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EXACT NEUMANN BOUNDARY CONTROLLABILITY FOR PROBLEMS OF TRANSMISSION OF THE WAVE EQUATION

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Abstract. Using the Hilbert Uniqueness Method, we study the problem of exact controllability in Neumann boundary conditions for problems of transmission of the wave equation. We prove that this system is exactly controllable for all initial states in $L^2(\Omega)\times(H^1(\Omega))'$.

1. Introduction. Throughout this paper, let Ω be a bounded domain (open, connected, and nonempty) in $\mathrm{R}^n (n\geq 1)$ with a boundary $\Gamma=\partial\,\Omega$ of class C^2, and Ω_1 given with $\bar{\Omega}_1\subset\Omega$ and $\Gamma_1=\partial\,\Omega_1$ of class C^2. Let $T>0$. Set $\Omega_2=\Omega-\Omega_1$, $Q=\Omega\times(0,T)$, $Q_1=\Omega_1\times(0,T)$, $Q_2=\Omega_2\times(0,T)$, $\Sigma=\Gamma\times(0,T)$, $\Sigma_1=\Gamma_1\times(0,T)$.

In [**6**], Lions studied the problem of exact controllability with Dirichlet boundary conditions for problems of transmission of the wave equation by introducing the Hilbert Uniqueness Method (HUM for short). Later, Nicaise [**10–12**] further considered this problem in R^2 with singularities.

In this paper, we consider the following Neumann boundary controllability problem in R^n: *For suitable times $T>0$ and every initial condition $\{y^0, y^1\}$, does there exist a control function g such that the solution $y=y(x,t;g)$ of the Neumann boundary value problem*

$$\begin{cases} y''-A(x)\Delta y=0 & \text{in } Q,\\ y(x,0)=y^0(x),\ y'(x,0)=y^1(x) & \text{in } \Omega,\\ \frac{\partial y_2}{\partial \nu}=g & \text{on } \Sigma,\\ y_1=y_2,\quad a_1=\frac{\partial y_1}{\partial\nu}=a_2\frac{\partial y_2}{\partial\nu} & \text{on } \Sigma_1,\end{cases}\tag{1.1}$$

satisfies

$$y(x,T;g)=y'(x,T;g)=0 \qquad \text{in } \Omega\ ?\tag{1.2}$$

In (1.1), $y_1=y|_{\Omega_1}$, $y_2=y|_{\Omega_2}$, ν is the unit normal of Γ or Γ_1 pointing towards the exterior of Ω or Ω_1, and $A(x)$ is given by

$$A(x)=\begin{cases} a_1, & x\in\Omega_1,\\ a_2, & x\in\Omega_2,\end{cases}$$

where a_1, and a_2 are positive constants.

We will prove that if Ω_1 is star-shaped and $a_2\leq a_1$, then for all initial states

$$\{y^0,y^1\}\in L^2(\Omega)\times(H^1(\Omega))',$$

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there exists a control function g such that the solution $y=y(x,t;g)$ of (1.1) satisfies (1.2). Here and in the sequel, $H^s(\Omega)$ always denotes the usual Sobolev space for $s\in R$.

The plan for the rest of this paper is as follows. In Section 2, we present the theorem about the existence and uniqueness of solutions of the problem of transmission. The estimates for the solutions (i.e., the so-called "inverse inequality") are given in Section 3. The main theorems of this paper are established in Section 4.

2. Homogeneous boundary problems. Consider the following homogeneous boundary problem

$$\begin{cases} u'' - A(x)\Delta u = f & \text{in } Q, \\ u(x,0) = u^0(x),\ u'(x,0) = u^1(x) & \text{in } \Omega, \\ \frac{\partial u_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ u_1 = u_2,\ a_1 = \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} & \text{on } \Sigma_1, \end{cases} \tag{2.1}$$

where $u_1=u|_{\Omega_1}$ and $u_2=u|_{\Omega_2}$. Set

$$H^2(\Omega_1,\Omega_2) = \{u : u \in H^1(\Omega);\, u_i = u|_{\Omega_i} \in H^2(\Omega_i),\, i=1,2; \\ a_1\frac{\partial u_1}{\partial \nu} = a_2\frac{\partial u_2}{\partial \nu} \text{ on } \Gamma_1; \frac{\partial u_2}{\partial \nu} = 0 \text{ on } \Gamma\} \tag{2.2}$$

with the norm

$$\| u \|_{H^2(\Omega_1,\Omega_2)} = [\, \| u \|^2_{H^1(\Omega)} + \| \Delta u_1 \|^2_{L^2(\Omega_1)} + \| \Delta u_2 \|^2_{L^2(\Omega_2)}]^{1/2}. \tag{2.3}$$

The well-posedness of (2.1) is by now well known ([**3**], Vol.5, Chap. XVIII] and [**4**]). We have the following result.

