residually finite. These groups are given by:

(1) the group of a single knot on a torus,

$$\langle x \rangle \underset{x^n = y^m}{*} \langle y \rangle;$$

(2) the group for the linkage of a torus knot with a circle,

$$\langle x, y : [x, y] \rangle \underset{x^n y^m = z^m}{*} \langle z \rangle;$$

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(3) the group for the linkage of a torus knot with circles within and outside the torus,

$$\langle x, y : [x, y]\rangle \underset{x^n y^m = a^m b^n}{*} \langle a, b : [a, b]\rangle;$$

(4) the group of the linkage of torus knots,

$$\langle a\rangle \underset{a^m = x_1^{\delta_1}}{*} \langle x_1, y_1 : [x_1, y_1]\rangle \underset{x_1^{\alpha_1} y_1^{\beta_1} = x_2^{\delta_2}}{*} \langle x_2, y_2 : [x_2, y_2]\rangle \underset{x_2^{\alpha_2} y_2^{\beta_2} = x_3^{\delta_3}}{*} \cdots$$
$$\cdots \underset{x_{n-1}^{\alpha_{n-1}} y_{n-1}^{\beta_{n-1}} = x_n^{\delta_n}}{*} \langle x_n, y_n : [x_n, y_n]\rangle \underset{x_n^{\alpha_n} y_n^{\beta_n} = x_{n+1}^{\delta_{n+1}}}{*} \langle x_{n+1}\rangle;$$

(5) the group of a hose knot,

$$A_r = \langle p_1\rangle \underset{p_1^{n_1} = q_1^{m_1}}{*} \langle q_1\rangle \underset{h_2 = q_2^{m_2}}{*} \langle q_2\rangle \underset{h_3 = q_3^{m_3}}{*} \cdots \underset{h_r = q_r^{m_r}}{*} \langle q_r\rangle,$$

where $h_i = p_i^{n_i - n_{i-1}m_i} q_{i-1}^{m_{i-1}m_i}$, $p_i = q_{i-1}^{-u_{i-1}} p_{i-1}^{k_{i-1} - n_{i-2}u_{i-1}} q_{i-2}^{m_{i-2}u_{i-1}}$, $q_0 = 1$, $n_0 = 0$ and $m_i k_i = n_i u_i + 1$ (see [5]).

Since the first four groups are tree products of finitely generated Abelian groups amalgamating cyclic subgroups, they are conjugacy separable [7]. By Theorem 3.5, they have Property E. Hence outer automorphism groups of those groups are residually finite (Theorem 3.6). Thus we can state the following theorem.

THEOREM 4.1. *Outer automorphism groups of groups given by (1)–(4) above are $\mathcal{RF}$.*

The only case left is the outer automorphism group of the group A_r of a hose knot in (5) above. The group A_r is obtained by repeated adjoining roots. Thus, applying Theorem 3.7, we shall show that A_r has Property E (Theorem 4.5). We first note some properties of A_r.

(1) Without loss of generality, we can assume $n_1 \neq \pm 1$ and $m_i \neq \pm 1$.
(2) $h_{i+1} = p_{i+1}^{n_{i+1} - n_i m_{i+1}} q_i^{m_i m_{i+1}}$ is cyclically reduced of length > 1 in $A_i = A_{i-1} *_{h_i = q_i^{m_i}} \langle q_i\rangle$ (see [11, p. 89]).
(3) Clearly $Z(A_1) = \langle p_1^{n_1}\rangle$ and $Z(A_r) = 1$ for $r \geq 2$.
(4) $C_{A_r}(q_i) = \langle q_i\rangle$ for each i.

