BULL. AUSTRAL. MATH. SOC. 34K15, 34C10
VOL. 49 (1994) [69–79]

OSCILLATION AND NONOSCILLATION OF A DELAY DIFFERENTIAL EQUATION

CHUNHAI KOU, WEIPING YAN AND JURANG YAN

In this paper, some necessary and sufficient conditions for oscillation of a first order delay differential equation with oscillating coefficients of the form

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0$$

are established. Several applications of our results improve and generalise some of the known results in the literature.

1. INTRODUCTION

The oscillatory behaviour of first order delay differential equation has been studied by many authors. In particular, several authors have studied the oscillations caused by deviating arguments. Besides its theoretical interest, this question is important from the viewpoint of practical applications. For some contributions in this area see [2] and references cited therein. Most of the known results, however, deal with equations having coefficients with fixed sign. The case where the coefficients are oscillating has been considered in only a few papers, for example, [5, 8] and [9].

Let us consider the first order differential equation with oscillating coefficients

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0, \tag{1}$$

where p_i, $\tau_i, i = 1, 2, \cdots, n$, are continuous functions on the interval $[t_0, \infty)$ and $\lim\limits_{t\to\infty}(t - \tau_i(t)) = \infty$, $i = 1, 2, \cdots, n$. If $T \geqslant t_0$, by a solution on $[T, \infty)$ of equation (1) we mean a real-valued function $x(t)$ defined on $[T, \infty)$, which is differentiable on $[T, \infty)$ and satisfies $x'(t) + \sum\limits_{i=1}^{n} p_i(t)\widetilde{x}(t - \tau_i(t)) = 0$ for $t \geqslant T$, where $\widetilde{x}$ is a continuous extension of $x(t)$ on $[T^*, \infty)$ with $T^* = \min\limits_{1\leqslant i\leqslant n}\{\inf(t - \tau_i(t))\}$. A solution of equation (1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is called non-oscillatory.

Received 4th February, 1993

We say that the conditions (C) hold if the following conditions are fulfilled:

(C_1) $\tau_i(t)$, $i = 1,2,\cdots,n$ are continuous on $[t_0,\infty)$, and there exist two constants α and β so that $0 < \alpha \leqslant \tau_n(t) \leqslant \tau_{n-1}(t) \leqslant \cdots \leqslant \tau_1(t) \leqslant \beta$.

(C_2) $p_i(t)$, $i = 1,2,\cdots,n$, are continuous on $[t_0,\infty)$, and for each m, $m = 1,2,\cdots,n$, $\sum_{i=1}^{m} p_i(t) \geqslant 0$; Furthermore, there exists a sequence of intervals $\{[a_k,b_k]\}$ with $b_k - a_k \geqslant \beta$ such that $p_i(t) \geqslant 0$, $i = 1,2,\cdots,n$, $t \in \bigcup_{k=1}^{\infty}(a_k,b_k)$.

Recently, under the conditions (C) Yongchao Chen [1] studied equation (1) with $\tau_i(t) \equiv \tau_i$ (constants), $i = 1,2,\cdots,n$ and gave the following main results.

THEOREM A. *Assume that the conditions (C) hold and*

$$\liminf_{t\to\infty} \int_{t-\tau_n^*/2}^{t} \sum_{i=1}^{n} p_i(s)ds > 0,$$

$$\lim_{t\to\infty} inf \int_{t-\tau_n^*}^{t} \sum_{i=1}^{n} p_i(s)ds > \frac{1}{e}.$$

Then all solutions of equation (1) with $\tau_i(t) \equiv \tau_i$ (constants), $i = 1,2,\cdots,n$, are oscillatory.

In this paper, by using a new technique we establish necessary and sufficient conditions, and explicit sufficient conditions for all solutions of equation (1) to be oscillatory. We also give conditions, under which equation (1) has at least one non-oscillatory solution and estimate of the asymptotic behaviour of the non-oscillatory solution.

