Bull. Aust. Math. Soc. **98** (2018), 277–285
doi:10.1017/S0004972718000370

A REACTION–DIFFUSION–ADVECTION EQUATION WITH COMBUSTION NONLINEARITY ON THE HALF-LINE

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(Received 2 April 2018; accepted 12 April 2018; first published online 2 July 2018)

Abstract

We study the dynamics of a reaction–diffusion–advection equation $u_t = u_{xx} - au_x + f(u)$ on the right half-line with Robin boundary condition $u_x = au$ at $x = 0$, where $f(u)$ is a combustion nonlinearity. We show that, when $0 < a < c$ (where c is the travelling wave speed of $u_t = u_{xx} + f(u)$), u converges in the $L^\infty_{loc}([0,\infty))$ topology either to 0 or to a positive steady state; when $a \geq c$, a solution u starting from a small initial datum tends to 0 in the $L^\infty([0,\infty))$ topology, but this is not true for a solution starting from a large initial datum; when $a > c$, such a solution converges to 0 in $L^\infty_{loc}([0,\infty))$ but not in $L^\infty([0,\infty))$ topology.

2010 *Mathematics subject classification*: primary 35K57; secondary 35K15, 35B40.

Keywords and phrases: reaction–diffusion equation, combustion equation, initial boundary value problem, asymptotic behaviour.

1. Introduction

In this paper, we study the asymptotic behaviour, as $t \to \infty$, of solutions of

$$\begin{cases} u_t = u_{xx} - au_x + f(u), & x > 0,\ t > 0, \\ u_x(t,0) = au(t,0), & t > 0, \\ u(0,x) = u_0(x), & x \geq 0, \end{cases} \tag{1.1}$$

where $a \geq 0$ denotes the advection strength, f is a combustion nonlinearity satisfying

$$f(s) = 0 \text{ in } [0,\theta], \quad f(s) > 0 \text{ in } (\theta,1), \quad f(s) < 0 \text{ in } (1,\infty) \tag{F}$$

for some $\theta \in (0,1)$ and u_0 is taken from the set

$$\mathcal{X} := \{\phi \mid \phi \in C([0,\infty)), \phi \geq 0, \phi \not\equiv 0 \text{ and the support of } \phi,\ \operatorname{spt}(\phi),\ \text{is compact}\}.$$

There have been many studies of reaction–diffusion equations with a nonlinearity f as in (F), where θ represents the ignition point in a combustion process, and so f is called a combustion nonlinearity. For example, Zlatoš [13], Du and Matano [7] and

This research was partly supported by NSFC (Nos. 11701374 and 11671262).

Du and Lou [5] gave a rather complete analysis of the dynamics of bounded solutions. More precisely, for the Cauchy problem and the free boundary problem of the equation $u_t = u_{xx} + f(u)$ (with f satisfying (F)), they proved trichotomy results: any bounded solution $u(t, x)$ converges as $t \to \infty$ in the $L^\infty_{loc}(\mathbb{R})$ topology to 0, 1 or θ, respectively, representing *vanishing, spreading* or *transition* of the solution. In this paper we consider the problem in an advective environment, with $-au_x$ denoting the effect of advection. Such problems arise in physics, chemistry and ecology. If the problem is considered in the whole space (that is, the Cauchy problem), then by transferring it into a moving frame with $z = x - at$, the equation is converted into one without advection. We consider the problem (1.1) on the half-line with nonflux boundary condition, $u_x = au$, and study the influence of the advection and the Robin boundary condition on the dynamics of bounded solutions of (1.1). We will see that the dynamics are quite different according to the sign of $a - c$. Similarly, [3, 4, 11] considered reaction–diffusion–advection equations in bounded domains with Robin boundary conditions; [2, 8] considered reaction–diffusion equations with bistable nonlinearities and Robin boundary conditions and [9, 10] considered reaction–diffusion–advection equations with free boundary conditions.

It is easily seen by the maximum principle that any positive solution of (1.1) is bounded. Following the ideas in [5, 7], we will show in Lemma 3.1 that this solution converges as $t \to \infty$ to a nonnegative steady state, that is, to a solution of

$$v'' - av' + f(v) = 0, \quad v(x) \geq 0 \text{ in } [0, \infty), \quad v'(0) = av(0), \quad v \in L^\infty((0, \infty)). \tag{1.2}$$

We will see that the structure of the set of such solutions depends sensitively on the sign of $a - c$, where $c > 0$ is the travelling wave speed of the equation $u_t = u_{xx} + f(u)$. More precisely, using the phase plane analysis, one can show that the solutions of (1.2) are classified as follows (see details in Section 2).