LEMMA 4.2. *Let A_1 be defined by (5). Then $\bigcap_{a \in A_1}(C_{A_1}(a)\,\langle h_2\rangle) = Z(A_1)\,\langle h_2\rangle$.*

PROOF. Consider $\overline{A}_1 = A_1/Z(A_1)$. Then $\overline{A}_1 = \langle \overline{p}_1, \overline{q}_1 : \overline{p}_1^{n_1}, \overline{q}_1^{m_1}\rangle = \langle \overline{p}_1 : \overline{p}_1^{n_1}\rangle * \langle \overline{q}_1 : \overline{q}_1^{m_1}\rangle$ and $\overline{h}_2 = (\overline{q}_1^{-u_1}\overline{p}_1^{k_1})^{n_2 - n_1 m_2}$. We shall first show that $\bigcap_{\overline{a} \in \overline{A}_1}(C_{\overline{A}_1}(\overline{a})\,\langle \overline{h}_2\rangle) = \langle \overline{h}_2\rangle$. For this, let $\overline{x} \in \bigcap_{\overline{a} \in \overline{A}_1}(C_{\overline{A}_1}(\overline{a})\,\langle \overline{h}_2\rangle)$. Suppose that $\overline{x} \notin \langle \overline{h}_2\rangle$. Since $\overline{x} \in C_{\overline{A}_1}(\overline{p}_1)\,\langle \overline{h}_2\rangle \cap C_{\overline{A}_1}(\overline{q}_1)\,\langle \overline{h}_2\rangle$, $\overline{x} = \overline{p}_1^{\alpha}\overline{h}_2^{i} = \overline{q}_1^{\beta}\overline{h}_2^{j}$ for some i, j and some $\overline{p}_1^{\alpha}, \overline{q}_1^{\beta} \notin \langle \overline{h}_2\rangle$. Then $\overline{q}_1^{-\beta}\overline{p}_1^{\alpha} = \overline{h}_2^{j-i}$. Since $\overline{h}_2 = (\overline{q}_1^{-u_1}\overline{p}_1^{k_1})^{n_2 - n_1 m_2}$ is cyclically reduced of length $2(n_2 - n_1m_2) \geq 2$, $\overline{q}_1^{-\beta}\overline{p}_1^{\alpha} = \overline{h}_2^{\pm 1}$. Moreover, $\overline{q}_1^{-\beta} = \overline{q}_1^{-u_1}$ and $\overline{p}_1^{\alpha} = \overline{p}_1^{k_1}$. Since $m_1k_1 = n_1u_1 + 1$, $(n_1, m_1) = 1$. Thus, either $|n_1| \neq 2$ or $|m_1| \neq 2$. We can choose non-trivial elements $\overline{p}_1^{d}, \overline{q}_1^{c}$ such that either $\overline{p}_1^{d} \neq \overline{p}_1^{\alpha}$ or $\overline{q}_1^{c} \neq \overline{q}_1^{-\beta}$. Let $\overline{w} = \overline{q}_1^{c}\overline{p}_1^{d}\overline{q}_1^{c}$. Since $\overline{x} \in \bigcap_{\overline{a}\in\overline{A}_1}(C_{\overline{A}_1}(\overline{a})\ \langle\overline{h}_2\rangle)$, $\overline{x} \in C_{\overline{A}_1}(\overline{w}^{-1}\overline{p}_1\overline{w})\ \langle\overline{h}_2\rangle$. Thus $\overline{x} = \overline{p}_1^{\alpha}\overline{h}_2^{i} = \overline{w}^{-1}\overline{p}_1^{\lambda}\overline{w}\overline{h}_2^{k}$ for some λ, k. Hence $\overline{p}_1^{-\alpha}\overline{w}^{-1}\overline{p}_1^{\lambda}\overline{w} = \overline{h}_2^{i-k}$, that is, $\overline{p}_1^{-\alpha}\overline{q}_1^{-c}\overline{p}_1^{-d}\overline{q}_1^{-c}\overline{p}_1^{\lambda}\overline{q}_1^{c}\overline{p}_1^{d}\overline{q}_1^{c} = \overline{h}_2^{i-k} = (\overline{p}_1^{-\alpha}\overline{q}_1^{\beta})\cdots(\overline{p}_1^{-\alpha}\overline{q}_1^{\beta})$. This implies that $\overline{q}_1^{-c} = \overline{q}_1^{\beta}$ and $\overline{p}_1^{-d} = \overline{p}_1^{-\alpha}$, contradicting our choice of either $\overline{q}_1^{c}$ or $\overline{p}_1^{d}$. Hence $\overline{x} \in \langle\overline{h}_2\rangle$, proving that $\bigcap_{\overline{a}\in\overline{A}_1}(C_{\overline{A}_1}(\overline{a})\ \langle\overline{h}_2\rangle) \subset \langle\overline{h}_2\rangle$. Therefore $\bigcap_{\overline{a}\in\overline{A}_1} C_{\overline{A}_1}(\overline{a})\ \langle\overline{h}_2\rangle = \langle\overline{h}_2\rangle$.

