J. Austral. Math. Soc. (*Series A*) **28** (1979), 452–460

SOME ASYMPTOTIC METHODS IN COMBINATORICS

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(Received 3 August 1978; revised 26 April 1979)

Communicated by W. D. Wallis

Abstract

Let $\langle f_n \rangle_{n \geqslant 0}$ be nonnegative real numbers with generating function $f(x) = \Sigma f_n x^n$. Assume $f(x)$ has the following properties: it has a finite nonzero radius of convergence x_0 with its only singularity on the circle of convergence at $x = x_0$ and $f(x_0)$ converges to y_0; $y = f(x)$ satisfies an analytic identity $F(x,y) = 0$ near (x_0, y_0); $F_{y^{(i)}}(x_0, y_0) = 0$, $0 \leqslant i < k$ and $F_{y^{(k)}}(x_0, y_0) \neq 0$. There are constants γ, a positive rational, and c such that $f_n \sim c x_0^{-n} n^{-(1+\gamma)}$. Furthermore, we show (i) in all cases how to determine γ and c from $f(x)$ and (ii) in certain cases how to determine them from $F(x,y)$.

1980 Mathematics subject classification (*Amer. Math. Soc.*): primary 05 A 15.

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Bender (1974) in his expository article and Harary, Robinson and Schwenk (1975) present similar methods of determining the asymptotic behavior of a sequence $\langle f_n \rangle_{n \geqslant 0}$ of nonnegative real numbers satisfying the following conditions.

Let $f(x) = \sum_{n \geqslant 0} f_n x^n$ be the generating function for $\langle f_n \rangle_{n \geqslant 0}$. Assume $f(x)$ has a finite nonzero radius of convergence x_0 with its only singularity on the circle of convergence at $x = x_0$ and furthermore that $f(x_0)$ converges to y_0. Assume also that there is a function $F(x, y)$ analytic near (x_0, y_0) such that

(i) $F(x, f(x)) \equiv 0$ near x_0,

(ii) $F(x_0, y_0) = 0$, $F_y(x_0, y_0) = 0$, $F_{yy}(x_0, y_0) \neq 0$.

(Often as in Harary, Robinson and Schwenk (1975), one can use F to show the only singularity of f on its circle of convergence is at $x = x_0$.)

Bender (1974) shows that if $F_x(x_0, y_0) \neq 0$ then

$$f_n \sim (x_0 F_x / 2\pi F_{yy})^{1/2} n^{-3/2} x_0^{-n},$$

where the partial derivatives are evaluated at (x_0, y_0). Harary, Robinson and

The research for this paper was done while the second author was visiting Michigan State University and the writing while the second author was visiting the University of Sydney.

Schwenk (1975) show that if

$$\lim_{x \to x_0^-} f'(x)(f(x_0) - f(x)) \neq 0$$

then

$$f_n \sim (b_1/2)(x_0/\pi)^{1/2} n^{-3/2} x_0^{-n},$$

where $b_1^2/2$ is the value of the above limit. They also discuss a case where

$$f_n \sim c \cdot n^{-5/2} x_0^{-n}$$

for some c which they indicate how to compute. Both results are generalizations of methods of Pólya (1937).

Recently Plotkin and Rosenthal found naturally occurring examples where instead of (ii) above holding we have for $k = 4$

(ii)′ $F_{y(j)}(x_0, y_0) = 0 \quad (0 \leqslant j \leqslant k-1)$,
$F_{y(k)}(x_0, y_0) \neq 0$,

where $F_{y(j)}$ is $\partial^j F/\partial y^j$. These raise the question: Do results similar to those in Bender (1974) and Harary, Robinson, and Schwenk (1975) hold for $k > 2$?

In Section 2 we derive results similar to those in Bender (1974) in that we read off the asymptotic behavior of $\langle f_n \rangle_{n \geqslant 0}$ from information on F for certain cases in which $k > 2$. In Section 3 we obtain results similar to those in Harary, Robinson and Schwenk (1975). But here we are able to read off the asymptotic behavior of $\langle f_n \rangle_{n \geqslant 0}$ from the information on f for all cases in which $k > 2$. Section 1 contains the necessary preliminaries for what follows.