2. MAIN THEOREMS

The following two lemmas are basic for all later discussions and Lemma 1 is a extension of Lemma in [1].

LEMMA 1. *Assume that the conditions (C) hold. Let $x(t)$ be an eventually positive solution of equation (1). Then for sufficiently large t, $x(t)$ is eventually non-increasing and*

$$\left(\sum_{i=1}^{n} p_i(t)\right) x(t-\tau_n(t)) \leqslant \sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)). \tag{2}$$

PROOF: Obviously, there exists $T \geqslant t_0$ so that

$$x(t-\tau_i(t)) > 0, \quad i = 1,2,\cdots,n, \qquad \text{for every} \quad t \geqslant T.$$

From (C), there exists $t_1 \geqslant T$ such that $p_i(t) \geqslant 0$, $i = 1,2,\cdots,n$, on $[t_1,t_1+\beta]$. Thus

$$x'(t) = -\sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)) \leqslant 0, \qquad \text{for all } t \in [t_1,t_1+\beta], \tag{3}$$

https://doi.org/10.1017/S0004972700016099 Published online by Cambridge University Press

which implies that $x(t)$ is non-increasing on $[t_1, t_1+\beta]$. We now claim that $x(t)$ is also non-increasing on $[t_1, t_1+\beta+\alpha]$. Indeed, from (C_1) and (3), for every $t \in [t_1+\beta, t_1+\beta+\alpha]$, we have that $t-\tau_i(t) \in [t_1, t_1+\beta]$, $i = 1, 2, \cdots, n$, and

$$x(t-\tau_1(t)) \geqslant x(t-\tau_2(t)) \geqslant \cdots \geqslant (t-\tau_n(t)). \tag{4}$$

By using the condition (C_2), (4) yields the inequality

$$\begin{aligned}
x'(t) &= -\sum_{i=1}^{n} p_i(t) x(t-\tau_i(t)) \\
&\leqslant -p_1(t)x(t-\tau_2(t)) - \sum_{i=2}^{n} p_i(t)x(t-\tau_i(t)) \\
&\leqslant -(p_1(t)+p_2(t))\,x(t-\tau_2(t)) - \sum_{i=3}^{n} p_i(t)x(t-\tau_i(t)) \leqslant \cdots\cdots \\
&\leqslant -\sum_{i=1}^{n} p_i(t)x(t-\tau_n(t)) \leqslant 0, \qquad \text{for} \quad t \in [t_1+\beta, t_1+\beta+\alpha].
\end{aligned}$$

This means $x(t)$ is also non-increasing on $[t_1+\beta, t_1+\beta+\alpha]$. By repeating the same procedure $k \geqslant 1$ times we are led to the following inequality:

$$\begin{aligned}
x'(t) &= -\sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)) \\
&\leqslant -\left(\sum_{i=1}^{n} p_i(t)\right) x(t-\tau_n(t)), \quad \text{for} \quad t \in [t_1+\beta, t_1+\beta+k\alpha].
\end{aligned}$$

Since k is aribitrary, the above inequality implies that for all $t \geqslant t_1$, $x(t)$ is non-increasing and

$$\left(\sum_{i=1}^{n} p_i(t)\right) x(t-\tau_n(t)) \leqslant \sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)).$$

The proof of Lemma 1 is complete. $\square$

In the sequel, unless otherwise specified, for sufficiently large $T \geqslant t_0$, we define

$$T_0 = \min_{1 \leqslant i \leqslant n} \{\inf_{t \geqslant T} (t-\tau_i(t))\}. \tag{5}$$

LEMMA 2. *Assume that the conditions (C) hold. Then for sufficiently large $T \geqslant t_0$, the differential inequality*

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)) \leqslant 0, \qquad t \geqslant T, \tag{6}$$

https://doi.org/10.1017/S0004972700016099 Published online by Cambridge University Press

has a positive continuous differentiable solution on $[T,\infty)$ if and only if the integral inequality

$$\sum_{i=1}^{n} p_i(t)\exp\left(\int_{t-\tau_i(t)}^{t} u(s)ds\right) \leqslant u(t), \qquad t \geqslant T, \tag{7}$$

has a non-negative solution on $[T,\infty)$.

