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ON STRONGLY NORMAL FUNCTIONS

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ABSTRACT. Loosely speaking, a function (meromorphic or harmonic) from the hyperbolic disk of the complex plane to the Riemann sphere is normal if its dilatation is bounded. We call a function strongly normal if its dilatation vanishes at the boundary. A sequential property of this class of functions is proved. Certain integral conditions, known to be sufficient for normality, are shown to be in fact sufficient for strong normality.

1. **Introduction.** Let $\mathbb{C}$ denote the complex plane, let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $D_r = \{z \in \mathbb{C} : |z| < r\}$. For a meromorphic function f, let

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denote its spherical derivative. We say that a complex-valued function f in D is *normal* if the family $\mathcal{F} = \{f(\gamma(z)) : \gamma \in \mathrm{Aut}(D)\}$ is a normal family in the sense of Montel, where $\mathrm{Aut}(D)$ denotes the collection of all one-to-one conformal mappings of D onto itself. It is known that a function f meromorphic in D is normal if and only if

$$\sup_{z\in D}(1 - |z|^2)f^{\#}(z) < \infty, \tag{1}$$

and that a real-valued function h harmonic in D is normal if and only if

$$\sup_{z\in D}(1 - |z|^2)\frac{|\mathrm{grad}\, h(z)|}{1 + h^2(z)} < \infty. \tag{2}$$

For $\gamma \in \mathrm{Aut}(D)$, setting $g(z) = f(\gamma(z))$, we have

$$(1 - |\gamma(z)|^2)f^{\#}(\gamma(z)) = (1 - |z|^2)g^{\#}(z).$$

In particular, if f is a meromorphic function automorphic with respect to a Fuchsian group Γ and $\gamma \in \Gamma$, then

$$\left(1 - |\gamma(z)|^2\right)f^{\#}(\gamma(z)) = (1 - |z|^2)f^{\#}(z).$$

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Thus, for an automorphic meromorphic function f, the above condition (1) can be replaced by

$$\sup_{z\in F}(1-|z|^2)f^{\#}(z)<\infty,$$

where F denotes a fundamental region for the Fuchsian group. Also, for an automorphic harmonic function h, the condition (2) can be replaced by

$$\sup_{z\in F}(1-|z|^2)\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}<\infty.$$

For the definitions and general properties of normal functions, see, for example, [5], [6] and [7].

There are some integral conditions for a function to be normal. Ch. Pommerenke [8] proved that an automorphic meromorphic function f in D which satisfies the condition

$$\iint_F\{f^{\#}(z)\}^2\,dx\,dy<\infty, \tag{3}$$

is normal, where F denotes a fundamental region for the corresponding Fuchsian group. In particular, a meromorphic function f in D with the property that

$$\iint_D\{f^{\#}(z)\}^2\,dx\,dy<\infty \tag{4}$$

is normal. R. Aulaskari, W. K. Hayman, and P. Lappan [1] proved that an automorphic meromorphic function f in D which satisfies

$$I=\iint_F(1-|z|^2)^{p-2}\{f^{\#}(z)\}^p\,dx\,dy<\infty, \tag{5}$$

for some $p>2$, is normal. Furthermore,

$$\sup_{z\in D}\big((1-|z|^2)f^{\#}(z)\big)\le 3\max(I^{1/p},I^{1/(p-2)}).$$

For harmonic functions, Aulaskari and Lappan [3] proved analogous results:

(1) A real-valued harmonic function h with the property

$$\iint_D\left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^2dx\,dy<\infty \tag{6}$$

is normal.

(2) If $f=h+i\tilde{h}$ is an automorphic holomorphic function such that

$$\iint_F\left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^2dx\,dy<\infty, \tag{7}$$

where F is a fundamental region, then h is normal.

In this paper, we distinguish a subclass of the class of normal meromorphic functions (Definitions 1 and 2). A function in this class is called a strongly normal function. We prove that the conditions (3) and (5) imply not only the normality of the function f but also its strong normality (Theorem 5). An example shows that the inverse is not true (Section 4). Similar definitions and results are given for harmonic functions (Definitions 3 and 4, Theorems 4 and 6). These Theorems strengthen the conclusions of the results mentioned above. Also, we obtain a sequential property of strongly normal functions. It turns out that, for a strongly normal function f, a sequence in the family $\mathcal{F} = \{f(\gamma(z)) : \gamma \in \mathrm{Aut}(D)\}$ contains a subsequence which converges to either $f(\gamma(z))$ with a $\gamma \in \mathrm{Aut}(D)$ or a constant identically (Theorems 1 and 2).