(1) If $a = 0$, then $v \equiv q \in [0, \theta]$ or $v = 1$.

(2) If $0 < a < c$, then $v \equiv 0$ or v is an *active state*: $v = V^0(x + z)$ for a unique $z > 0$, where V^0 is the unique solution of

$$v'' - av' + f(v) = 0, \quad v'(x) > 0 \text{ in } [0, \infty), \quad v(0) = 0, \quad v(\infty) = 1. \tag{1.3}$$

(3) If $a = c$, then $v \equiv 0$ or v is an *active state*: $v \in \mathcal{V}^* := \{V^*(x - z) \mid z \geq 0\}$, where V^* is the unique solution of

$$v'' - av' + f(v) = 0, \quad v'(x) > 0 \text{ in } \mathbb{R}, \quad v(-\infty) = 0, \quad v(0) = \theta, \quad v(\infty) = 1. \tag{1.4}$$

(4) If $a > c$, then $v \equiv 0$.

Our first main result is the following theorem.

Theorem 1.1. *Assume (F) and $\phi \in \mathcal{X}$. Let $u_\sigma(t, x)$ be the time-global classical solution of (1.1) with initial data $u_0 = \sigma\phi$ ($\sigma > 0$).*

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(i) *When* $a = 0$*, there exists* $\sigma^* \in (0, \infty]$ *such that spreading happens* (*that is,* $u_\sigma \to 1$ *in the* $L^\infty_{loc}([0, \infty))$ *topology*) *when* $\sigma > \sigma^*$*; vanishing happens* (*that is,* $u_\sigma \to 0$ *in the* $L^\infty_{loc}([0, \infty))$ *topology*) *when* $\sigma < \sigma^*$ *and transition happens* (*that is,* $u_\sigma \to \theta$ *in the* $L^\infty_{loc}([0, \infty))$ *topology*) *when* $\sigma = \sigma^*$.

(ii) *When* $0 < a < c$*, there exist* $\sigma_*, \sigma^* \in (0, \infty]$ *with* $\sigma_* \le \sigma^*$ *such that spreading happens* (*that is,* $u_\sigma \to V^0(x + z)$ *in the* $L^\infty_{loc}([0, \infty))$ *topology*) *when* $\sigma > \sigma^*$ *and vanishing happens when* $\sigma \le \sigma^*$*. Furthermore, when* $\sigma < \sigma_*$*,* $u_\sigma \to 0$ *in the* $L^\infty([0, \infty))$ *topology and, when* $\sigma \in [\sigma_*, \sigma^*]$*,* $u_\sigma \to 0$ *in* $L^\infty_{loc}([0, \infty))$ *but not in the* $L^\infty([0, \infty))$ *topology.*

(iii) *When* $a > c$*, vanishing always happens. Furthermore, there exists* $\sigma_* \in (0, \infty]$ *such that* $u_\sigma \to 0$ *in the* $L^\infty([0, \infty))$ *topology when* $\sigma < \sigma_*$ *and* $u_\sigma \to 0$ *in* $L^\infty_{loc}([0, \infty))$ *but not in the* $L^\infty([0, \infty))$ *topology when* $\sigma \ge \sigma_*$.

(iv) *When* $a = c$*, the solution* u_σ *converges as* $t \to \infty$ *to* 0 *or an active state in* $\mathcal{V}^*$*. Furthermore, there exists* $\sigma_* \in (0, \infty]$ *such that* $u_\sigma \to 0$ *in the* $L^\infty([0, \infty))$ *topology when* $\sigma < \sigma_*$ *and* $u_\sigma \not\to 0$ *in the* $L^\infty([0, \infty))$ *topology when* $\sigma \ge \sigma_*$.

We remark that in the critical case $a = c$, the problem (1.2) has a continuum of active states $V^*(x - z) \in \mathcal{V}^*$ and it is hard to distinguish one of these from the others. So, in this case, we cannot yet say which active state should be the real ω-limit of u.

This paper is organised as follows. In Section 2 we present some steady states of the equation by using the phase plane method. In Section 3 we study the asymptotic behaviour of the solutions, including some common convergence results and the detailed long time dynamics of bounded solutions in the cases $0 < a < c$, $a > c$ and $a = c$.