PROOF: Let $x(t)$ be a positive solution on $[T,\infty)$ of equation (1) and for all $t \geqslant T_0$, $x(t-\tau_i(t)) > 0$, $i = 1,2,\cdots,n$, where T_0 is as defined by (5). Set

$$u(t) = -\frac{x'(t)}{x(t)} \quad \text{for} \quad t \geqslant T_0. \tag{8}$$

From Lemma 1, without loss of generality, for $t \geqslant T_0$ we can suppose that $u(t) \geqslant 0$. By integrating both sides of (8) from T to t we obtain

$$x(t) = x(T)\exp\left(-\int_T^t u(s)ds\right). \tag{9}$$

Substituting (9) into (6), we can see that $u(t)$ is a non-negative continuous solution of (7) on $[T,\infty)$.

Conversely, let $u(t)$ be a non-negative continuous solution of (7) on $[T,\infty)$. It is easy to verify that $x(t) = \exp\left(-\int_T^t u(s)ds\right)$ is a positive continuous differentiable solution of (6) on $[T,\infty)$. The proof is complete. $\square$

With (1) and for every $T \geqslant t_0$, we introduce the sequence of functions:

$$u_0(t) = 0 \qquad \text{for} \quad t \geqslant T,$$

and for $k = 1,2,\cdots$.

$$u_k(t) = \begin{cases} \sum\limits_{i=1}^{n} p_i(t)\exp\left(\int_{t-\tau_i(t)}^{t} u_{k-1}(s)ds\right) & \text{for} \quad t \geqslant T, \\ 0 & \text{for} \quad T_0 \leqslant t < T, \end{cases} \tag{10}$$

where T_0 is as defined by (5). It is obvious that $u_k(t)$, $k = 1,2,\cdots$, is continuous on $[T_0,T)$ and $[T_1,\infty)$, respectively.

Since $u_0(t) = 0$, $t \geqslant T_0$, $u_1(t) = \sum\limits_{i=1}^{n} p_i(t) \geqslant 0$, $t \geqslant T$, and $u_1(t) \geqslant 0$, $t \geqslant T_0$,

https://doi.org/10.1017/S0004972700016099 Published online by Cambridge University Press

hence $u_1(t) \geqslant u_0(t)$ for $t \geqslant T_0$. From (C) and (10) we have

$$\begin{aligned} u_2(t) &\geqslant p_1(t)\exp\left(\int_{t-\tau_2(t)}^{t} u_1(s)ds\right) + \sum_{i=2}^{n} p_i(t)\exp\left(\int_{t-\tau_i(t)}^{t} u_1(s)ds\right) \\ &\geqslant (p_1(t)+p_2(t))\exp\left(\int_{t-\tau_2(t)}^{t} u_1(s)ds\right) + \sum_{i=3}^{n} p_i(t)\exp\left(\int_{t-\tau_i(t)}^{t} u_1(s)ds\right) \\ &\geqslant \cdots \geqslant \sum_{i=1}^{n} p_i(t)\exp\left(\int_{t-\tau_n(t)}^{t} u_1(s)ds\right) \\ &\geqslant \sum_{i=1}^{n} p_i(t) = u_1(t), \qquad\qquad \text{for} \quad t \geqslant T, \end{aligned}$$

which means that $u_2(t) \geqslant u_1(t)$ for all $t \geqslant T_0$. By induction it is easy to see that for $t \geqslant T_0$

$$u_{k+1}(t) \geqslant u_k(t) \geqslant 0, \qquad k = 1, 2, \cdots. \tag{11}$$

Now we are ready to establish the following theorem.