2. **Strongly normal functions.**

DEFINITION 1. A function f meromorphic in D is called a *strongly normal function* if

$$(1 - |z|^2)f^{\#}(z) \longrightarrow 0, \quad z \longrightarrow \partial D.$$

There are many normal meromorphic functions which are not strongly normal. In fact, every meromorphic function automorphic with respect to an infinite group is not strongly normal since $(1 - |z|^2)f^{\#}(z)$ has the same value at equivalent points. For example, the function $\exp[(z-1)/(z+1)]$ cannot be strongly normal.

DEFINITION 2. A function f meromorphic in D and automorphic with respect to a Fuchsian group Γ is called a *strongly normal function with respect to* Γ if

$$(1 - |z|^2)f^{\#}(z) \longrightarrow 0, \quad z \longrightarrow \partial D, \quad z \in F,$$

where F is a fundamental region of Γ.

DEFINITION 3. A real-valued function h harmonic in D is called a *strongly normal function* if

$$(1 - |z|^2)\frac{|\mathrm{grad}\, h(z)|}{1 + h^2(z)} \longrightarrow 0, \quad z \longrightarrow \partial D.$$

DEFINITION 4. A real-valued function h harmonic in D and automorphic with respect to a Fuchsian group Γ is called a *strongly normal function with respect to* Γ if

$$(1 - |z|^2)\frac{|\mathrm{grad}\, h(z)|}{1 + h^2(z)} \longrightarrow 0, \quad z \longrightarrow \partial D, \quad z \in F,$$

where F is a fundamental region for Γ.

As indicated above, even a bounded holomorphic function need not be strongly normal. However, if a function f meromorphic in D satisfies condition (4), that is, if its image has a finite spherical area, counted according to multiplicity, then it turns out that f is strongly normal. Also, condition (6) implies the strong normality of the harmonic

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function h. For automorphic meromorphic functions, conditions (3) and (5) imply the strong normality, with respect to the corresponding group, of the automorphic meromorphic function f, and condition (7) implies the strong normality of not only f but also the real part h. We will prove these propositions in the next section.

Returning to the normality of the family $\{f(\gamma(z)) : \gamma \in \mathrm{Aut}(D)\}$, we prove a sequential property of strongly normal functions.

THEOREM 1. *If a function f, meromorphic or harmonic in D, is strongly normal, then, for every sequence $\{\gamma_n\} \subset \mathrm{Aut}(D)$, there exists a subsequence $\{\gamma_{n_k}\}$ such that $\{f(\gamma_{n_k}(z))\}$ converges, locally uniformly in D, to either $f(\gamma_0(z))$ with $\gamma_0 \in \mathrm{Aut}(D)$ or a constant identically.*

PROOF. Assume first that f is a meromorphic function. Let

$$\gamma_n(z) = e^{i\theta_n} \cdot \frac{z + z_n}{1 + \bar{z}_n z}.$$

If $z_n \longrightarrow z_0 \in D$, then we may choose a subsequence $\{\gamma_{n_k}\}$ such that $\theta_{n_k} \longrightarrow \theta_0$. Consequently, $\gamma_{n_k} \longrightarrow \gamma_0$, where

$$\gamma_0 = e^{i\theta_0} \cdot \frac{z + z_0}{1 + \bar{z}_0 z},$$

and $f(\gamma_{n_k}(z)) \longrightarrow f(\gamma_0(z))$ locally uniformly in D. Otherwise, we have a subsequence γ_{n_k} such that $z_{n_k} \longrightarrow \partial D$ and $f(z_{n_k}) \longrightarrow w_0$.

