Math. Proc. Camb. Phil. Soc.: page 1 of 26
doi:10.1017/S0305004122000159

Higher rank motivic Donaldson–Thomas invariants of $\mathbb{A}^3$ via wall-crossing, and asymptotics

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(*Received* 20 *April* 2020*; revised* 16 *December* 2021*; accepted* 24 *March* 2022)

In memory of Professor Marco Andrea Garuti

Abstract

We compute, via motivic wall-crossing, the generating function of *virtual motives* of the Quot scheme of points on $\mathbb{A}^3$, generalising to higher rank a result of Behrend–Bryan–Szendrői. We show that this motivic partition function converges to a Gaussian distribution, extending a result of Morrison.

2020 Mathematics Subject Classification: 14N35 (Primary); 11N45 (Secondary)

0. *Introduction*

This paper has a two-fold goal: to compute, and to study the asymptotic behaviour of the generating function of rank r motivic Donaldson–Thomas (DT in short) invariants of $\mathbb{A}^3$, namely the generating series

$$\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3,q)=\sum_{n\geq 0}\left[\text{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)\right]_{\text{vir}}\cdot q^n\ \in\ \mathcal{M}_{\mathbb{C}}[\![q]\!].$$

Here $\mathcal{M}_{\mathbb{C}}$ is a suitable motivic ring and $[\cdot]_{\text{vir}}\in\mathcal{M}_{\mathbb{C}}$ is the *virtual motive* (cf. Section 1·1), induced by the critical locus structure on the Quot scheme $\text{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)$ parametrising 0-dimensional quotients of length n of the free sheaf $\mathscr{O}^{\oplus r}$.

The following is our first main result.

THEOREM 0·1. *There is an identity*

$$\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3, q) = \prod_{m\geq 1}\prod_{k=0}^{rm-1}\left(1-\mathbb{L}^{2+k-\frac{rm}{2}}q^m\right)^{-1}. \tag{0·1}$$

Moreover, this series factors as r copies of shifted rank 1 contributions: there is an identity

$$\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3, q) = \prod_{i=1}^{r}\mathsf{DT}_1^{\text{points}}\left(\mathbb{A}^3, q\mathbb{L}^{\frac{-r-1}{2}+i}\right). \tag{0·2}$$

The result was first obtained in the case $r=1$ by Behrend, Bryan and Szendrői [3] via an explicit motivic vanishing cycle calculation. Formula (0·2) follows by combining Formula (0·1) and Lemma 2.5 with one another. The approach of Section 2·3, where we prove Formula (0·1), is based on the techniques of *motivic wall-crossing* for framed objects developed by Mozgovoy [27, 28] building on the Kontsevich–Soibelman wall-crossing [20]. This approach allows us us to express the invariants for $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$, which we view as '$r$-framed' DT invariants, in terms of the universal series of the invariants of *unframed* representations of the 3-loop quiver in a critical chamber. These ideas can be employed to compute *framed motivic DT invariants* of small crepant resolutions of affine toric Calabi–Yau 3-folds [8], which also exhibit similar factorisation properties.

The fact that partition functions of rank r invariants factor as r copies of partition functions of rank 1 invariants, shifted just as in Formula (0·2), has also been observed in the context of K-theoretic DT theory of $\mathbb{A}^3$ [14], as well as in string theory [29] and other motivic settings, such as [8, 23]. Recent work of Feyzbakhsh–Thomas [16] shows that general higher rank DT invariants can be linked back to genuine rank 1 DT invariants: we believe our Formula (0·2) is an explicit (motivic) shadow of this general principle. The exponential form of Formula (0·1) has been exploited in [35] to define higher rank motivic DT invariants for an arbitrary smooth quasiprojective 3-fold.