THEOREM 1. *Assume that the conditions (C) hold. Then the following statements are equivalent:*

(a) *Equation (1) has a non-oscillatory solution.*
(b) *The delay differential inequality (6) has an eventually positive solution.*
(c) *There exists a sufficiently large $T \geqslant t_0$ such that the sequence $\{u_k(t)\}$ which is defined by (10) converges pointwise to a finite limit for each $t \geqslant T_0$.*

PROOF: (a) $\Rightarrow$ (b). Let $x(t)$ be a non-oscillatory solution of equation (1). As the negative of a solution of equation (1) is also a solution of the same equation, we can assume that $x(t)$ is eventually positive. Clearly, $x(t)$ is also an eventually positive solution of (6).

(b) $\Rightarrow$ (c). Assume that (6) has an eventually positive solution $x(t)$ and there exists $T^* \geqslant t_0$ such that $x(t) > 0$, $t \geqslant T^*$. From (5) we can choose T such that $T_0 \geqslant T^*$. Set $u(t) = -x'(t)/x(t)$. By using Lemma 2, it follows that the inequality (7) has a non-negative solution $u(t)$ which is defined in $[T_0, \infty)$. Consider the sequence

(10). Noting $u(t) \geqslant u_0(t)$, $t \geqslant T_0$ and using (10) and (C), we find that

$$\begin{aligned} u_1(t) &= \sum_{i=1}^{n} p_i(t) \leqslant \left(\sum_{i=1}^{n} p_i(t)\right) \exp\left(\int_{t-\tau_n(t)}^{t} u(s)ds\right) \\ &\leqslant \left(\sum_{i=1}^{n-1} p_i(t)\right) \exp\left(\int_{t-\tau_{n-1}(t)}^{t} u(s)ds\right) + p_n(t)\exp\left(\int_{t-\tau_n(t)}^{t} u(s)ds\right) \\ &\leqslant \left(\sum_{i=1}^{n-2} p_i(t)\right) \exp\left(\int_{t-\tau_{n-2}(t)}^{t} u(s)ds\right) + p_{n-1}(t)\exp\left(\int_{t-\tau_{n-1}(t)}^{t} u(s)ds\right) \\ &\quad + p_n(t)\exp\left(\int_{t-\tau_n(t)}^{t} u(s)ds\right) \leqslant \cdots \leqslant \sum_{i=1}^{n} p_i(t)\exp\left(\int_{t-\tau_i(t)}^{t} u(s)ds\right) \\ &\leqslant u(t), \qquad \text{for} \quad t \geqslant T_0. \end{aligned}$$

So, $u_1(t) \leqslant u(t)$, $t \geqslant T_0$. By a simple induction, we can show that

$$0 \leqslant u_k(t) \leqslant u(t), \qquad k = 0, 1, 2, \cdots,$$

for $t \geqslant T_0$. Then from (11) it follows that

$$0 \leqslant u_k(t) \leqslant u_{k+1}(t) \leqslant u(t), \qquad k = 0, 1, 2, \cdots,$$

for $t \geqslant T_0$. Therefore, the sequence $\{u_k(t)\}$ converges pointwise to a finite limit for all $t \geqslant T_0$.

(c)$\Rightarrow$ (a). Set $\widetilde{u}(t) = \lim_{k\to\infty} u_k(t)$, $t \geqslant T_0$. By Lebesgue's monotone convergence theorem, $\widetilde{u}(t)$ satisfies

$$\widetilde{u}(t) = \begin{cases} \sum_{i=1}^{n} p_i(t)\exp\left(\int_{t-\tau_i(t)}^{t} \widetilde{u}(s)ds\right), & t \geqslant T, \\ 0, & T_0 \leqslant t < T. \end{cases} \tag{12}$$

which implies that $\widetilde{u}(t)$ is a continuous solution of the integral equation

$$u(t) = \sum_{i=1}^{n} p_i(t)\exp\left(\int_{t-\tau_i(t)}^{t} u(s)ds\right).$$

Let $x(t) = \exp\left(-\int_T^t \widetilde{u}(s)ds\right)$. Then it is easy to verify that $x(t)$ is a positive solution of equation (1).