1

Let $\langle f_n \rangle_{n \geqslant 0}$ be nonnegative real numbers and let $f(x) = \sum_{n \geqslant 0} f_n x^n$. Assume $f(x)$ has a finite nonzero radius of convergence x_0 with its only singularity on the circle of convergence at $x = x_0$ and that $f(x_0)$ converges to y_0. Assume there is a function $F(x, y)$ analytic near (x_0, y_0) such that

(i) $F(x, f(x)) \equiv 0$ near x_0,
(ii) $F_{y(j)}(x_0, y_0) = 0 \quad (0 \leqslant j \leqslant k-1)$,
$F_{y(k)}(x_0, y_0) \neq 0$.

We generalize the proof of Bender (1974), p. 505. By the Weierstrass preparation theorem (see, for example, Hörmander (1966), p. 144) near (x_0, y_0)

$$F(x, y) = U(x, y) P(x, y)$$

where

(*a*) $U(x, y)$ is analytic, nonvanishing near (x_0, y_0); and
(*b*) $P(x, y)$ is a function which is a monic polynomial of degree k in $(y - y_0)$ such

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that for each $j<k$ the coefficient $p_j(x)$ of $(y-y_0)^j$ is a function vanishing at and analytic near x_0. $P(x,y)$ is called a Weierstrass polynomial.

From the factorization of F and (i) and (a)

$$P(x,f(x)) \equiv 0 \quad \text{near } x_0.$$

Hence $f(x)-y_0$ can be expressed as a fractional power series (see, for example, Walker (1950), p. 98)

$$\sum_{i=1}^{\infty} f_i(x)\left(1-\frac{x}{x_0}\right)^{r_i} \quad \left(\text{or } \sum_{i=1}^{m} f_i(x)\left(1-\frac{x}{x_0}\right)^{r_i}\right),$$

where

(i) r_i is an increasing sequence of rational numbers and

(ii) $f_i(x)$ is analytic nonvanishing near $x=x_0$.

(The branch of $(1-(x/x_0))^{r_i}$ we are using may be chosen to be the one for which $(1-(x/x_0))^{r_i}$ is real for x real, near to and less than x_0.)

Thus $f(x)$ can be expressed as

$$\sum_{i=1}^{\infty} a_i\left(1-\frac{x}{x_0}\right)^{s_i} \quad \left(\text{or } \sum_{i=1}^{m} a_i\left(1-\frac{x}{x_0}\right)^{s_i}\right)$$

where

(i) s_i is an increasing sequence of rational numbers and

(ii) a_i are nonzero.

By a special case of a theorem of Darboux (see Bender (1974), Theorem 4) we have

$$f_n \sim (a_p/\Gamma(-s_p))x_0^{-n}\, n^{-(1+s_p)},$$

where

(i) s_p is the least noninteger amongst the s_i's; and

(ii) Γ is the classical gamma function.

(We know some r_i, and hence some s_i, is nonintegral as f has a singularity at x_0. Furthermore, as $f(x_0)$ converges all r_i are positive.)

Alternately we have

$$f_n \sim (f_q(x_0)/\Gamma(-r_q))\, x_0^{-n}\, n^{-(1+r_q)}$$

where r_q is the least noninteger amongst the r_i's.

To generalize the results in Bender (1974) and Harary, Robinson and Schwenk (1975) it suffices to indicate how to find s_p and how to compute a_p. This will be done by examining more carefully how the fractional power series is obtained using the method of Newton polygons.

2

Let n_j be the order of the zero at $x = x_0$ of $p_j(x)$, the coefficient of $(y-y_0)^j$ in the Weierstrass polynomial $P(x,y)$. Say $p_j(x) = \sum_{i \geqslant n_j} p_{ji}(x-x_0)^i$ where $p_{jn_j} \neq 0$ (if $p_j(x) \equiv 0$, we let $n_j = +\infty$). Let $n_k = 0$.