Formula (0·1) allows us to interpret the refined DT invariants of $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ in terms of a weighted count of r-tuples of plane partitions $\overline{\pi} = (\pi_1, \ldots, \pi_r)$ of total size n (also known in the physics literature as *r-coloured plane partitions*). Setting $T = \mathbb{L}^{1/2}$, the coefficient of q^n in $\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3, q)$ can be written as

$$M_{n,r}(T) = \sum_{\overline{\pi}} T^{S_{n,r}(\overline{\pi})}, \tag{0·3}$$

where $S_{n,r}$ is a certain explicit random variable on the space of r-tuples of plane partitions. In Section 3, we describe the asymptotic behaviour of (a renormalisation of) the refined DT generating series, generalising A. Morrison's result for $r=1$ [24]. We discuss the relationship with Morrison's work in Section 3·1.

The following is our second main result. It will be proved in Theorem 3.2 in the main body.

THEOREM 0·2. *As $n\to\infty$, the normalised random variable $n^{-2/3}S_{n,r}$ converges in distribution to $\mathcal{N}(\mu, \sigma^2)$ with*

$$\mu = \frac{r^{1/3}\pi^2}{2^{5/3}(\zeta(3))^{2/3}} \quad \text{and} \quad \sigma^2 = \frac{r^{5/3}}{(2\zeta(3))^{1/3}},$$

where $\zeta(s)$ is Riemann's zeta function.

In the final part of the introduction, we discuss some relation with existing work on asymptotics of plane partitions and counting invariants, from a mathematical and physics perspective.

The asymptotics of (various refinements of) the generating function of plane partitions has attracted quite a lot of attention in the mathematical community, since the objects involved sit in the intersection between several areas of mathematics: random variables and distributions, combinatorics, algebraic geometry, string theory. See [36] for E. M. Wright's seminal work on the asymptotics of actual partitions, without refinements. Morrison based his work [24] on Wright's analysis, and our analysis in turn builds on Morrison's. As far as the ubiquity of Gaussian distributions is concerned, besides Morrison's paper we would like to mention the work by Panario–Richmond–Young [30] who, on the other hand, refine the plane partition counts by considering a colouring of the parts $\pi_{i,j}$ of a plane partition π according to the parity of $i-j$, in order to study the asymptotic behaviour of the difference between the 0-coloured and 1-coloured partition functions: the result is (asymptotycally) Gaussian, and geometrically corresponds to the 3-dimensional orbifold $\mathbb{A}^1 \times [\mathbb{A}^2/\mathbb{Z}_2]$. From the physics literature, we refer the reader to [37] and the references therein for a related analysis of the asymptotics of BPS states starting from a quiver perspective.

The higher rank motivic DT partition function of $\mathbb{A}^3$ is rather similar in shape to the K-theoretic partition function [14, theorem A], although the latter is defined via torus localisation, and no torus-equivariance is present on the motivic side. However, given the similarity, one could still ask: what is the K-theoretic counterpart of our Theorem 0·2? Although we do not have a definite answer, it is worth noting that the K-theoretic DT invariants also satisfy a remarkable 'regularity' property: the coefficient of q^n in the K-theoretic DT series can be written as

$$M^{\mathrm{K}}_{n,r}(t_1,t_2,t_3) = \sum_{\overline{\pi}} \mathsf{Y}_{\overline{\pi}}(t_1,t_2,t_3,w_1,\ldots,w_r,\kappa), \qquad \kappa = (t_1t_2t_3)^{\frac{1}{2}},$$

where t_i and w_j are equivariant parameters of a torus $\mathbf{T} = \mathbb{G}_m^3 \times \mathbb{G}_m^r$ acting on $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$, and $\mathsf{Y}_{\overline{\pi}}$ is a (rather complicated) function of these parameters. The remarkable property (see [14, theorem 6.5]) is that – as suggested by the notation – the right – hand side *does not depend on* $w_1,\ldots,w_r$, even though the individual summands do. This phenomenon is known in string theory as 'independence on Coulomb moduli' [12].

Whether this symmetry property, used crucially for the computation of K-theoretic DT invariants, is an actual shadow of a Gaussian distribution, is a mistery that we leave for the future to explore.

1. *Background material*

1·1. *Rings of motives*

Let $K_0(St_{\mathbb{C}})$ be the Grothendieck ring of stacks. It can be defined as the localisation of the ordinary Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$ at the classes $[\mathrm{GL}_k]$ of general linear groups [4]. The invariants we want to study will live in the extended ring

$$\mathcal{M}_{\mathbb{C}} = K_0(\mathrm{St}_{\mathbb{C}})\big[\mathbb{L}^{-\frac{1}{2}}\big],$$

where $\mathbb{L} = [\mathbb{A}^1] \in K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{St}_{\mathbb{C}})$ is the Lefschetz motive.