2. Steady states

We use phase plane analysis to study nonnegative and bounded steady states of the equation in (1.1), that is, nonnegative and bounded solutions of

$$v'' - av' + f(v) = 0, \qquad x \in I, \tag{2.1}$$

where I is some interval in $\mathbb{R}$. The equation (2.1) is equivalent to the system

$$\begin{cases} v' = w, \\ w' = aw - f(v). \end{cases} \tag{2.2}$$

A solution (v, w) of this system traces out a trajectory in the vw-phase plane. Such a trajectory has slope

$$\frac{dw}{dv} = a - \frac{f(v)}{w}$$

at any point where $w \ne 0$. It is easily seen that $(q, 0)$ (with $q \in [0, \theta]$) and $(1, 0)$ are all equilibrium points of (2.2) on the phase plane. Clearly, $(1, 0)$ is a saddle point and the equation is reduced to $dw/dv = a$ for $v \in [0, \theta]$. By a simple phase analysis, it is not difficult to give all nonnegative solutions of (2.1) which correspond to trajectories.

https://doi.org/10.1017/S0004972718000370 Published online by Cambridge University Press

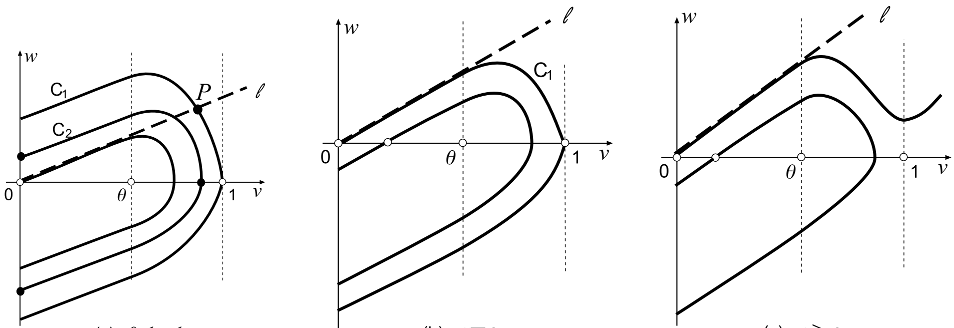

FIGURE 1. Trajectories corresponding to the equation $v'' - av' + f(v) = 0$.

Furthermore, the boundary condition $v'(0) = av(0)$ in (1.2) is satisfied whenever the corresponding trajectory intersects the line $\ell: \ w = av$.

In case $0 < a < c$ (see Figure 1(a)), $v \equiv 0$ is a trivial solution of (1.2). There is a trajectory C_1 connecting a point $(0, \omega_+^a)$ (for some $\omega_+^a > 0$) and $(1, 0)$, which corresponds to a solution V^0 of (1.3). The trajectory C_1 contacts the line $\ell: \ w = av$ at a unique point $P(q_0, aq_0)$ for some $q_0 \in (\theta, 1)$. Assume that $x = z > 0$ is the unique point where $V^0(z) = q_0$. Then $v(x) := V^0(x + z)$ is a solution of (1.2). Clearly, (1.2) has no other solutions besides 0 and $V^0(x + z)$.

On the other hand, it is easily seen that, for each $\omega_+ \in (0, \omega_+^a)$, there is a trajectory C_2 connecting $(0, \omega_+)$ and $(0, \omega_-)$ (for some $\omega_- < 0$) through $(q, 0)$ (for some $q \in (\theta, 1)$). This trajectory corresponds to a solution $V(x; a, \omega_+)$ of the equation in (1.2). With a suitable shift of coordinates we can assume that $V(0; a, \omega_+) = V(L; a, \omega_+) = 0$ for some $L = L(a, \omega_+) > 0$. Denote the set of such solutions by

$$\mathcal{V}(a) := \{V(x; a, \omega_+) \mid \omega_+ \in (0, \omega_+^a)\}. \tag{2.3}$$

In case $a = c$ (see Figure 1(b)), $v \equiv 0$ is also a trivial solution of (1.2). Moreover, there is a heteroclinic orbit C_1 connecting the points $(0, 0)$ and $(1, 0)$, which corresponds to a steady-state solution V^* of (1.4). The orbit C_1 coincides with the line $\ell: \ w = av$ in the interval $v \in [0, \theta]$, so we can obtain a continuum of solutions of (1.2): $v(x) = V^*(x - z)$ for any $z \geq 0$.