THEOREM 2.1. *(i) Suppose* Γ *and* Γ_1 *are Lipschitz. Then, for any initial condition* $(u^0,u^1)\in H^1(\Omega)\times L^2(\Omega)$ *and* $f\in L^1(0,T;L^2(\Omega))$, *problem (2.1) has a unique weak solution u with*

$$u \in C([0,T];H^1(\Omega)) \cap C^1([0,T];L^2(\Omega)). \tag{2.4}$$

Moreover, there exists a constant $C>0$ *such that for every* $t\in[0,T]$

$$\| u(t) \|_{H^1(\Omega)} + \| u'(t) \|_{L^2(\Omega)} \le C\,[\, \| u^0 \|_{H^1(\Omega)} + \| u^1 \|_{L^2(\Omega)} + \| f \|_{L^1(0,T;L^2(\Omega))}]. \tag{2.5}$$

(ii) *Suppose* Γ *and* Γ_1 *are of class* C^2. *Then for any initial condition* $(u^0,u^1)\in H^2(\Omega_1,\Omega_2)\times H^1(\Omega)$ *and* $f\in L^1(0,T;H^1(\Omega))$, *problem (2.1) has a unique strong solution u with*

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$$u \in C([0, T]; H^2(\Omega_1, \Omega_2)) \cap C^1([0, T]; H^1(\Omega)). \tag{2.6}$$

Moreover, there exists a constant $C>0$ such that for every $t\in[0,T]$

$$\begin{aligned} &\| u'(t) \|_{H^1(\Omega)} + \| u(t) \|_{H^2(\Omega_1,\Omega_2)} \\ &\leq C [\| u^1 \|_{H^1(\Omega)} + \| u^0 \|_{H^2(\Omega_1,\Omega_2)} + \| f \|_{L^1(0,T;H^1(\Omega))}]. \end{aligned} \tag{2.7}$$

3. Basic inequalities. We adopt the notation used in [**6**,**7**] as follows. Let $x^0\in R^n$, and set

$$m(x) = x - x^0 = (x_k - x_k^0).$$

$$\Gamma(x^0) = \{x \in \Gamma : m(x) \cdot \nu(x) = m_k(x) \cdot \nu_k(x) > 0\}$$

$$\Gamma_*(x^0) = \Gamma - \Gamma(x^0) = \{x \in \Gamma : m(x) \cdot \nu(x) \leq 0\}$$

$$\Sigma(x^0) = \Gamma(x^0) \times (0, T)$$

$$\Sigma_*(x^0) = \Gamma_*(x^0) \times (0, T)$$

$$R(x^0) = \max_{x\in\bar{\Omega}}|m(x)| = \max_{x\in\bar{\Omega}}|\Sigma_{k=1}^n (x_k - x_k^0)^2|^{\frac{1}{2}}.$$

where ν denotes the outward unit normal to Γ.

We define the energy of the solution u of (2.1) by

$$E(t) = \frac{1}{2}\int_\Omega [\, |u'(x,t)|^2 + A(x)|\nabla u|^2 \,]\, dx,$$

If $f=0$, then we have the classical result (see [**6**,**9**])

$$E(t) \equiv E(0).$$

The following identities are essential for establishing the follow-up inverse inequalities.

LEMMA 3.1. *Let $q=(q_k)$ a vector field in $[C^1(\bar{\Omega})]^n$. Suppose u is the strong solution of (2.1) in the sense of (ii) of Theorem 2.1. Then the following identity holds*:

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$$
\begin{aligned}
&\frac{1}{2}\int_{\Sigma} q_k \nu_k\big(|u'_2|^2 - a_2|\nabla_\sigma u_2|^2\big) d\Sigma \\
&= \left(u'(t), q_k \frac{\partial u(t)}{\partial x_k}\right)\Big|_0^T + \int_Q A(x) \frac{\partial u}{\partial x_j}\frac{\partial q_k}{\partial x_i}\frac{\partial u}{\partial x_k} dxdt \\
&+ \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\big(|u'|^2 - A(x)|\nabla u|^2\big) dxdt \\
&- a_1\left(1 - \frac{a_1}{a_2}\right)\int_{\Sigma_1} q_k \nu_k \left|\frac{\partial u_1}{\partial \nu}\right|^2 d\Sigma \\
&- \frac{1}{2}\int_{\Sigma_1} q_k \nu_k (a_2|\nabla u_2|^2 - a_1|\nabla u_1|^2) d\Sigma - \int_Q q_k \frac{\partial u}{\partial x_k} f dxdt,
\end{aligned}
\tag{3.1}
$$

where

$$
\left(u'(t), q_k \frac{\partial u(t)}{\partial x_k}\right) = \int_\Omega u'(t) q_k \frac{\partial u(t)}{\partial x_k} dx,
$$

and $\nabla_\sigma u = \{\sigma_j u\}_{j=1}^n$ *denotes the tangential gradient of* u *on* Γ. (See [**6**, p.137].)

REMARK 3.2. If $n = 1$, then (3.1) becomes

$$
\begin{aligned}
&\frac{1}{2}\int_{\Sigma} q\nu |u'_2|^2 d\Sigma \\
&= \left(u'(t), q_k \frac{\partial u(t)}{\partial x}\right)\Big|_0^T + \int_Q A(x)\left|\frac{\partial u}{\partial x}\right|^2 \frac{\partial q}{\partial x} dxdt \\
&+ \frac{1}{2}\int_Q \frac{\partial q}{\partial x}\left(|u'|^2 - A(x)\left|\frac{\partial u}{\partial x}\right|^2\right) dxdt - a_1\left(1 - \frac{a_1}{a_2}\right)\int_{\Sigma_1} q\nu \left|\frac{\partial u_1}{\partial \nu}\right|^2 d\Sigma \\
&- \frac{1}{2}\int_{\Sigma_1} q\nu (a_2|\nabla u_2|^2 - a_1|\nabla u_1|^2) d\Sigma - \int_Q q \frac{\partial u}{\partial x} f dxdt.
\end{aligned}
\tag{3.1'}
$$

This is a generalisation of the identity in Remark 1.5 of [**6**].