To prove $\bigcap_{a\in A_1}(C_{A_1}(a)\ \langle h_2\rangle) = Z(A_1)\ \langle h_2\rangle$, let $x \notin Z(A_1)\ \langle h_2\rangle$. Then $\overline{x} \notin \langle\overline{h}_2\rangle$ in $\overline{A}_1 = A_1/Z(A_1)$. Since $\bigcap_{\overline{a}\in\overline{A}_1} C_{\overline{A}_1}(\overline{a})\ \langle\overline{h}_2\rangle = \langle\overline{h}_2\rangle$ by the above, there exists $\overline{a} \in \overline{A}_1$ such that $\overline{x} \notin C_{\overline{A}_1}(\overline{a})\ \langle\overline{h}_2\rangle$. Then $x \notin C_{A_1}(a)\ \langle h_2\rangle$. Thus $x \notin \bigcap_{a\in A_1}(C_{A_1}(a)\ \langle h_2\rangle)$. This shows that $\bigcap_{a\in A_1}(C_{A_1}(a)\ \langle h_2\rangle) \subset Z(A_1)\ \langle h_2\rangle$. Hence $\bigcap_{a\in A_1}(C_{A_1}(a)\ \langle h_2\rangle) = Z(A_1)\ \langle h_2\rangle$. □

LEMMA 4.3. *For $r \geq 2$, there exists $c \in A_{r-1}\backslash\langle q_{r-1}\rangle\langle h_r\rangle$.*

PROOF. We have two cases.

CASE 1 ($r = 2$). Let $\overline{A}_1 = A_1/Z(A_1)$ as before. Then $\overline{A}_1 = \langle\overline{p}_1 : \overline{p}_1^{n_1}\rangle * \langle\overline{q}_1 : \overline{q}_1^{m_1}\rangle$ and $\overline{h}_2 = (\overline{q}_1^{-u_1}\overline{p}_1^{k_1})^{n_2-n_1m_2}$. Since $(n_1, m_1) = 1$, either $|n_1| \neq 2$ or $|m_1| \neq 2$. Then we can choose non-trivial elements $\overline{p}_1^{\alpha}, \overline{q}_1^{\beta}$ such that $\overline{p}_1^{\alpha}\overline{q}_1^{\beta} \neq \overline{p}_1^{-k_1}\overline{q}_1^{u_1}$. We shall show that $\overline{p}_1^{\alpha}\overline{q}_1^{\beta} \notin \langle\overline{q}_1\rangle\langle\overline{h}_2\rangle$. Suppose that $\overline{p}_1^{\alpha}\overline{q}_1^{\beta} = \overline{q}_1^{i}\overline{h}_2^{j}$ for some i, j. Then

$$\overline{p}_1^{\alpha}\overline{q}_1^{\beta} = \overline{q}_1^{i}(\overline{q}_1^{-u_1}\overline{p}_1^{k_1})^{(n_2-n_1m_2)j}. \tag{4.1}$$

If $|(n_2 - n_1m_2)j| \geq 2$, then the right-hand side of (4.1) is of length at least 3. Since the left-hand side of (4.1) is of length 2, (4.1) does not hold. Hence $|(n_2 - n_1m_2)j| = 1$. Then the only possible case of (4.1) is $i = 0$ and $(n_2 - n_1m_2)j = -1$, that is, $\overline{p}_1^{\alpha}\overline{q}_1^{\beta} = \overline{p}_1^{-k_1}\overline{q}_1^{u_1}$. This clearly contradicts the choice of $\overline{p}_1^{\alpha}\overline{q}_1^{\beta}$. Hence $\overline{p}_1^{\alpha}\overline{q}_1^{\beta} \notin \langle\overline{q}_1\rangle\langle\overline{h}_2\rangle$. Let $c = p_1^{\alpha}q_1^{\beta}$. Then $c \notin \langle q_1\rangle\langle h_2\rangle$ as required.