Set $f(\gamma_{n_k}(z)) = g_{n_k}(z)$, we have

$$(1 - |z|^2) g^{\#}_{n_k}(z) = (1 - |\gamma_{n_k}(z)|^2) f^{\#}(\gamma_{n_k}(z)).$$

Given a positive number $r' < 1$, the above equality gives

$$\max_{z \in D_{r'}} (1 - |z|^2) g^{\#}_{n_k}(z) = \max_{z \in \gamma_{n_k}(D_{r'})} (1 - |z|^2) f^{\#}(z).$$

It is obvious that, for a fixed r', $\gamma_{n_k}(D_{r'})$ goes to the boundary ∂D of the unit disk as $k \longrightarrow \infty$. Thus,

$$\max_{z \in \gamma_{n_k}(D_{r'})} (1 - |z|^2) f^{\#}(z) \longrightarrow 0, \quad k \longrightarrow \infty,$$

and, consequently,

$$\max_{z \in D_{r'}} (1 - |z|^2) g^{\#}_{n_k}(z) \longrightarrow 0, \quad k \longrightarrow \infty.$$

Note that since $g_{n_k}(0) = f(\gamma_{n_k}(0)) = f(z_{n_k}) \longrightarrow w_0$, we know that $g_{n_k}(z)$ converges to w_0 uniformly for $z \in D_{r'}$. Since r' can be arbitrarily close to 1, this proves Theorem 1 for a meromorphic function f.

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The proof for a harmonic function f is exactly a repetition of the above reasoning with $f^{\#}$ and $g^{\#}_{n_k}$ replaced by $|\operatorname{grad} f|/(1+f^2)$ and $|\operatorname{grad} g_{n_k}|/(1+g^2_{n_k})$ respectively.

The next result asserts that the conclusion of Theorem 1 remains valid for a function which is strongly normal with respect to a Fuchsian group.

THEOREM 2. *If a function f, meromorphic or harmonic in D, is automorphic and strongly normal with respect to a Fuchsian group Γ, then, for every sequence $\{\gamma_n\} \in$ Aut(D), there exists a subsequence $\{\gamma_{n_k}\}$ such that $\{f(\gamma_{n_k}(z))\}$ converges, locally uniformly in D, to either $f(\gamma_0(z))$ with $\gamma_0 \in$ Aut(D) or a constant identically.*

PROOF. We consider the meromorphic case only. Let F be a fundamental region of Γ. Set $z_n = \gamma_n(0)$. Let z'_n be the equivalent point in F of z_n, say $\lambda_n(z_n) = z'_n$. Set $\phi_n(z) = \lambda_n(\gamma_n(z))$, we have

$$f(\gamma_n(z)) = f\Big(\lambda_n\big(\gamma_n(z)\big)\Big) = f\big(\phi_n(z)\big).$$

Only the case that $z'_n \longrightarrow \partial D$ needs a more careful discussion.

Fix a positive number r' with $0 < r' < 1$. Now let $\epsilon > 0$ and choose $0 < r_1 < 1$ such that $(1-|z|^2)f^{\#}(z) < \epsilon$ for $z \in F \setminus D_{r_1}$. Choose a positive number r_2 such that $r_1 < r_2 < 1$ and $d(r_1, r_2) = d(0, r')$, where $d(\cdot,\cdot)$ denotes the non-Euclidian distance. The set of points in F which are equivalent to points in D_{r_2} has a positive Euclidian distance to ∂D. Therefore, there exists a positive number r_3 such that $r_2 < r_3 < 1$ and there is no point in $F \setminus D_{r_3}$ equivalent to a point in D_{r_2}.

For sufficiently large n, we have $z'_n \in F \setminus D_{r_3}$. Assume that z' is an arbitrary point in the non-Euclidian disk $\delta_n = \{z \in D : d(z, z'_n) \le d(0, r')\}$. The unique point z'' in F equivalent to z' must lie outside of D_{r_1}. To show this, let $\lambda(z') = z''$ and $\lambda(z'_n) = z''_n$, where $\lambda \in \Gamma$. We have $d(z'', z''_n) = d(z', z'_n) \le d(0, r')$. Therefore, if $z'' \in D_{r_1}$, then $z''_n \in D_{r_2}$ for $d(r_1, r_2) = d(0, r')$. However, z'_n cannot be equivalent to a point in D_{r_2} because of the property of r_3. This proves that $z'' \in F \setminus D_{r_1}$ and consequently,

$$(1-|z'|^2)f^{\#}(z') = (1-|z''|^2)f^{\#}(z'') < \epsilon.$$

Now, set $g_n(z) = f\big(\phi_n(z)\big)$. Then $g_n(0) = f(z'_n)$. We have

$$\max_{z \in D_{r'}} (1-|z|^2)g^{\#}_n(z) = \max_{z \in \delta_n} (1-|z|^2)f^{\#}(z) < \epsilon.$$