Proof. Multiplying (2.1) by $q_k \frac{\partial u}{\partial x_k}$ and integrating on Q, we have

$$
\int_Q q_k \frac{\partial u}{\partial x_k} u'' dxdt - \int_Q q_k \frac{\partial u}{\partial x_k} A(x)\Delta u dxdt = \int_Q q_k \frac{\partial u}{\partial x_k} f dxdt. \tag{3.2}
$$

Integrating by parts, we obtain

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$$\begin{aligned}
&\int_Q q_k \frac{\partial u}{\partial_k} u'' dxdt \\
&= \left(u'(t), q_k \frac{\partial u(t)}{\partial x_k}\right)\Big|_0^T - \frac{1}{2}\int_{\Sigma_1} q_k \nu_k |u_1'|^2 d\Sigma + \frac{1}{2}\int_{Q_1} \frac{\partial q_k}{\partial x_k} |u_1'|^2 dxdt \\
&+ \frac{1}{2}\int_{\Sigma_1} q_k \nu_k |u_2'|^2 d\Sigma - \frac{1}{2}\int_{\Sigma} q_k \nu_k |u_2'|^2 d\Sigma + \frac{1}{2}\int_{Q_2} \frac{\partial q_k}{\partial x_k} |u_2'|^2 dxdt \\
&= \left(u'(t), q_k \frac{\partial u(t)}{\partial x_k}\right)\Big|_0^T - \frac{1}{2}\int_{\Sigma} q_k \nu_k |u_2'|^2 d\Sigma + \frac{1}{2}\int_{Q} \frac{\partial q_k}{\partial x_k} |u'|^2 dxdt,
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\int_Q A(x) q_k \frac{\partial u}{\partial x_k} \Delta u dxdt \\
&= \int_{\Sigma_1} a_1 \frac{\partial u_1}{\partial \nu} q_k \frac{\partial u_1}{\partial x_k} d\Sigma - \int_{Q_1} a_1 \frac{\partial u_1}{\partial x_j} \frac{\partial}{\partial x_j}\left(q_k \frac{\partial u_1}{\partial x_k}\right) dxdt \\
&- \int_{\Sigma_1} a_2 \frac{\partial u_2}{\partial \nu} q_k \frac{\partial u_2}{\partial x_k} d\Sigma - \int_{Q_2} a_2 \frac{\partial u_2}{\partial x_j} \frac{\partial}{\partial x_j}\left(q_k \frac{\partial u_2}{\partial x_k}\right) dxdt.
\end{aligned} \tag{3.4}$$

But,

$$\begin{aligned}
&\int_{Q_1} a_1 q_k \frac{\partial u_1}{\partial x_j} \frac{\partial^2 u_1}{\partial x_k \partial x_j} dxdt \\
&= \frac{1}{2}\int_{Q_1} a_1 q_k \frac{\partial}{\partial x_k} |\nabla u_1|^2 dxdt \\
&= \frac{1}{2}\int_{\Sigma_1} a_1 q_k \nu_k |\nabla u_1|^2 d\Sigma - \frac{1}{2}\int_{Q_1} a_1 |\nabla u_1|^2 \frac{\partial q_k}{\partial x_k} dxdt,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\int_{Q_2} a_2 q_k \frac{\partial u_2}{\partial x_j} \frac{\partial^2 u_2}{\partial x_k \partial x_i} dxdt \\
&= \frac{1}{2}\int_{Q_2} a_2 q_k \frac{\partial}{\partial x_k} |\nabla u_2|^2 dxdt \\
&= -\frac{1}{2}\int_{\Sigma_1} a_2 q_k \nu_k |\nabla u_2|^2 d\Sigma + \frac{1}{2}\int_{\Sigma} a_2 q_k \nu_k |\nabla u_2|^2 d\Sigma \\
&- \frac{1}{2}\int_{Q_1} a_2 |\nabla u_2|^2 \frac{\partial q_k}{\partial x_k} dxdt.
\end{aligned} \tag{3.6}$$

Noting that $|\nabla u_2|^2 = |\nabla_\sigma u_2|^2$ on Σ, it follows from (3.4), (3.5), and (3.6) that

$$
\begin{aligned}
&\int_Q A(x) q_k \frac{\partial u}{\partial x_k} \Delta u dx dt \\
&= \int_{\Sigma_1} a_1 \frac{\partial u_1}{\partial \nu} q_k \left(\frac{\partial u_1}{\partial x_k} - \frac{\partial u_2}{\partial x_k} \right) d\Sigma - \int_Q A(x) \frac{\partial u}{\partial x_j} \frac{\partial q_k}{\partial x_j} \frac{\partial u}{\partial x_k} dx dt \\
&+ \frac{1}{2} \int_{\Sigma_1} q_k \nu_k (a_2 |\nabla u_2|^2 - a_1 |\nabla u_1|^2) d\Sigma \\
&- \frac{1}{2} \int_{\Sigma} a_2 q_k \nu_k |\nabla_\sigma u_2|^2 d\Sigma + \frac{1}{2} \int_Q A(x) |\nabla u|^2 \frac{\partial q_k}{\partial x_k} dx dt.
\end{aligned}
\tag{3.7}
$$