Since

$$0 = P(x, f(x)) = (f(x)-y_0)^k + \sum_{j=0}^{k-1} p_j(x)(f(x)-y_0)^j,$$

in order that the lowest degree term in $(x-x_0)$ cancels when we substitute $\sum a_i(1-(x/x_0))^{s_i}$ for $f(x)-y_0$ we must have that two of the $\{n_i + is_1 \mid 0 \leqslant i \leqslant k\}$ are equal and all others are at least as large. Thus $s_1 = (n_{i_1} - n_{i_2})/(i_2 - i_1)$ for some $0 \leqslant i_1 < i_2 \leqslant k$. So s_1 is expressible as a rational number with denominator $\leqslant k$.

We have shown:

LEMMA 2.1. *If $P(x,f(x)) \equiv 0$ near x_0 where $P(x,y)$ is a function which is a monic polynomial of degree k in $(y-y_0)$, then the denominator of the lowest degree term in a fractional power series expansion of $f(x)$ about x_0 can be taken to be at most k.*

In fact from the above representation of s_1 we see that the denominator is k if and only if $n_0 = ks_1$ and $n_i + is_1 \geqslant ks_1$, $0<i<k$, if and only if $n_i \geqslant (1-i/k)n_0$, $0<i<k$. The n_i's are determined by $P(x,y)$ as follows:

LEMMA 2.2. *n_i is the least p such that*

$$P_{x^{(p)}y^{(i)}}(x_0, y_0) \neq 0.$$

COROLLARY 2.3. *$s_1 = n_0/k$ if and only if*

(i) $$P_{x^{(p)}}(x_0, y_0) \begin{cases} = 0, & p < n_0, \\ \neq 0, & p = n_0, \end{cases}$$

and

(ii) *For $0<i<k$ and $p<(1-i/k)n_0$*

$$P_{x^{(p)}y^{(i)}}(x_0, y_0) = 0.$$

Even though $P(x,y)$ cannot be effectively obtained from $F(x,y)$, F yields significant information about P.

PROPOSITION 2.4. *$s_1 = n_0/k$ if and only if*

(i) $$F_{x^{(p)}}(x_0, y_0) \begin{cases} = 0, & p < n_0, \\ \neq 0, & p = n_0, \end{cases}$$

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and

(ii) *For* $0<i<k$ *and* $p<(1-i/k)n_0$

$$F_{x^{(p)}y^{(i)}}(x_0, y_0)=0.$$

PROOF. Assume $s_1=n_0/k$. By Leibniz's rule for derivatives of products we can express $F_{x^{(p)}y^{(i)}}(x_0, y_0)$ in terms of $P_{x^{(q)}y^{(j)}}(x_0, y_0)$ for $q \leqslant p$, $j \leqslant i$ and U and its derivatives. Inspection then shows (i) and (ii) of 2.4 hold as (i) and (ii) of 2.3 hold. (We use the fact that $(1-i/k)n_0$ is decreasing in i.)

Conversely (i) and (ii) of 2.4 imply (i) and (ii) of 2.3. If (i) of 2.3 fails, then we have

$$P_{x^{(p)}}(x_0, y_0)\begin{cases}=0, & p<n_0',\\ \neq 0, & p=n_0',\end{cases}$$

where $n_0' \neq n_0$; and hence by what we have just proved the analogous statement will be true for $F_{x^{(p)}}(x_0, y_0)$. If (i) of 2.3 holds but (ii) of 2.3 fails, let i_0 be the least integer for which it fails and let r be the least integer for which

$$P_{x^{(r)}y^{(i(0))}}(x_0, y_0) \neq 0.$$

Inspection of $F_{x^{(r)}y^{(i(0))}}(x_0, y_0)$ expressed in terms of

$$P_{x^{(q)}y^{(j)}}(x_0, y_0),\ q \leqslant r, j \leqslant i_0 \text{ and } U$$

and its derivatives shows it is also nonzero.