The proof of Theorem 1 is complete. $\square$

The next result is an immediate consequence of Theorem 1.

https://doi.org/10.1017/S0004972700016099 Published online by Cambridge University Press

Theorem 2. *Assume that the conditions (C) hold. Then the following statements are equivalent.*

(a_1) All solutions of equation (1) are oscillatory.

(b_1) The inequality (6) has no eventually positive solutions.

(c_1) For every large $T \geqslant t_0$ there exists $t^* \geqslant T$ such that the sequence (10) satisfy

$$\lim_{k\to\infty} u_k(t^*) = \infty.$$

3. Applications of main theorem

In this section we apply the above main theorems to give some sufficient conditions for equation (1) to have at least one non-oscillatory solution and for all solutions of equation (1) to be oscillatory.

Theorem 3. *Assume that the conditions (C) hold, and there exists sufficiently large $T \geqslant t_0$ such that for all $t \geqslant T$,*

$$\int_{t-\tau_1(t)}^{t} \sum_{i=1}^{n} p_i^+(s)\,ds \leqslant \frac{1}{e}, \tag{13}$$

where $p_i^+(t) = \max\{p_i(t), 0\}$. Then equation (1) has a non-oscillatory solution $x(t)$ with

$$\exp\left(-e\int_T^t \sum_{i=1}^{n} p_i^+(s)\,ds\right) \leqslant x(t) \leqslant \exp\left(-\int_T^t \sum_{i=1}^{n} p_i(s)\,ds\right) \tag{14}$$

for all $t \geqslant T$.

Proof: Consider the sequence (10) on $[T_0, \infty)$, where T_0 is as defined by (5). From (10) and (C_2) it follows that $0 \leqslant u_1(t) = \sum_{i=1}^{n} p_i(t) \leqslant e\sum_{i=1}^{n} p_i^+(t)$, $t \geqslant T$. Thus $0 \leqslant u_1(t) \leqslant e\sum_{i=1}^{n} p_i^+(t)$, $t \geqslant T_0$. In view of (10) and (13) we obtain

$$u_2(t) \leqslant \sum_{i=1}^{n} p_i^+(t) \exp\left(\int_{t-\tau_1(t)}^{t} u_1(s)\,ds\right) \leqslant e\sum_{i=1}^{n} p_i^+(t), \qquad t \geqslant T.$$

Hence $u_2(t) \leqslant e\sum_{i=1}^{n} p_i^+(t)$, $t \geqslant T_0$. By (10), (13) and a simple induction, it is easy to show that

$$u_k(t) \leqslant e\sum_{i=1}^{n} p_i^+(t), \quad t \geqslant T_0,\ k = 0, 1, 2, \cdots. \tag{15}$$

Thus, by applying Theorem 1, we know that equation (1) has a non-oscillatory solution.

Let us now prove (14). Set

$$\tilde{u}(t) = \lim_{k\to\infty} u_k(t), \qquad \text{pointwise on} \quad [T,\infty), \tag{16}$$

then $\tilde{u}(t)$ satisfies (12). Then it is easy to verify that

$$x(t) = \exp\left(-\int_T^t \tilde{u}(s)ds\right)$$

is a positive solution on $[T,\infty)$ of equation (1). From (10) and (15), we have that $\sum_{i=1}^n p_i(t) \leqslant \tilde{u}(t) \leqslant e\sum_{i=1}^n p_i^+(t)$, $t \geqslant T$, which means that the estimate (14) hold. The proof is complete. ∎

Theorem 4. *Assume that the conditions (C) hold and for sufficiently large T there exists $\lambda > 0$ such that*

$$\sup_{t\geqslant T}\left\{\frac{1}{\lambda}\sum_{i=1}^n p_i^+(t)\exp\left(\lambda\tau_i(t)\right)\right\} \leqslant 1, \tag{17}$$

where $p_i^+(t) = \max\{p_i(t), 0\}$. Then equation (1) has a non-oscillatory solution.