Since

$$
a_1 \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} \quad \text{and} \quad \sigma_k u_1 = \sigma_k u_2 \quad \text{on} \quad \Sigma_1,
$$

and

$$
\frac{\partial u_1}{\partial x_k} = \nu_k \frac{\partial u_1}{\partial \nu} + \sigma_k u_1, \quad \frac{\partial u_2}{\partial x_k} = \nu_k \frac{\partial u_2}{\partial \nu} + \sigma_k u_2,
$$

it follows from (3.2), (3.3), and (3.7) that

$$
\begin{aligned}
&\int_Q f q_k \frac{\partial u}{\partial x_k} dx dt \\
&= \left(u'(t), q_k \frac{\partial u(t)}{\partial x_k} \right) \Big|_0^T - \frac{1}{2} \int_\Sigma q_k \nu_k |u_2'|^2 d\Sigma + \frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} |u'|^2 dx dt \\
&- \int_{\Sigma_1} a_1 \left(1 - \frac{a_1}{a_2} \right) q_k \nu_k \left| \frac{\partial u_1}{\partial \nu} \right|^2 d\Sigma + \int_Q A(x) \frac{\partial u}{\partial x_j} \frac{\partial q_k}{\partial x_j} \frac{\partial u}{\partial x_k} dx dt \\
&- \frac{1}{2} \int_{\Sigma_1} q_k \nu_k (a_2 |\nabla u_2|^2 - a_1 |\nabla u_1|^2) + \frac{1}{2} \int_\Sigma a_2 q_k \nu_k |\nabla_\sigma u_2|^2 d\Sigma \\
&- \frac{1}{2} \int_Q A(x) |\nabla u|^2 \frac{\partial q_k}{\partial x_k} dx dt.
\end{aligned}
$$

This is (3.1).

LEMMA 3.3. *Suppose there exists* $x^0 \in \Omega_1$ *such that* $m(x) \cdot \nu(x) \geq 0$ *on* Γ_1 *where* ν *is directed towards the exterior of* Ω_1. *Assume* $a_2 \leq a_1$ *and* $T > \frac{2R(x^0)}{\sqrt{a_2}}$. *Then for all weak solutions* u *of (2.1) with initial conditions* $(u^0, u^1) \in H^1(\Omega) \times L^2(\Omega)$ *and* $f = 0$, *there exists* $C(T) > 0$ *such that*

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$$\int_{\Sigma} m_k \nu_k \left(|u_2'|^2 - a_2 |\nabla_\sigma u_2|^2 \right) d\Sigma + \int_{\Gamma} m_k \nu_k \left(|u_2(0)|^2 + |u_2(T)|^2 \right) d\Gamma \\ \geq C(T) \left(\| u^0 \|^2_{H^1(\Omega)} + \| u^1 \|^2_{L^2(\Omega)} \right). \tag{3.8}$$

In the case $n = 1$, the term $a_2 |\nabla_\sigma u_2|^2$ on the left-hand side of (3.8) disappears.

Remark 3.4. If Ω_1 is star-shaped (see [**14**], p.294), then the condition on Ω_1 in the lemma is fulfilled.

Proof. We prove the lemma only in the case of $n > 1$. We omit the proof in the case of $n = 1$ because it is just a combination of the following proof with Lemma 1.4 of ([**6**], chap. 3, p. 142).

Taking $q_k = m_k$ in Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{2} \int_{\Sigma} m_k \nu_k \left(|u_2'|^2 - a_2 |\nabla_\sigma u_2|^2 \right) d\Sigma \\
&= \left(u'(t), m_k \frac{\partial u(t)}{\partial x_k} \right) \Big|_0^T + \int_Q A(x) |\nabla u|^2 dx dt \\
&+ \frac{n}{2} \int_Q \left(|u'|^2 - A(x) |\nabla u|^2 \right) dx dt - a_1 \left(1 - \frac{a_1}{a_2} \right) \int_{\Sigma_1} m_k \nu_k \left| \frac{\partial u_1}{\partial \nu} \right|^2 d\Sigma \\
&- \frac{1}{2} \int_{\Sigma_1} m_k \nu_k \left(a_2 |\nabla u_2|^2 - a_1 |\nabla u_1|^2 \right) d\Sigma \\
&= \left(u'(t), m_k \frac{\partial u(t)}{\partial x_k} \right) \Big|_0^T + \frac{n-1}{2} \int_Q \left(|u'|^2 - A(x) |\nabla u|^2 \right) dx dt \\
&+ \int_0^T E(t) dt - a_1 \left(1 - \frac{a_1}{a_2} \right) \int_{\Sigma_1} m_k \nu_k \left| \frac{\partial u_1}{\partial \nu} \right|^2 d\Sigma \\
&- \frac{1}{2} \int_{\Sigma_1} m_k \nu_k \left(a_2 |\nabla u_2|^2 - a_1 |\nabla u_1|^2 \right) d\Sigma.
\end{aligned} \tag{3.9}$$