Letting $\epsilon \longrightarrow 0$, we conclude that $g^{\#}_n(z)$ tends to 0 uniformly for $z \in D_{r'}$. Note that we may assume that $g_n(0) = f(z'_n) \longrightarrow w_0$ and r' can be arbitrarily close to 1. We arrive at the conclusion that g_n converges to w_0 locally uniformly in D. This completes the proof.

3. **Main results.**

To prove our main results, we make essential use of the following lemma due to Dufresnoy [4].

LEMMA 1. *Let f be a function meromorphic in the disk $D_r = \{z \in \mathbb{C} : |z| < r\}$. If*

$$\iint_{D_r} \{f^{\#}(z)\}^2 \, dx \, dy \leq \sigma\pi$$

with $0 \leq \sigma < 1$, then

$$f^{\#}(0) \leq \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

As a consequence of Lemma 1, we have the following.

LEMMA 2. *Let h be a real-valued function harmonic in D_r. If*

$$\iint_{D_r} \left\{\frac{|\operatorname{grad} h(z)|}{1 + h^2(z)}\right\}^2 dx \, dy \leq \sigma\pi,$$

with $0 \leq \sigma < 1$, then

$$\frac{|\operatorname{grad} h(0)|}{1 + h^2(0)} \leq \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

PROOF. Let $f = h + i\tilde{h}$ be a holomorphic function and $\tilde{h}(0) = 0$. Since

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \leq \frac{|\operatorname{grad} h(z)|}{1 + h^2(z)},$$

we have

$$\iint_{D_r} \{f^{\#}(z)\}^2 \, dx \, dy \leq \sigma\pi.$$

Thus, Lemma 1 gives

$$f^{\#}(0) \leq \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

Since

$$f^{\#}(0) = \frac{|\operatorname{grad} h(0)|}{1 + h^2(0)},$$

the conclusion of Lemma 2 follows.

Lemma 3 is a stronger version of Lemma 1 also proved by Dufresnoy.

LEMMA 3. *Let f be a function meromorphic in the disk D_r and let A denote the spherical area of $f(D_r)$, counted without consideration of multiplicity. If $A \leq \sigma\pi$ with $0 \leq \sigma < 1$, then*

$$f^{\#}(0) \leq \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

Now, we state and prove the main results.

THEOREM 3. *If f is a function meromorphic in D which satisfies condition (4), then f is a strongly normal function.*

PROOF. Fix a small positive number $\sigma < 1$. Let

$$\Delta(z') = \{z \in \mathbb{C} : |z - z'| < 1 - |z'|\}.$$

We have

$$\iint_{\Delta(z')} \{f^{\#}(z)\}^2 \, dx \, dy \leq \sigma\pi,$$

provided that z' is sufficiently close to ∂D. Applying Lemma 1 to the function $f(z)$ and the disk $\Delta(z')$, we obtain

$$f^{\#}(z') \leq \frac{1}{1 - |z'|} \left\{ \frac{\sigma}{1 - \sigma} \right\}^{1/2}.$$

Consequently, for z' sufficiently close to ∂D, we have

$$(1 - |z'|^2) f^{\#}(z') \leq 2 \left\{ \frac{\sigma}{1 - \sigma} \right\}^{1/2}.$$

Letting $\sigma \longrightarrow 0$ we obtain the conclusion of Theorem 3.

The same reasoning, using Lemma 2 instead of Lemma 1, gives the following similar result on harmonic functions.

THEOREM 4. *If h is a real-valued function harmonic in D which satisfies condition (6), then h is a strongly normal function.*

THEOREM 5. *Let f be a function meromorphic in D, which is automorphic with respect to a Fuchsian group Γ. If*

$$\iint_F (1 - |z|^2)^{p-2} \{f^{\#}(z)\}^p \, dx \, dy < \infty$$

for some $p \geq 2$, where F is a fundamental region of Γ, then f is a strongly normal function with respect to Γ.