In case $a > c$ (see Figure 1(c)), $v \equiv 0$ is a trivial solution of (1.2) and there are no other bounded, nonnegative solutions of (1.2) defined on $[0, \infty)$.

3. Asymptotic behaviour of bounded solutions

In this section we consider the asymptotic behaviour of the bounded solutions of (1.1) and prove our main theorem. The conclusions in Theorem 1.1(i) have been proved in [7, Theorem 1.4]. We now consider the other three cases. In the first part we give some common convergence results and in the other three parts we consider the cases $0 < a < c$, $a > c$ and $a = c$, respectively.

3.1. Some common convergence results. First we present a general convergence result (in the $L^\infty_{loc}([0,\infty))$ topology).

Lemma 3.1. *For any $u_0 \in \mathcal{X}$, the problem (1.1) has a time-global solution u which is positive and bounded:*

$$0 < u(t,x) \le M, \quad x > 0, \ t > 0.$$

Moreover, $u(t,x)$ converges as $t \to \infty$ to a solution of (1.2) in the $C^2_{loc}([0,\infty))$ topology.

Proof. By the comparison principle, it is easy to show that

$$0 < u(t,x) \le M := 1 + \|u_0\|_{L^\infty}, \quad x > 0, \ t > 0,$$

since 0 is a lower solution and M is an upper solution of (1.1). The global existence then follows from the standard parabolic theory.

Now we prove the convergence of u as $t \to \infty$ to a steady state. For any time sequence $\{t_n\}$ increasing to ∞, by the parabolic estimate, there are a subsequence of $\{t_n\}$ (denoted again by $\{t_n\}$) and a solution $W(t,x)$ of (1.1) defined over $t \in \mathbb{R}$ such that $u(t+t_n,x) \to W(t,x)$ as $n \to \infty$ in the $C^{1,2}_{loc}(\mathbb{R}\times[0,\infty))$ topology. We first show that $W(t,x)$ is actually a steady state of (1.1). If $W(t_1,x_1) = 0$ for some $t_1 \in \mathbb{R}, x_1 \ge 0$, then $W(t,x) \equiv 0$ by the maximum principle and so it is a steady state. In what follows, we assume that $W(t,x) > 0$ for all $t \in \mathbb{R}, x \ge 0$. Construct a steady state $v(x)$ as follows:

$$\begin{cases} v'' - av' + f(v) = 0, & x > 0,\\ v(0) = W(0,0), & v'(0) = W_x(0,0). \end{cases}$$

Since $W_x(0,0) = aW(0,0) > 0$, by the phase plane analysis in Section 2, we see that there are two possible cases: in the first case, $v(x) > 0$ in some interval $[0,X]$ and $v(X) = 0$ and so $u(t,X) - v(X) > 0$ for all $t > 0$; in the second case, $v(x) > 0$ for $x \ge 0$ and $v(x) > \frac{1}{2}$ for large $x > 0$. (In fact, when the trajectory of v tends to $(1,0)$, we have $v(x) \to 1$ and, when the trajectory of v goes rightward across $v = 1$, we have $v(x) > 1$ for large x.) So, for each $t > 0$, we have $u(t,x) - v(x) < 0$ for all large x, since $u(t,x) \to 0$ as $x \to \infty$. Now we use the zero-number argument (cf. [1, 5–7]) to study the zero-number $\mathcal{Z}(\eta(t,\cdot))$ of the function $\eta(t,x) := u(t,x) - v(x)$ on $[0,X]$ (in the first case) or $[0,\infty)$ (in the second case). Both u and v satisfy the Robin boundary conditions:

$$u_x(t,0) = au(t,0), \quad v_x(0) = W_x(0,0) = aW(0,0) = av(0).$$

Both $\eta(t,0) \ne 0$ and $\eta(t,0) = 0$ are possible. When $\eta(t,0) \ne 0$, we can use the zero-number results in [1] directly to conclude that $\mathcal{Z}(\eta(t,\cdot))$ is nonincreasing in t and it is strictly decreasing at the time moment s when $\eta(s,\cdot)$ has degenerate zeros. As t increases, if $\eta(t,0)$ reaches 0 at some time $t = s$, that is, if $\eta(s,0) = 0$ for some $s \in \mathbb{R}$ and $\eta(t,0) \ne 0$ for $t < s$ with $s - t$ small, the results in [1] cannot be used directly in the time interval $[t,s]$. However, by the Robin boundary condition, $x = 0$ is a degenerate zero of $\eta(s,\cdot)$. So, we can use the same argument as in the proof of [6, Lemma 2.4] to conclude that $\mathcal{Z}(\eta(t,\cdot))$ is also strictly decreasing at the time moment s. In summary, $\mathcal{Z}(\eta(t,\cdot))$ is nonincreasing in t and is strictly decreasing at the time moment when $\eta(t,\cdot)$