Multiplying (2.1) by u and integrating over Q, we obtain

$$\begin{aligned}
0 = (u', u)\Big|_0^T - \int_Q |u'|^2 - \int_{\Sigma_1} a_1 \frac{\partial u_1}{\partial \nu} u_1 d\Sigma + \int_{Q_1} a_1 |\nabla u_1|^2 dx dt \\
+ \int_{\Sigma_1} a_2 \frac{\partial u_2}{\partial \nu} u_2 d\Sigma + \int_{Q_2} a_2 |\nabla u_2|^2 dx dt.
\end{aligned}$$

The transmission conditions give

$$(u'(t), u(t))|_0^T = \int_Q \left(| u'|^2 - A(x)| \nabla u |^2\right) dxdt.$$

Therefore, (3.9) becomes

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma m_k \nu_k\left(| u_2'|^2 - a_2| \nabla_\sigma u_2|^2\right) d\Sigma \\
&= \left(u'(t), m_k \frac{\partial u(t)}{\partial x_k} + \frac{n-1}{2} u(t)\right)\Big|_0^T + TE(0) \\
&- a_1\left(1 - \frac{a_1}{a_2}\right) \int_{\Sigma_1} m_k \nu_k \left| \frac{\partial u_1}{\partial \nu}\right|^2 d\Sigma \\
&- \frac{1}{2}\int_{\Sigma_1} m_k \nu_k (a_2| \nabla u_2|^2 - a_1| \nabla u_1|^2) d\Sigma.
\end{aligned} \tag{3.10}$$

To prove (3.8), we have to estimate the right hand of (3.10). First, from the Cauchy-Schwarz's inequality we have

$$\begin{aligned}
&\left|\left(u'(t), m_k \frac{\partial u(t)}{\partial x_k} + \frac{n-1}{2} u(t)\right)\right| \\
&\leq \frac{R(x^0)}{2\sqrt{a_2}} \int_\Omega | u'(t)|^2 dx + \frac{a_2}{2R(x^0)\sqrt{a_2}} \int_\Omega \left| m_k \frac{\partial u(t)}{\partial x_k} + \frac{n-1}{2} u(t) \right|^2 dx.
\end{aligned} \tag{3.11}$$

Moreover,

$$\begin{aligned}
&\int_\Omega \left| m_k \frac{\partial u(t)}{\partial x_k} + \frac{n-1}{2} u(t) \right|^2 dx \\
&= \int_\Omega \left| m_k \frac{\partial u}{\partial x_k}\right|^2 dx + \frac{(n-1)^2}{4} \int_\Omega | u(t)|^2 dx + (n-1)\left(m_k \frac{\partial u}{\partial x_k}, u(t)\right).
\end{aligned}$$

Since

$$\begin{aligned}
\left(m_k \frac{\partial u}{\partial x_k}, u(t)\right) &= \frac{1}{2}\int_\Omega m_k \frac{\partial}{\partial x_k}(| u(t)|^2 dx \\
&= \frac{1}{2}\int_{\Gamma_1} m_k \nu_k | u_1(t)|^2 d\Gamma - \frac{n}{2}\int_{\Omega_1} | u_1(t)|^2 dx \\
&- \frac{1}{2}\int_{\Gamma_1} m_k \nu_k | u_2(t)|^2 d\Gamma + \frac{1}{2}\int_\Gamma m_k \nu_k | u_2(t)|^2 d\Gamma \\
&- \frac{n}{2}\int_{\Omega_2} | u_2(t)|^2 dx \\
&= \frac{1}{2}\int_\Gamma m_k \nu_k | u_2(t)|^2 d\Gamma - \frac{n}{2}\int_\Omega | u(t)|^2 dx,
\end{aligned}$$

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then,

$$\begin{aligned}
&\int_\Omega |\, m_k \frac{\partial u(t)}{\partial x_k} + \frac{n-1}{2} u(t)|^2 dx \\
&= \int_\Omega |\, m_k \frac{\partial u(t)}{\partial x_k} |^2 dx + \frac{1-n^2}{4} \int_\Omega |\, u(t)|^2 dx \\
&+ \frac{n-1}{2} \int_\Gamma m_k \nu_k |\, u_2(t)|^2 d\Gamma \\
&\le R_0^2 \int_\Omega |\, \nabla u(t)|^2 dx + \frac{1-n^2}{4} \int_\Omega |\, u(t)|^2 dx \\
&+ \frac{n-1}{2} \int_\Gamma m_k \nu_k |\, u(t)|^2 d\Gamma.
\end{aligned}$$

Thus, (3.11) becomes

$$\begin{aligned}
&\left| \left(u'(t), m_k \frac{\partial u(t)}{\partial x_k} + \frac{n-1}{2} u(t) \right) \right| \\
&\le \frac{R(x^0)}{\sqrt{a_2}} E(t) + \frac{\sqrt{a_2}(1-n^2)}{8R(x^0)} \int_\Omega |\, u(t)|^2 dx \\
&+ \frac{\sqrt{a_2}(n-1)}{4R(x^0)} \int_\Gamma m_k \nu_k |\, u_2(t)|^2 d\Gamma.
\end{aligned}$$