PROOF. We may assume that $\bar{F}$ is not a compact subset of D. Fix a small positive number σ. There exists a number r_1 such that $0 < r_1 < 1$ and

$$\iint_{F \setminus D_{r_1}} (1 - |z|^2)^{p-2} \{f^{\#}(z)\}^p \, dx \, dy \leq \sigma\pi.$$

Let r_2 be the number such that $r_1 < r_2 < 1$ and $d(r_1, r_2) = 1$. There is a number r_3 with the property that no point in $F \setminus D_{r_3}$ is equivalent to a point in D_{r_2}.

For a point $z' \in F \setminus D_{r_3}$, set $\Delta(z') = \{z \in D : d(z, z') < 1\}$. It is easy to find a measurable set $E' \subset \Delta(z')$ such that no points in E' are equivalent and, for each point $z \in \Delta(z')$, there is a point $\zeta \in E'$ equivalent to z. We have $f\big(\Delta(z')\big) = f(E')$ for f is automorphic. Let E be the subset of F equivalent to E'. The same reasoning as in the proof of Theorem 2 shows that no point of $F \bigcap D_{r_1}$ is equivalent to any point in $\Delta(z')$ and, in particular, it follows that $E \subset F \setminus D_{r_1}$. Denoting by A the spherical area of $f\big(\Delta(z')\big) = f(E')$, without consideration of multiplicity, we have, by Hölder's inequality with respect to the measure $(1-|z|^2)^{-2}\,dx\,dy$,

$$\begin{aligned}
A &\leq \iint_{E'} \{f^{\#}(z)\}^2\,dx\,dy = \iint_{E'} \{(1-|z|^2)f^{\#}(z)\}^2 \frac{dx\,dy}{(1-|z|^2)^2} \\
&\leq \left\{\iint_{E'} \{(1-|z|^2)f^{\#}(z)\}^p \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{2/p} \left\{\iint_{E'} \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{1-2/p} \\
&= \left\{\iint_{E} \{(1-|z|^2)f^{\#}(z)\}^p \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{2/p} \left\{\iint_{E'} \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{1-2/p} \\
&\leq \left\{\iint_{F\setminus D_{r_1}} (1-|z|^2)^{p-2}\{f^{\#}(z)\}^p\,dx\,dy\right\}^{2/p} \left\{\iint_{\Delta(z')} \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{1-2/p} \\
&\leq (\sigma\pi)^{2/p} C^{1-2/p},
\end{aligned}$$

where the absolute constant C is the non-Euclidian area of a disk of non-Euclidian radius 1.

Now, let $\gamma \in \operatorname{Aut}(D)$ be a transformation satisfying $\gamma(0) = z'$. Then γ maps some disk D_r onto $\Delta(z')$, where r is the absolute constant such that $d(0,r) = 1$. Applying Lemma 3 to the function $g(z) = f\big(\gamma(z)\big)$, we obtain

$$(1-|z'|^2)f^{\#}(z') = g^{\#}(0) \leq \frac{1}{r}\left\{\frac{\sigma^{2/p}(C/\pi)^{1-2/p}}{1-\sigma^{2/p}(C/\pi)^{1-2/p}}\right\}^{1/2}.$$

Since σ may be arbitrarily small, the proof of Theorem 5 is complete.

THEOREM 6. *Let $f = h + i\tilde{h}$ be a function holomorphic in D, which is automorphic with respect to a Fuchsian group Γ. If*

$$\iint_F (1-|z|^2)^{p-2}\left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^p dx\,dy < \infty,$$

where F is a fundamental region of Γ, then h is a strongly normal function with respect to Γ, and consequently, f is a strongly normal function with respect to Γ.