Proof: Consider the sequence (10) on $[T_0,\infty)$, where T_0 is as defined by (5). $u_1(t) = \sum_{i=1}^n p_i(t) \leqslant \lambda_1$ where $\lambda_1 = \sup_{t\geqslant T}\sum_{i=1}^n p_i^+(t)$. Since $u_1(t) = 0$, $T_0 \leqslant t < T$, $u_1(t) \leqslant \lambda_1$, $t \geqslant T_0$. Also $u_2(t) \leqslant \sum_{i=1}^n p_i^+(t)\exp\left(\int_{t-\tau_i(t)}^t u_1(s)ds\right) \leqslant \sum_{i=1}^n p_i^+(t)\exp\left(\lambda_1\tau_i(t)\right) \leqslant \lambda_2$ where $\lambda_2 = \sup_{t\geqslant T}\left\{\sum_{i=1}^n p_i^+(t)\exp\left(\lambda_1\tau_i(t)\right)\right\}$. Obviously, $\lambda_1 \leqslant \lambda_2$. By induction it is easy to see that

$$u_{k+1}(t) \leqslant \sum_{i=1}^n p_i^+(t)\exp\left(\lambda_k\tau_i(t)\right) \leqslant \lambda_{k+1},$$

where

$$\lambda_{k+1} = \sup_{t\geqslant T}\left\{\sum_{i=1}^n p_i^+(t)\exp\left(\lambda_k\tau_i(t)\right)\right\},$$

and $\lambda_k \leqslant \lambda_{k+1}$. By using (17), we derive that

$$\lambda_1 \leqslant \sum_{i=1}^n p_i^+(t)\exp\left(\lambda\tau_i(t)\right) \leqslant \lambda, \qquad t \geqslant T.$$

Hence, it is easy to show that $\lambda_k \leqslant \lambda$, $k = 1, 2, \cdots$, and $u_k(t) \leqslant \lambda$, $t \geqslant T$. By applying Theorem 1, we conclude that equation (1) has a non-oscillatory solution. The proof is complete. ∎

Now, we establish the following oscillation comparison theorem.

https://doi.org/10.1017/S0004972700016099 Published online by Cambridge University Press

Theorem 5. *Assume that the condition (C) hold and all solutions of the delay differential equation*

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_n(t)) = 0 \tag{18}$$

are oscillatory. Then all solutions of equation (1) are also oscillatory.

Proof: Suppose that equation (1) has a non-oscillatory solution $x(t)$. As the opposite of a solution of equation (1) is also a solution of the same equation, we may suppose that $x(t) > 0$ for all $t \geqslant t_1 \geqslant t_0$. By Lemma 1, for all sufficiently large t, we have

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_n(t)) \leqslant 0.$$

From Theorem 2 of [6], it follows that equation (18) has an eventually positive solution. This is a contradiction. The proof of the theorem is complete. ☐

Combining Theorem 5 with known oscillation criteria for equation (18) (see [2, 3, 4, 7]) we may obtain various results for oscillation of all solutions of equation (1).

Corollary 1. *Assume that the conditions (C) hold and each one of the following conditions is satisfied,*

$$(i) \qquad \lim_{t\to\infty} inf \int_{t-\tau_n(t)}^{t} \sum_{i=1}^{n} p_i(s)ds > \frac{1}{e},$$

$$(ii) \qquad \lim_{t\to\infty} inf\{\tau_n(t) \sum_{i=1}^{n} p_i(t)dt\} > \frac{1}{e}.$$

Then all solitions of equation (1) are oscillatory.