Secondly, we estimate the last two terms of (3.10). Since

$$|\, \nabla u_1|^2 = \left| \frac{\partial u_1}{\partial \nu} \right|^2 + |\, \nabla_\sigma u_1|^2, \quad |\, \nabla u_2|^2 = |\frac{\partial u_2}{\partial \nu}|^2 + |\, \nabla_\sigma u_2|^2,$$

and $\nabla_\sigma u_1 = \nabla_\sigma u_2$ on Σ_1, we deduce that

$$\begin{aligned}
&- a_1 \left(1 - \frac{a_1}{a_2} \right) \int_{\Sigma_1} m_k \nu_k |\frac{\partial u_1}{\partial \nu}|^2 d\Sigma - \frac{1}{2} \int_{\Sigma_1} m_k \nu_k (a_2 |\, \nabla u_2|^2 - a_1 |\, \nabla u_1|^2) \\
&= -a_1 \left(1 - \frac{a_1}{a_2} \right) \int_{\Sigma_1} m_k \nu_k |\frac{\partial u_1}{\partial \nu}|^2 d\Sigma \\
&- \frac{1}{2} \int_{\Sigma_1} m_k \nu_k \left[\left(\frac{a_1^2}{a_2} - a_1 \right) |\frac{\partial u_1}{\partial \nu}|^2 + (a_2 - a_1) |\, \nabla_\sigma u_1|^2 \right] d\Sigma \\
&= \frac{a_1(a_1 - a_2)}{2a_2} \int_{\Sigma_1} m_k \nu_k |\frac{\partial u_1}{\partial \nu}|^2 d\Sigma + \frac{a_1 - a_2}{2} \int_{\Sigma_1} m_k \nu_k |\nabla_\sigma u_1|^2 d\Sigma \ge 0,
\end{aligned} \tag{3.12}$$

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since $a_1 \le a_2$ and $m_k \nu_k \ge 0$ on Γ_1. Therefore, it follows from (3.10), (3.11), and (3.12) that

$$\frac{1}{2}\int_{\Sigma} m_k \nu_k \left(|u_2'|^2 - a_2 |\nabla_\sigma u_2|^2\right) d\Sigma$$
$$\ge TE(0) - \frac{2R(x^0)}{\sqrt{a_2}} E(0) - \frac{\sqrt{a_2}(1-n^2)}{8R(x^0)} \int_{\Omega} |u(0)|^2 dx$$
$$- \frac{\sqrt{a_2}(n-1)}{4R(x^0)} \int_{\Gamma} m_k \nu_k \left(|u(0)|^2 + |u(T)|^2\right) d\Gamma.$$

This implies (i).

Note that there exists $C > 0$ such that

$$\int_{\Gamma(x^0)} (|u(0)|^2 + |u(T)|^2) d\Gamma \le C \int_{\Sigma(x^0)} (|u'|^2 + |u|^2) d\Sigma.$$

This is because

$$\int_{\Gamma(x^0)} T|u(T)|^2 d\Gamma = \int_{\Gamma(x^0)} \int_0^T u^2 dt d\Gamma + \int_{\Gamma(x^0)} \int_0^T t du^2 d\Gamma$$
$$\le (T+1) \int_{\Sigma(x^0)} (|u'|^2 + |u|^2) d\Sigma,$$

and

$$\int_{\Gamma(x^0)} T|u(0)|^2 d\Gamma = \int_{\Gamma(x^0)} \int_0^T u^2 dt d\Gamma + \int_{\Gamma(x^0)} \int_0^T (t-T) du^2 d\Gamma$$
$$\le (T+1) \int_{\Sigma(x^0)} (|u'|^2 + |u|^2) d\Sigma.$$

Therefore, Lemma 3.3 gives the following result.

LEMMA 3.5. (Inverse inequality) *Suppose there exists* $x^0 \in \Omega_1$ *such that* $m(x) \cdot \nu(x) \ge 0$ *on* Γ_1, *where* ν *is directed towards the exterior of* Ω_1. *Assume* $a_2 \le a_1$ *and* $T > \frac{2R(x^0)}{\sqrt{a_2}}$. *Then for all strong solutions u of (2.1) with initial conditions* $(u^0, u^1) \in H^1(\Omega) \times L^2(\Omega)$ *and* $f = 0$, *there exists a constant* $C(T) > 0$ *such that*

$$\int_{\Sigma(x^0)} (|u_2'|^2 + |u_2|^2) d\Sigma + \int_{\Sigma_*(x^0)} |\nabla_\sigma u_2|^2 d\Sigma$$
$$\ge C(T)\left(\|u^0\|^2_{H^1(\Omega)} + \|u'\|^2_{L^2(\Omega)}\right).$$

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4. Main theorem. The main theorem of this paper is as follows.

THEOREM 4.1. *Suppose there exists* $x^0 \in \Omega_1$ such that $m(x) \cdot \nu(x) \geq 0$ *on* Γ_1, *where* ν *is directed towards the exterior of* Ω_1. *Assume* $a_2 \leq a_1$ *and* $T > \frac{2R(x^0)}{\sqrt{a_2}}$. *Then for all initial states*

$$\{y^0, y^1\} \in L^2(\Omega) \times (H^1(\Omega))',$$

there exists a control function

$$g = \begin{cases} g_0 & on\ \Sigma(x^0), \\ g_1 & on\ \Sigma_*(x^0), \end{cases}$$

with $g_0 \in (H^1(\Sigma(x^0)))'$ *and* $g_1 \in (H^1(\Sigma_*(x^0)))'$ *such that the solution* $y = y(x,t;g)$ *of* (1.1) *satisfies* (1.2).