The proof of Theorem 6 is almost the same as that of Theorem 5. First, we may assume that $\bar{F}$ is not a compact subset of D and, for a fixed positive number $\sigma < 1$, we have positive numbers r_1, r_2 and r_3 such that

$$\iint_{F\setminus D_{r_1}} (1-|z|^2)^{p-2}\left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^p dx\,dy \leq \sigma\pi,$$

$r_1 < r_2 < 1$, $d(r_1, r_2) = 1$, and no point in $F \setminus D_{r_3}$ is equivalent to a point in D_{r_2}. For $z' \in F \setminus D_{r_3}$, set $f_1(z) = f(z) - i\tilde{h}(z')$ and $\Delta(z') = \{z \in D : d(z,z') < 1\}$. We have E' and E such that $E' \subset \Delta(z')$, $f_1\big(\Delta(z')\big) = f_1(E')$, and $E \subset F \setminus D_{r_1}$ is equivalent to E'. Denoting

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by A the spherical area of $f_1\big(\Delta(z')\big) = f_1(E')$, without consideration of multiplicity, we have

$$\begin{aligned}
A &\leq \iint_{E'} \{f_1^{\#}(z)\}^2\, dx\, dy \\
&\leq \left\{ \iint_{F\setminus D_{r_1}} (1-|z|^2)^{p-2} \{f_1^{\#}(z)\}^p\, dx\, dy \right\}^{2/p} \left\{ \iint_{\Delta(z')} \frac{dx\, dy}{(1-|z|^2)^2} \right\}^{1-2/p} \\
&\leq \left\{ \iint_{F\setminus D_{r_1}} (1-|z|^2)^{p-2} \left\{ \frac{|\mathrm{grad}\, h(z)|}{1+h^2(z)} \right\}^p dx\, dy \right\}^{2/p} \left\{ \iint_{\Delta(z')} \frac{dx\, dy}{(1-|z|^2)^2} \right\}^{1-2/p} \\
&\leq (\sigma\pi)^{2/p} C^{1-2/p}.
\end{aligned}$$

Now, just as in the proof of Theorem 5, we have

$$(1-|z'|^2) f_1^{\#}(z') \leq \frac{1}{r} \left\{ \frac{\sigma^{2/p}(C/\pi)^{1-2/p}}{1-\sigma^{2/p}(C/\pi)^{1-2/p}} \right\}^{1/2}.$$

Note that, since

$$f_1^{\#}(z') = \frac{|\mathrm{grad}\, h(z')|}{1+h^2(z')},$$

we have

$$(1-|z'|^2)\frac{|\mathrm{grad}\, h(z')|}{1+h^2(z')} \leq \frac{1}{r} \left\{ \frac{\sigma^{2/p}(C/\pi)^{1-2/p}}{1-\sigma^{2/p}(C/\pi)^{1-2/p}} \right\}^{1/2}.$$

Letting $\sigma \longrightarrow 0$ completes the proof of Theorem 6.

4. **Remarks.** (1) Theorem 3 is a special case of Theorem 5. In fact, Theorem 3 is also a special case of a theorem proved by Aulaskari, Hayman, and Lappan[1], which states that if f is a rotation automorphic function with respect to a finitely generated Fuchsian group Γ and

$$\iint_F (1-|z|^2)^{p-2} \{f^{\#}(z)\}^p\, dx\, dy < \infty$$

for some $p \geq 2$, where F is a fundamental region of Γ, then f is a normal function and also

$$\lim_{|z|\to 1, z\in F} (1-|z|^2) f^{\#}(z) = 0.$$

(2) The examples $f_n(z) = (z-1)^{1/n}$, for $n = 1, 2, \cdots$, show that the conclusion of Theorem 3 is sharp in the sense that $(1-|z|^2)f^{\#}(z)$ cannot be replaced by $(1-|z|^2)^{1-\epsilon} f^{\#}(z)$ with an arbitrarily small $\epsilon > 0$. In fact, the functions $f_n(z)$ are all bounded univalent functions, so they have images of finite spherical area. On the other hand, a simple calculation gives, for $0 < x < 1$,

$$(1-x^2)^{1-\epsilon} f_n^{\#}(x) \geq \frac{1}{5n}(1-x)^{1/n-\epsilon}.$$

For a given ϵ, taking n so large that $1/n < \epsilon$, we have

$$(1-x^2)^{1-\epsilon} f_n^{\#}(x) \longrightarrow \infty, \quad x \longrightarrow 1.$$