has degenerate zeros for both $x \in (0, \infty)$ and at $x = 0$. Consequently, we can follow the ideas in [7] and [5] to conclude that $W(t, x) \equiv v(x)$, that is, the ω-limit of u consists only of steady states. Finally, we prove that $\omega(u)$ is a singleton (this implies that $u(t, \cdot)$ converges to a steady state). Assume, by contradiction, that both $v_1(x), v_2(x) \in \omega(u)$ and $v_1(0) < v_2(0)$. Since $\omega(u)$ is connected, there exists another steady state $v_3(x)$ satisfying $v_1(0) < v_3(0) < v_2(0)$. Using an argument similar to that in [12], one can derive a contradiction. This completes the proof of the lemma. □

Next, we present a sufficient condition which guarantees the convergence of u to 0 even in the $L^\infty([0, \infty))$ topology.

Lemma 3.2. *Let u be the solution of (1.1). If $u(t_0, x) \le \theta$ for some $t_0 > 0$, then $\|u(t, \cdot)\|_{L^\infty([0,\infty))} \to 0$ as $t \to \infty$.*

Proof. Assume that $M := 1 + \|u_0\|_\infty$ and $L > 0$ is a number satisfying $\mathrm{spt}(u_0) \subset [0, L]$. By the comparison principle,

$$0 \le u(t, x) \le M \text{ for } x \ge 0, \quad t \in [0, t_0]; \quad 0 < u(t, x) < \theta \text{ for } x \ge 0, \quad t > t_0.$$

Assume further that $f(u) \le k_1 u$ for all $u \ge 0$ and some $k_1 > 0$. Set

$$k(t) := \begin{cases} k_1, & t \in [0, t_0], \\ \text{a positive smooth function}, & t \in [t_0, 2t_0], \\ 0, & t \ge 2t_0 \end{cases} \quad \text{and} \quad K(t) := \int_0^t k(s)\, ds.$$

Then $K(t) \le K^* := \int_0^{2t_0} k(s)\, ds$. We now consider the following problem:

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} - a\tilde{u}_x + k(t)\tilde{u}, & x > 0,\ t > 0, \\ \tilde{u}_x(t, 0) = a\tilde{u}(t, 0), & t > 0, \\ \tilde{u}(0, x) = u_0(x), & x \ge 0. \end{cases} \tag{3.1}$$

Since

$$f(u(t, x)) \begin{cases} \le k_1 u(t, x) = k(t)u(t, x), & x \ge 0,\ t \in [0, t_0], \\ = 0 \le k(t)u(t, x), & x \ge 0,\ t \ge t_0, \end{cases}$$

we see that u is a lower solution of (3.1) and so $u(t, x) \le \tilde{u}(t, x)$ for $x \ge 0, t > 0$.

Set $w(t, x) := \tilde{u}e^{-(1/2)ax}e^{(1/4)a^2 t - K(t)}$; then w solves

$$\begin{cases} w_t = w_{xx}, & x > 0,\ t > 0, \\ w_x(t, 0) = \frac{1}{2}aw(t, 0), & t > 0, \\ w(0, x) = u_0(x)e^{-(1/2)ax} \ge 0, & x \ge 0. \end{cases}$$

Furthermore, we assume that $\hat{w}$ is the solution of

$$\begin{cases} \hat{w}_t = \hat{w}_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ \hat{w}(0, x) = w(0, x) \le \begin{cases} M, & |x| \le L, \\ 0, & |x| > L. \end{cases} \end{cases}$$

Then $\hat{w}$ is expressed explicitly as

$$\hat{w}(t,x) = \frac{1}{2\sqrt{\pi t}}\int_0^\infty [e^{-(x-y)^2/4t} + e^{-(x+y)^2/4t}]w(0,y)\,dy \le \frac{M}{\sqrt{\pi t}}\int_0^L e^{-(x-y)^2/4t}\,dy.$$