CASE 2 ($r \geq 3$). Let $N = \langle p_1^{n_1}, q_2, \ldots, q_{r-1}\rangle^{A_{r-1}}$ and let $\overline{A}_{r-1} = A_{r-1}/N$. Then

$$\overline{A}_{r-1} = \langle\overline{p}_1, \overline{q}_1 : \overline{p}_1^{n_1}, \overline{q}_1^{m_1}, \overline{h}_2\rangle = \langle\overline{p}_1, \overline{q}_1 : \overline{p}_1^{n_1}, \overline{q}_1^{m_1}, (\overline{q}_1^{-u_1}\overline{p}_1^{k_1})^{n_2-n_1m_2}\rangle.$$

Then $\overline{p}_i = \overline{p}_{i-1}^{k_{i-1}n_{i-2}u_{i-1}}$ and $\overline{h}_i = \overline{p}_i^{n_i-n_{i-1}m_i}$ for $i \geq 3$. Hence $\overline{h}_r \in \langle\overline{p}_r\rangle \subset \langle\overline{p}_2\rangle$. Thus $\langle\overline{q}_{r-1}\rangle\langle\overline{h}_r\rangle = \langle\overline{h}_r\rangle \subset \langle\overline{p}_2\rangle$, where $\overline{p}_2 = \overline{q}_1^{-u_1}\overline{p}_1^{k_1}$. Since $\overline{A}_{r-1} \neq \langle\overline{p}_2\rangle$, there exists $\overline{c} \in \overline{A}_{r-1}\backslash\langle\overline{p}_2\rangle$. Then $\overline{c} \notin \langle\overline{h}_r\rangle$. Let c be a preimage of $\overline{c}$ in A_{r-1}. Then $c \in A_{r-1}\backslash\langle q_{r-1}\rangle\langle h_r\rangle$, as required. □

LEMMA 4.4. *For $r \geq 1$, $\bigcap_{a \in A_r} (C_{A_r}(a)\, \langle h_{r+1} \rangle) = Z(A_r)\, \langle h_{r+1} \rangle$.*

PROOF. The case of $r = 1$ is proved in Lemma 4.2. So we consider $r \geq 2$. Since $Z(A_r) = 1$ for $r \geq 2$, we shall show that $\bigcap_{a \in A_r} (C_{A_r}(a)\, \langle h_{r+1} \rangle) = \langle h_{r+1} \rangle$. Let $x \in \bigcap_{a \in A_r} (C_{A_r}(a)\, \langle h_{r+1} \rangle)$. Suppose that $x \notin \langle h_{r+1} \rangle$. Then $x \in C_{A_r}(q_r)\, \langle h_{r+1} \rangle \cap C_{A_r}(q_{r-1})\, \langle h_{r+1} \rangle$. Since $C_{A_r}(q_r) = \langle q_r \rangle$, $C_{A_r}(q_{r-1}) = \langle q_{r-1} \rangle$ and $x \notin \langle h_{r+1} \rangle$, we have $x = q_r^i h_{r+1}^{s_1} = q_{r-1}^j h_{r+1}^{s_2}$ for some i, j, s_1, s_2, where $q_r^i, q_{r-1}^j \notin \langle h_{r+1} \rangle$. Hence