Proof. We apply HUM. To do so, we consider the problem:

$$\begin{cases} u'' - A(x)\Delta u = 0 & & \text{in } Q, \\ u(x,0) = u^0(x), & u'(x,0) = u^1(x) & \text{in } \Omega, \\ \frac{\partial u_2}{\partial \nu} = 0 & & \text{on } \Sigma, \\ u_1 = u_2, & a_1 \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{4.1}$$

For any$\{u^0, u^1\} \in (C^\infty(\overline{\Omega}) \cap H^2(\Omega_1, \Omega_2)) \times C^\infty(\overline{\Omega})$, by Theorem 2.1, problem (4.1) has a unique solution u with

$$C([0,T]; H^2(\Omega_1, \Omega_2)) \cap C([0,T]; H^1(\Omega)).$$

Define

$$\| \{u^0, u^1\} \|_F = \left(\int_{\Sigma(x^0)} (|u_2'|^2 + |u_2|^2) d\Sigma + \int_{\Sigma_*(x^0)} |\nabla_\sigma u_2|^2 d\Sigma \right)^{1/2},$$

which is a norm on $(C^\infty(\overline{\Omega}) \cap H^2(\Omega_1, \Omega_2)) \times C^\infty(\overline{\Omega})$, due to Lemma 3.5. Let F be the completion of $(C^\infty(\overline{\Omega}) \cap H^2(\Omega_1, \Omega_2)) \times C^\infty(\overline{\Omega})$ with respect to the norm $\| \cdot \|_F$. Then Lemma 3.5 implies that

$$F \subset H^1(\Omega) \times L^2(\Omega),$$

consequently

$$(H^1(\Omega))' \times L^2(\Omega) \subset F'.$$

According to the definition of F, we have for any $\{u^0,u^1\}\in F$,

$$u|_{\Sigma(x^0)}, \quad u'|_{\Sigma(x^0)} \in L^2(\Sigma(x^0)), \quad \nabla_\sigma u|_{\Sigma_*(x^0)} \in (L^2(\Sigma_*(x^0)))^n.$$

To apply the HUM, we need to consider the backward problem:

$$\begin{cases} \phi'' - A(x)\Delta\phi = 0 & \text{in } Q, \\ \phi(T) = \phi'(T) = 0 & \text{in } \Omega, \\ \phi_1 = \phi_2, \;\; a_1 \frac{\partial \phi_1}{\partial \nu} = a_2 \frac{\partial \phi_2}{\partial \nu} & \text{on } \Sigma_1, \\ \frac{\partial \phi}{\partial \nu} = \begin{cases} -u_2 + \frac{\partial}{\partial t} u'_2 & \text{on } \Sigma(x^0), \\ \Delta_{\Gamma_*(x^0)} u_2 & \text{on } \Sigma_*(x^0). \end{cases} \end{cases} \tag{4.2}$$

For the definition of the operator $\Delta_{\Gamma_*}(x^0)$, see [**6**, p.138]. The solution of (4.2) can be defined by the transposition method (see [**8**]) as follows.

Definition 4.2. ϕ is said to be a *weak solution* of (4.2) if there exist $\{\rho^1,-\rho^0\}\in F'$ such that ϕ satisfies

$$\begin{aligned} &\int_Q f\phi dxdt - (\rho^0, \theta^1) + (\rho^1, \rho^0) \\ &= \int_{\Sigma(x^0)} (\theta_2 u_2 + \theta'_2 u'_2) d\Sigma + \int_{\Sigma_*(x^0)} a_2 \nabla_\sigma \theta_2 \nabla_\sigma u_2 d\Sigma, \end{aligned} \tag{4.3}$$

for any $\{\theta^0,\theta^1\}\in F$, $f\in L^1(0,T;H^1(\Omega,0))$, and where θ is the solution of the following problem:

$$\begin{cases} \theta'' - A(x)\Delta\theta = f & \text{in } Q, \\ \theta(0) = \theta^0, \theta'(0) = \theta^1 & \text{in } \Omega, \\ \frac{\partial \theta_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ \theta_1 = \theta_2, \;\; a_1 \frac{\partial \theta_1}{\partial \nu} = a_2 \frac{\partial \theta_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{4.4}$$

We define $\phi(0) = \rho^0$, $\phi'(0) = \rho^1$.