By comparison, we have $w(t,x) \le \hat{w}(t,x)$ and so

$$\begin{aligned} u(t,x) &\le e^{(1/2)ax+K(t)-(1/4)a^2t}\hat{w}(t,x) \le e^{(1/2)ax+K^*-(1/4)a^2t}\frac{M}{\sqrt{\pi t}}\int_0^L e^{-(x-y)^2/4t}\,dy \\ &\le \overline{M}(t,x) := \begin{cases} \frac{C}{\sqrt{t}}e^{-(1/4)a^2t}, & 0 \le x \le L, \quad t > 0, \\ \frac{C}{\sqrt{t}}e^{(1/2)ax-(1/4t)(x-L)^2-(1/4)a^2t}, & x > L, \ t > 0 \end{cases} \end{aligned}$$

for some C depending on a, M, K^*, L. Since

$$\frac{ax}{2} - \frac{(x-L)^2}{4t} - \frac{a^2t}{4} \le \frac{ax}{2} - \frac{a(x-L)}{2} = \frac{aL}{2},$$

we have, for some constant C_1 independent of t and x,

$$u(t,x) \le \overline{M}(t,x) \le \frac{C_1}{\sqrt{t}} \to 0 \quad \text{as } t \to \infty.$$

This proves the lemma. □

3.2. The case $0 < a < c$. In this case the problem (1.2) has only two solutions: 0 and $V^0(x+z)$. So, we have a dichotomy result by Lemma 3.1: any solution converges as $t \to \infty$ to either 0 or $V^0(x+z)$. For any given $\phi \in \mathcal{X}$, the set $(0,\infty)$ of the parameter σ is divided into three subsets:

$$\begin{cases} \Sigma_0 := \{\sigma > 0 \mid \|u_\sigma(t,\cdot)\|_{L^\infty([0,\infty))} \to 0 \text{ as } t \to \infty\}, \\ \Sigma_* := \{\sigma > 0 \mid u_\sigma(t,\cdot) \to 0 \text{ as } t \to \infty \text{ in } L^\infty_{loc}([0,\infty)) \\ \qquad\qquad \text{but not in the } L^\infty([0,\infty)) \text{ topology}\}, \\ \Sigma_1 := \{\sigma > 0 \mid u_\sigma(t,\cdot) \to V^0(\cdot + z) \text{ as } t \to \infty \text{ in } L^\infty_{loc}([0,\infty))\}. \end{cases} \tag{3.2}$$

Claim 1: Σ_0 is a nonempty open interval. In fact, Σ_0 consists of all small positive σ, since $0 \le \sigma\phi \le \theta$ for any small σ, and so $\|u_\sigma(t,\cdot)\|_{L^\infty([0,\infty))} \to 0$ by Lemma 3.2. Next, we show that Σ_0 is an open set. Assume that $\sigma_1 \in \Sigma_0$; then there exists a time t_1 such that $u_{\sigma_1}(t_1,x) < \theta/2$ for $x \ge 0$ (by Lemma 3.2). Since $u_\sigma(t_1,x)$ depends continuously on σ in the L^∞ topology, we see that

$$u_{\sigma_1+\epsilon}(t_1,x) \le \theta, \quad x \ge 0,$$

for any ϵ provided $|\epsilon|$ is small. This means that $\sigma_1 + \epsilon \in \Sigma_0$ and so Σ_0 is an open set. Finally, Σ_0 is an interval by the comparison principle.

Claim 2: If Σ_1 is not empty, then it is an open interval. In fact, for any $\sigma_2 \in \Sigma_1$, since $u_{\sigma_2}(t,\cdot) \to V^0(\cdot + z)$, there exists a time t_2 such that

$$u_{\sigma_2}(t_2,x) > V(x-x_0;a,\omega_+), \quad x \in [x_0, x_0 + L(a,\omega_+)],$$

for some V taken from $\mathcal{V}(a)$ in (2.3) and any large x_0. By the continuous dependence of $u_\sigma(t_2, x)$ on σ and by the comparison principle,

$$u_\sigma(t_2, x) > V(x - x_0; a, \omega_+), \quad x \in [x_0, x_0 + L(a, \omega_+)],$$

provided $\sigma > \sigma_2$ or $\sigma < \sigma_2$ with $|\sigma - \sigma_2|$ small. Hence, $u_\sigma(t, x) > V(x - x_0; a, \omega_+)$ for all $t > t_2$ and so $\sigma \in \Sigma_1$ by the dichotomy result. This implies that Σ_1 is an open interval of the form (σ^*, ∞). Note that $\Sigma_1 \neq \emptyset$ if $\phi(x) > 0$ in some interval $[x_0, x_0 + L(a, \omega_+)]$, since in this case $\sigma\phi > V(x - x_0; a; \omega_+)$ in this interval provided σ is sufficiently large.