In the case where $s_1=n_0/k$ is not an integer it remains to compute a_1. If we substitute $f(x)$ for y in the power series expansion of $F(x,y)$ about (x_0, y_0), by 2.4 the lowest power of $(x-x_0)$ obtained is $(x-x_0)^{n_0}$. Since $F(x,f(x)) \equiv 0$ near x_0, we have by computing the coefficient of $(x-x_0)^n$ that

$$(*)\quad F_{x^{n_0}}\cdot\frac{1}{n_0!}+\sum\left\{F_{x^{((1-j/k)n_0}y^{(j))}}\frac{1}{((1-j/k)n_0)!j!}\left(\frac{a_1}{(-x_0)^{s_1}}\right)^j \middle| (1-j/k)\,n_0 \text{ is an integer, } 1 \leqslant j \leqslant k-1\right\}$$
$$+F_{y^{(k)}}\frac{1}{k!}\left(\frac{a_1}{(-x_0)^{s_1}}\right)^k=0,$$

where the derivatives are evaluated at (x_0, y_0).

It remains to solve this equation for a_1. In general we cannot determine which solution is the desired one. However, if there is only one solution for which $a_1/\Gamma(-s_1)$ is positive real, we may conclude

$$f_n \sim (a_1/\Gamma(-s_1))\, x_0^{-n}\, n^{-(1+s_1)}$$

For example:

THEOREM 2.5. *Let $\langle f_n \rangle_{n \geqslant 0}$ be nonnegative real numbers such that*

(i) *$f(x) = \sum_{n \geqslant 0} f_n x^n$ has a finite nonzero radius of convergence x_0 with its only singularity on the circle of convergence at $x = x_0$.*

(ii) *$f(x_0)$ converges to y_0.*

(iii) *There is a function $F(x, y)$ analytic near (x_0, y_0) and relatively prime integers n_0 and k such that*

$$(a)\ F(x, f(x)) \equiv 0 \quad \text{near } x_0,$$

$$(b)\ F_{y^{(j)}}(x_0, y_0) \begin{cases} = 0, & 0 \leqslant j \leqslant k-1, \\ \neq 0, & j = k, \end{cases}$$

$$(c)\ F_{x^{(p)}}(x_0, y_0) \begin{cases} = 0, & 0 \leqslant p < n_0, \\ \neq 0, & p = n_0, \end{cases}$$

$$(d)\ F_{x^{(p)} y^{(j)}}(x_0, y_0) = 0, \quad p < (1 - j/k) n_0,$$

then

$$f_n \sim (a_1 / \Gamma(-n_0/k)) \cdot x_0^{-n} n^{-(1+n_0/k)}.$$

where a_1 is the solution of

$$F_{x^{(n(0))}} \frac{1}{n_0!} + F_{y^{(k)}} \frac{1}{k!} \frac{a_1^k}{(-x_0)^{n_0}} = 0$$

such that $a_1/\Gamma(-n_0/k)$ is positive real.

PROOF. As k and n_0 are relatively prime $(1 - j/k)n_0$ is not an integer for $1 \leqslant j \leqslant k-1$. Thus (*) reduces to the above equation.

3

As in Section 2 we know s_1 is a positive rational number expressible with denominator $\leqslant k$. Say $s_1 = m/n$ in reduced form. s_1 is characterized by

LEMMA 3.1.

$$\lim_{x \to x_0^-} \frac{d^p}{dx^p} ((f(x) - y_0)^q) = \begin{cases} 0, & 0 < p/q < m/n, \\ \infty, & p/q > m/n, \\ \textit{finite, nonzero}, & p/q = m/n, \end{cases}$$

where p, q are positive integers. In particular if $p = m$ and $q = n$ the limit is $a_1^n m!/(-x_0)^m$.

PROOF. This may be proven by using the expansion of $f(x)-y_0$ as

$$\sum a_i\left(1-\frac{x}{x_0}\right)^{s_i}$$

to obtain a similar fractional power series expansion of

$$\frac{d^p}{dx^p}((f(x)-y_0)^q).$$

To determine if s_1 is nonintegral and in that case to obtain the asymptotic behavior of $\langle f_n\rangle_{n\geq 0}$ one may proceed as follows:

Step 1. Compute $F_{y^{(k)}}(x_0, y_0)$ until we find the least k for which it is nonzero. This gives the maximum possible denominator for s_1.