Lemma 4.3. *Problem (4.2) has a unique solution in the sense of Definition 4.2 satisfying*

$$\phi \in L^\infty(0, T; (H^1(\Omega, 0))'),$$

$$\{\phi'(0), -\phi(0)\} \in F'.$$

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Moreover, there exists $C>0$ such that

$$\| \{\phi'(0), -\phi(0)\} \|_{F'} \leq C \| \{u^0, u^1\} \|_F . \tag{4.5}$$

We admit this lemma for the moment. We now define a linear operator Λ by

$$\Lambda\{u^0, u^1\} = \{\phi'(0), -\phi(0)\} \tag{4.6}$$

Taking $f=0$ in (4.3), we find

$$\begin{aligned} &\langle \Lambda\{u^0, u^1\}, \{u^0, u^1\}\rangle \\ &= (\phi'(0), u^0) - (\phi(0), u^1) \\ &= \int_{\Sigma(x^0)} (| u_2'|^2 + | u_2|^2) d\Sigma + \int_{\Sigma_*(x^0)} a_2 |\nabla_\sigma u_2|^2 d\Sigma. \end{aligned} \tag{4.7}$$

Lemma 3.5, Lemma, 4.3, and the Lax-Milgram Theorem show that Λ is an isomorphism from F to F'. This means that for all $\{y^1,-y^0\}\in F'$, the equation

$$\Lambda\{u^0, u^1\} = \{y^1, -y^0\}$$

has a unique solution $\{u^0,u^1\}$. With this initial condition we solve problem (4.1), and then solve problem (4.2). Then we have found the control function

$$g = \begin{cases} -u_2 + \frac{\partial}{\partial t} u_2', & on\ \Sigma(x^0) \\ \Delta_{\Gamma_*(x^0)} u_2, & on\ \Sigma_*(x^0). \end{cases}$$

with $g_0 = -u_2 + \frac{\partial}{\partial \tau} u_2' \in (H^1(\Sigma(x^0)))'$ and $g_1 = \Delta_{\Gamma_*(x^0)} u_2 \in (H^1(\Sigma_*(x^0)))'$ such that

$$y(x, t; g) = \phi(x, t; g)$$

is the solution of (1.1) satisfying (1.2). Thus, we have proved Theorem 4.1 provided we can prove Lemma 4.3.

Proof of Lemma 4.3. The solution θ of problem (4.4) can be written as $\theta = \upsilon + w$, where υ and w are, respectively, solutions of the following problems:

$$\begin{cases} \upsilon'' - A(x)\Delta \upsilon = 0 & \text{in } Q, \\ \upsilon(x, 0) = \theta^0(x), \upsilon'(x, 0) = \theta^1(x) & \text{in } \Omega, \\ \frac{\partial \upsilon_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ \upsilon_1 = \upsilon_2, \quad a_1 \frac{\partial \upsilon_1}{\partial \nu} = a_2 \frac{\partial \upsilon_2}{\partial \nu} & \text{on } \Sigma_1, \end{cases} \tag{4.8}$$

and

$$\begin{cases} w'' - A(x)\Delta w = f & \text{in } Q, \\ w(x,0) = u'(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial w_2}{\partial \nu} = 0 & \text{on } \Sigma, \\ w_1 = w_2, \quad a_1 \frac{\partial w_1}{\partial \nu} = a_2 \frac{\partial w_2}{\partial \nu} & \text{on } \Sigma_1. \end{cases} \tag{4.9}$$

Since $\{\theta^0, \theta^1\} \in F$, we have

$$\| \{\theta^0, \theta^1\} \|_F = \left(\int_{\Sigma(x^0)} (|\upsilon'_2|^2 + |\upsilon_2|^2) d\Sigma + \int_{\Sigma_*(x^0)} |\nabla_\sigma \upsilon_2|^2 d\Sigma. \right.$$

On the other hand, by Theorem 2.1 and the trace theorem (see [**7**, Chapter 1]), we have

$$\left(\int_{\Sigma(x^0)} (|w'_2|^2 + |w_2|^2) d\Sigma + \int_{\Sigma_*(x^0)} |\nabla_\sigma w_2|^2 d\Sigma \right)^{1/2} \leq C \| f \|_{L^1(0,T;H^1(\Omega))} .$$

Therefore,

$$\begin{aligned} &| \int_Q f\phi dx dt - (\rho^0, \theta^1) + (\rho^1, \theta^0) | \\ &= | \int_{\Sigma(x^0)} (\theta_2 u_2 + \theta'_2 u'_2) d\Sigma + \int_{\Sigma_*(x^0)} a_2 \nabla_\sigma \theta_2 \nabla_\sigma u_2 d\Sigma | \\ &\leq | \int_{\Sigma(x^0)} (\upsilon_2 u_2 + \upsilon'_2 u'_2) d\Sigma + \int_{\Sigma_*(x^0)} a_2 \nabla_\sigma \upsilon_2 \nabla_\sigma u_2 d\Sigma | \\ &+ | \int_{\Sigma(x^0)} (w_2 u_2 + w'_2 u'_2) d\Sigma + \int_{\Sigma_*(x^0)} a_2 \nabla_\sigma w_2 \nabla_\sigma u_2 d\Sigma | \\ &\leq C \big(\| \{\theta^0, \theta^1\} \|_F + \| f \|_{L^1(0,T;H^1(\Omega))} \big) \| \{u^0, u^1\} \|_F . \end{aligned} \tag{4.10}$$

Thus, there exist $\phi \in L^\infty(0,T;(H^1(\Omega,0))')$ and $\{\rho^0, -\rho^1\} \in F'$ such that (4.3) holds. That is, ϕ is a weak solution of (4.2) and $\{\phi(0), -\phi'(0)\} \in F'$. Taking $f = 0$, (4.10) gives (4.5).

REMARK 4.4. If Ω is star-shaped with respect to x^0, then $\Sigma_*(x^0) = \phi$. In this case, we obtain a control function g with $g \in (H^1(0,T;L^2(\Gamma)))'$.

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