If $\Sigma_1 = \emptyset$, then either $\Sigma_0 = (0, \infty)$ or $\Sigma_0 = (0, \sigma_*)$ and $\Sigma_* = [\sigma_*, \infty)$ for some $\sigma_* > 0$. In both cases, this completes the proof of Theorem 1.1(ii). If $\Sigma_1 \neq \emptyset$, then by Claims 1 and 2, there exist $\sigma^* \geq \sigma_* > 0$ such that $\Sigma_0 = (0, \sigma_*)$, $\Sigma_1 = (\sigma^*, \infty)$ and $\Sigma_* = [\sigma_*, \sigma^*]$. Again, this completes the proof of Theorem 1.1(ii).

3.3. The case $a > c$. In this case, the only bounded solution of (1.2) is 0, so any solution u of (1.1) converges to 0 in the $L^\infty_{loc}([0, \infty))$ topology. Consequently, for any given $\phi \in \mathcal{X}$, the set $(0, \infty)$ of the parameter σ is divided into two parts: Σ_0 and Σ_*, defined as in (3.2). The above Claim 1 remains valid in the present case: Σ_0 is a nonempty open interval. If $\Sigma_0 = (0, \infty)$, then there is nothing left to prove. If $\Sigma_0 = (0, \sigma_*)$ for some $\sigma_* > 0$, then $\Sigma_* = [\sigma_*, \infty)$ and the proof for Theorem 1.1(iii) is then completed. The following claim shows that $\Sigma_* \neq \emptyset$ is possible for some $\phi \in \mathcal{X}$.

Claim 3: If $\phi(x) > 0$ in $[x_1, x_2]$ and if $x_2 - x_1$ is sufficiently large, then $\Sigma_* \neq \emptyset$. If fact, for any given $s \in (a - c, a]$, we have $a - s \in [0, c)$. By the analysis in Section 2 (for the case $0 < a < c$), the equation $v'' - (a - s)v' + f(v) = 0$ has a family of compactly supported solutions $V(x; a - s, \omega_+) \in \mathcal{V}(a - s)$, with $V(x; a - s, \omega_+)$ defined on some interval $I := [0, L(a - s, \omega_+)]$. For any such function and any $x_0 \geq 0$, we see that $\underline{u}(t, x) := V(x - st - x_0; a - s, \omega_+)$ is a lower solution of (1.1) and so

$$u(t, x) \geq \underline{u}(t, x), \quad x \in [st + x_0, st + x_0 + L(a - s, \omega_+)],\ t > 0,$$

provided this inequality holds at $t = 0$. Such a possibility can occur if $x_1 < x_0$, $x_2 - x_1 > L(a - s; \omega_+)$ and $\sigma > 0$ is large. Therefore, any such σ is in Σ_*.

3.4. The case $a = c$. This is the critical case and the problem (1.2) has a continuum of solutions: $V^*(x - z)$ for $z \geq 0$ as well as 0. For any given $\phi \in \mathcal{X}$, we define $\Sigma_0, \Sigma_*, \Sigma_1$ as in (3.2), with $V^0(x + z)$ replaced by $V^*(x - z)$ for some $z > 0$. Then, by Lemma 3.2, the set Σ_0 is an open interval. As above, there is nothing left to prove if $\Sigma_0 = (0, \infty)$. We now consider the case $\Sigma_0 = (0, \sigma_*)$ for some $\sigma_* > 0$, that is, $\Sigma_* \cup \Sigma_1 = [\sigma_*, \infty)$. This is possible for some $\phi \in \mathcal{X}$. In fact, the above Claim 3 remains valid in the current case, so, for some $\phi \in \mathcal{X}$, u_σ does not converge to 0 in the L^∞ topology (that is, $\sigma \notin \Sigma_0$) when σ is large. However, whether $\Sigma_* = \emptyset$ or not is still an open problem.

Acknowledgement

The authors would like to thank Professor Bendong Lou for helpful discussions.

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https://doi.org/10.1017/S0004972718000370 Published online by Cambridge University Press