Step 2. Compute

$$\lim_{x\to x_0^-}\frac{d^l}{dx^l}(f(x)-y_0)$$

for each nonnegative integer l until either (i) the value is finite nonzero or (ii) the value is infinite. In case (i) we have s_1 is integral. We return to this case later. In case (ii) s_1 lies strictly between $l-1$ and l.

Step 3. There are only finitely many rational numbers between $l-1$ and l with denominator $\leqslant k$. Find out for which one, m/n, of these

$$\lim_{x\to x_0^-}\frac{d^m}{dx^m}((f(x)-y_0)^n)$$

is finite nonzero. This is s_1. a_1 is a solution of

$$\frac{a_1^n\, m!}{(-x_0)^m}=\lim_{x\to x_0^-}\frac{d^m}{dx^m}((f(x)-y_0)^n).$$

As in Section 2, since $f_n \sim (a_1/\Gamma(-s_1))\, x_0^{-n}x^{-(1+s_1)}$, a_1 must be the unique solution such that $a_1/\Gamma(-s_1)$ is positive real.

Observe in Step 2, if s_1 turns out to be an integer l, then a_1 is determined by

$$\frac{a_1\, l!}{(-x_0)^l}=\lim_{x\to x_0^-}\frac{d^l}{dx^l}((f(x)-y_0)).$$

In the case where s_1 is an integer we call the above three steps Stage 1.

Now let us assume for the sake of induction that $s_1,\ldots,s_t$ are integers and that we have found $a_1,\ldots,a_t$. We show how to determine whether s_{t+1} is an integer and how to find a_{t+1}.

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Let

$$L(x) = \sum_{i=1}^{t} a_i\left(1-\frac{x}{x_0}\right)^{s_i}$$

and let $g(x) = f(x) - L(x)$.

LEMMA 3.2. *s_{t+1} can be written with a denominator $\leqslant k$.*

PROOF. Let $P(x,y)$ be the Weierstrass polynomial of degree k in $(y-y_0)$ obtained in Section 1. Since $P(x,f(x)) \equiv 0$ near x_0 we have $P(x, g(x)+L(x)) \equiv 0$ near x_0. Since $s_1, \ldots, s_t$ are integers, by the binomial expansion of $(g(x)+L(x))^j$ we can obtain a similar function $Q(x,y)$ which is a monic polynomial of degree k in $(y-y_0)$ and such that $Q(x,g(x)) \equiv 0$ near x_0. We are now done by 2.1.

LEMMA 3.3.

$$\lim_{x \to x_0^-} \frac{d^p}{dx^p}((g(x)-y_0)^q) = \begin{cases} 0, & 0<p/q<m/n, \\ \infty, & p/q>m/n, \\ \text{finite, nonzero}, & p/q=m/n, \end{cases}$$

where p, q are integers. In particular if $p=m$, $q=n$ the limit is

$$\frac{(a_{t+1})^n m!}{(-x_0)^m}.$$

PROOF. This is proven just as in 3.1.

Stage $t+1$

Step 1. Compute

$$\lim_{x \to x_0^-} \frac{d^l}{dx^l}(g(x)-y_0)$$

for each nonnegative integer $l > s_t$ until either (i) the value is finite, nonzero or (ii) the value is infinite. In case (i) s_{t+1} is integral and we can find a_{t+1} from

$$\frac{a_{t+1}\, l!}{(-x_0)^l} = \lim_{x \to x_0^-} \frac{d^l}{dx^l}(g(x)-y_0).$$

In case (ii) s_{t+1} lies strictly between $l-1$ and l.

Step 2. This is just like Step 3 of Stage 1 except we use $g(x)$ throughout instead of $f(x)$ and we are determining s_{t+1} and a_{t+1} instead of s_1 and a_1.

https://doi.org/10.1017/S1446788700012593 Published online by Cambridge University Press

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https://doi.org/10.1017/S1446788700012593 Published online by Cambridge University Press