Corollary 2. *Assume that the conditions (C) hold and the function $t-\tau_n(t)$ is non-decreasing. Moreover, assume that each one of the following conditions is satisfied:*

$$(i) \quad \lim_{t\to\infty} sup \int_{t-\tau_n(t)}^{t} \sum_{i=1}^{n} p_i(s)ds > 1,$$

$$(ii) \quad \lim_{t\to\infty} inf \int_{t-\tau_n(t)}^{t} \sum_{i=1}^{n} p_i(s)ds = \alpha \leqslant \frac{1}{e}, \quad \text{and}$$

$$\lim_{t\to\infty} inf \int_{t-\tau_n(t)}^{t} \sum_{i=1}^{n} p_i(s)ds > \frac{\ln \lambda_\alpha + 1}{\lambda_\alpha},$$

where λ_α is the smaller solution of the equation $\lambda = e^{\alpha\lambda}$. Then all solutions of equation (1) are oscillatory.

https://doi.org/10.1017/S0004972700016099 Published online by Cambridge University Press

REMARK. Obviously, the condition (i) of Corollary 1 improves and generalises Theorem A.

EXAMPLE. Consider the differential equation

$$x'(t) + ax(t-\tau_1) + b\sin tx(t-\tau_2) + c\sin\frac{t}{2}x(t-\tau_3) = 0, \qquad t \geqslant 0, \tag{19}$$

where a, b, c, τ_1, τ_2, and τ_3 are positive constants with $a \geqslant b + c$ and $\tau_3 < \tau_2 < \tau_1 \leqslant \pi$. It is easy to check that equation (19) satisfies the condition (C). If

$$\int_{t-\tau_1}^{t} [a + b(\sin s)^+ + c\left(\sin\frac{s}{2}\right)^+]ds \leqslant \frac{1}{e},$$

then, by Theorem 3, equation (19) has a non-oscillatory solution. If

$$\lim_{t\to\infty} \inf \tau_3\left(a + b\sin t + c\sin\frac{t}{2}\right) > \frac{1}{e},$$

then, by Corollary 1, all solutions of equation (19) are oscillatory.

REFERENCES

[1] Yongshao Chen, 'Oscillation and asymptotic behavior of solutions of first order linear functional differential equations with oscillatory coefficients', (in Chinese), *Acta Math. Appl. Sinica* **12** (1989), 96–104.

[2] I. Győri and G. Ladas, *Oscillation theory of delay differential equations with applications* (Clarendon Press, Oxford, 1991).

[3] B.R. Hunt and J.A. Yorke, 'When all solutions of $x' = -\sum q_i(t)x(t-\tau_i(t))$ oscillate', *J. Differential Equations* **53** (1984), 139–145.

[4] R.G. Koplatadze and T.A. Chanturia, 'On the oscillatory and monotone solutions of first order differential equations with deviating arguments', (in Russian), *Differensial'nye Uravneniya* **18** (1982), 1463–1465.

[5] M.R. Kulenovic and M.K. Grammatikopoulos, 'First order functional differential inequalities with oscillating coefficients', *Nonlinear Anal.* **8** (1984), 1043–1054.

[6] M.R. Kulenovic and M.K. Grammatikopoulos, 'Some comparison and oscillation results for first order differential equations and inequalities with a deviating argument', *J. Math. Anal. Appl.* **131** (1988), 67–84.

[7] M.K. Kwong, 'Oscillation of first order delay equations', *J. Math. Anal. Appl.* **156** (1991), 274–286.

[8] G. Ladas, Y.G. Sficas and I.P. Stavroulakis, 'Functional differential inequalities and equations with oscillating coefficients', in *Trends in theory and practice of nonlinear differential equations*, (V. Lakshmikantham, Editor) (Marcel Dekker, New York, Basel, 1984).

[9] J.S. Yu, Z.C. Wang, B.G. Zhang and X.Z. Qian, 'Oscillations of differential equations with deviating arguments', *Pan. Math. J.* **2** (1992), 59–78.

Department of Information
Shanxi Economic Management Institute
Taiyuan, Shanxi 030006
People's Republic of China
(Chunhai Kou)

Department of Mathematics
Shanxi University
Taiyuan, Shanxi 030006
People's Republic of China
(Weiping Yan and Jurang Yan)

Department of Mathematics
Shanxi University
Taiyuan, Shanxi 030006
People's Republic of China