(3) The following example shows that the converse of Theorem 3 is not true. Consider the functions

$$g(z) = \exp\left(\frac{z+1}{z-1}\right), \quad f(z) = (z-1)^{1/2} g(z).$$

We have

$$g'(z) = -\frac{2g(z)}{(z-1)^2},$$

$$f'(z) = (z-1)^{1/2} g'(z) + \frac{1}{2}(z-1)^{-1/2} g(z) = \frac{1}{2}(z-5)(z-1)^{-3/2} g(z).$$

Thus, for $z \in D$,

$$f^{\#}(z) \leq |f'(z)| = \frac{3}{|z-1|^{3/2}} \exp\left(-\frac{1-|z|^2}{|z-1|^2}\right)$$

$$(1-|z|^2)^{3/4} f^{\#}(z) \leq 3\left(\frac{1-|z|^2}{|z-1|^2}\right)^{3/4} \exp\left(-\frac{1-|z|^2}{|z-1|^2}\right).$$

Hence, letting

$$M = \max_{r>0} r^{3/4} e^{-r},$$

we have

$$(1-|z|^2)^{3/4} f^{\#}(z) \leq 3M$$

for $z \in D$.

On the other hand, $f(z)$ has an image of infinite spherical area. To show this, we go to the left half plane $L = \{\zeta \in \mathbb{C} : \Re\zeta < 0\}$. Set

$$\zeta = \frac{z+1}{z-1}.$$

Then

$$z - 1 = \frac{2}{\zeta - 1}, \quad f(z) = \phi(\zeta) = \frac{2^{1/2} e^{\zeta}}{(\zeta-1)^{1/2}}.$$

What we want to prove is that $\phi(\zeta)$ maps L onto a covering surface with infinite spherical area, that is,

$$A = \iint_L \{\phi^{\#}(\zeta)\}^2 \, d\xi \, d\eta = \infty.$$

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For $\zeta \in L$, we have

$$\phi'(\zeta) = \frac{e^{\zeta}}{2^{1/2}(\zeta-1)^{1/2}}\left(2 - \frac{1}{\zeta-1}\right),$$

$$|\phi'(\zeta)| \geq \frac{|e^{\zeta}|}{2^{1/2}|\zeta-1|^{1/2}}, \quad |\phi^{\#}(\zeta)| \geq \frac{|e^{\zeta}|}{5|\zeta-1|^{1/2}}.$$

Let $\zeta = \xi + i\eta$ and let

$$G_n = \{\zeta \in \mathbb{C} : -1 < \xi < 0, 2n\pi < \eta < 2(n+1)\pi\}$$

for $n = 0, 1, \cdots$. Then,

$$\begin{aligned} A &\geq \sum_{n=0}^{\infty} \iint_{G_n} \{\phi^{\#}(\zeta)\}^2 \, d\xi \, d\eta \geq \frac{1}{25} \sum_{n=0}^{\infty} \iint_{G_n} \frac{|e^{\zeta}|^2}{|\zeta-1|} \, d\xi \, d\eta \\ &\geq \frac{1}{50} \sum_{n=0}^{\infty} \frac{1}{\sqrt{1+(n+1)^2\pi^2}} \iint_{G_n} |e^{\zeta}|^2 \, d\xi \, d\eta \\ &= \frac{1}{50}\pi(1-e^{-2}) \sum_{n=0}^{\infty} \frac{1}{\sqrt{1+(n+1)^2\pi^2}} = \infty. \end{aligned}$$

(4) We cannot omit the assumption that $\tilde{h}$ is also automorphic with respect to Γ in Theorem 6. Indeed, Aulaskari and Lappan [2] constructed a holomorphic function $W(z)$ with the following properties:

(1) the real part $h(z)$ is automorphic with respect to a Fuchsian group Γ;
(2) $\iint_F |W'(z)|^2 \, dx \, dy < \infty$, where F is a fundamental region of Γ;
(3) $W(z)$ and consequently $h(z)$ is not normal.

NOTE. Professor Joel Schiff, after receiving a preliminary version of the present paper, wrote us that it was proved on pages 39 and 180 of his book [9], that Dirichlet-finite functions on D are normal and that it would appear from the proof that such functions are actually strongly normal, and in fact, strongly Bloch.

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