$$q_{r-1}^{-j} q_r^i = h_{r+1}^{s_2 - s_1}. \tag{4.2}$$

Since h_{r+1} is cyclically reduced of length > 1 in $A_r = A_{r-1} *_{\langle h_r \rangle} \langle q_r \rangle$ and the left-hand side of (4.2) is of length at most 2, equation (4.2) implies that $s_2 - s_1 = \pm 1$ and $h_{r+1}^{\pm 1} = q_{r-1}^{-j} q_r^i$ is cyclically reduced of length 2. Hence $q_r^i \notin \langle h_r \rangle = \langle q_r^{m_r} \rangle$. By Lemma 4.3, there exists $c \in A_{r-1} \backslash \langle q_{r-1} \rangle \langle h_r \rangle$. Let $w = c q_r c$. Since $x \in \bigcap_{a \in A_r} (C_{A_r}(a)\, \langle h_{r+1} \rangle)$, $x \in C_{A_r}(w q_{r-1} w^{-1})\, \langle h_{r+1} \rangle$. Hence $x = q_r^i h_{r+1}^{s_1} = w q_{r-1}^{\epsilon} w^{-1} h_{r+1}^{s_3}$ for some ϵ, s_3, where $q_{r-1}^{\epsilon} \neq 1$. Since h_r is cyclically reduced of length > 1 in $A_{r-1} = A_{r-2} *_{h_{r-1} = q_{r-1}^{m_{r-1}}} \langle q_{r-1} \rangle$, we have $q_{r-1}^{\epsilon} \notin \langle h_r \rangle$. This implies that $h_{r+1}^{s_1 - s_3} = q_r^{-i} w q_{r-1}^{\epsilon} w^{-1} = q_r^{-i} c q_r c q_{r-1}^{\epsilon} c^{-1} q_r^{-1} c^{-1}$, which is cyclically reduced in A_r. Since $h_{r+1}^{\pm 1} = q_{r-1}^{-j} q_r^i$,

$$q_r^{-i} c q_r c q_{r-1}^{\epsilon} c^{-1} q_r^{-1} c^{-1} = (q_r^{-i} q_{r-1}^j)\, (q_r^{-i} q_{r-1}^j)\, (q_r^{-i} q_{r-1}^j)(q_r^{-i} q_{r-1}^j). \tag{4.3}$$

Hence there exist $h_r^{\epsilon_1}, \ldots, h_r^{\epsilon_6}$ such that $c = q_{r-1}^j h_r^{\epsilon_1}$, $q_r = h_r^{-\epsilon_1} q_r^{-i} h_r^{\epsilon_2}$, $c = h_r^{-\epsilon_2} q_{r-1}^j h_r^{\epsilon_3}, \ldots, q_r^{-1} = h_r^{-\epsilon_5} q_r^{-i} h_r^{\epsilon_6}$, and $c^{-1} = h_r^{-\epsilon_6} q_{r-1}^j$. Thus $c \in \langle q_{r-1} \rangle \langle h_r \rangle$, which contradicts the choice of c. Therefore $x \in \langle h_{r+1} \rangle$. This implies that $\bigcap_{a \in A_r} (C_{A_r}(a)\, \langle h_{r+1} \rangle) \subset \langle h_{r+1} \rangle$. Hence $\bigcap_{a \in A_r} (C_{A_r}(a)\, \langle h_{r+1} \rangle) = \langle h_{r+1} \rangle = Z(A_r)\, \langle h_{r+1} \rangle$ for $r \geq 2$. □

Applying Theorem 3.7, we have the following theorem.

THEOREM 4.5. *The group of a hose knot has Property E.*

PROOF. Clearly A_1 has Property E by Corollary 3.2. By Lemma 4.2 and Theorem 3.7, $A_2 = A_1 *_{h_2 = q_2^{m_2}} \langle q_2 \rangle$ has Property E. Inductively, suppose that A_{r-1} has Property E. Then $A_r = A_{r-1} *_{h_r = q_r^{m_r}} \langle q_r \rangle$ has Property E by Lemma 4.4 and Theorem 3.7. □

We note that the group of a hose knot is conjugacy separable [11]. Thus, applying Theorems 2.4 and 4.5, we have the following theorem.

THEOREM 4.6. *Outer automorphism groups of hose knot groups are $\mathcal{RF}$.*

Consequently, we have shown that the outer automorphism groups of all Brauner groups stated at the beginning of this section are residually finite.

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PROBLEM. Are outer automorphism groups of conjugacy separable 1-relator groups residually finite?

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