1·1·1. *The virtual motive of a critical locus*

Let U be a smooth d-dimensional $\mathbb{C}$-scheme, $f: U \to \mathbb{A}^1$ a regular function. The *virtual motive* of the critical locus $\operatorname{crit} f = Z(\mathrm{d}f) \subset U$, depending on the pair (U, f), is defined in [3] as the motivic class

$$[\operatorname{crit} f]_{\mathrm{vir}} = -\mathbb{L}^{-\frac{d}{2}} \cdot [\phi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}},$$

where $[\phi_f] \in K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ is the (absolute) motivic vanishing cycle class defined by Denef and Loeser [13]. The '$\hat{\mu}$' decoration means that we are considering $\hat{\mu}$-equivariant motives, where $\hat{\mu}$ is the group of all roots of unity. However, the motivic invariants studied here will live in the subring $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ of classes carrying the trivial $\hat{\mu}$-action.

Example 1·1. Set $f = 0$. Then $\operatorname{crit} f = U$ and $[U]_{\mathrm{vir}} = \mathbb{L}^{-(\dim U)/2} \cdot [U]$. For instance, $[\mathrm{GL}_k]_{\mathrm{vir}} = \mathbb{L}^{-k^2/2} \cdot [\mathrm{GL}_k]$.

Remark 1·2. We use lambda-ring conventions on $\mathcal{M}_{\mathbb{C}}$ from [3, 11]. In particular we use the definition of $[\operatorname{crit} f]_{\mathrm{vir}}$ from [3, section 2.8], which differs (slightly) from the one in [25]. The difference amounts to the substitution $\mathbb{L}^{1/2} \to -\mathbb{L}^{1/2}$. The Euler number specialisation with our conventions is $\mathbb{L}^{1/2} \to -1$.

1·2. *Quivers: framings, and motivic quantum torus*

A quiver Q is a finite directed graph, determined by its sets Q_0 and Q_1 of vertices and edges, respectively, along with the maps h, $t: Q_1 \to Q_0$ specifying where an edge starts or ends. We use the notation

$$t(a) \bullet \xrightarrow{\quad a \quad} \bullet\, h(a)$$

to denote the *tail* and the *head* of an edge $a \in Q_1$.

All quivers in this paper will be assumed connected. The *path algebra* $\mathbb{C}Q$ of a quiver Q is defined, as a $\mathbb{C}$-vector space, by using as a $\mathbb{C}$-basis the set of all paths in the quiver, including a trivial path ϵ_i for each $i \in Q_0$. The product is defined by concatenation of paths whenever the operation is possible, and is set to be 0 otherwise. The identity element is $\sum_{i \in Q_0} \epsilon_i \in \mathbb{C}Q$.

On a quiver Q one can define the *Euler–Ringel form* $\chi_Q(-,-): \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ by

$$\chi_Q(\alpha, \beta) = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)},$$

as well as the skew-symmetric form

$$\langle \alpha, \beta \rangle_Q = \chi_Q(\alpha, \beta) - \chi_Q(\beta, \alpha).$$

Definition 1·3. (r-framing) Let Q be a quiver with a distinguished vertex $0 \in Q_0$, and let r be a positive integer. We define the quiver $\widetilde{Q}$ by adding one vertex, labelled ∞, to the original vertices in Q_0, and r edges $\infty \to 0$. We refer to $\widetilde{Q}$ as the r-framed quiver obtained out of $(Q, 0)$.

https://doi.org/10.1017/S0305004122000159 Published online by Cambridge University Press

Let Q be a quiver. Define its *motivic quantum torus* (or *twisted motivic algebra*) as

$$\mathcal{T}_Q = \prod_{\alpha\in\mathbb{N}^{Q_0}} \mathcal{M}_{\mathbb{C}} \cdot y^{\alpha}$$

with product

$$y^{\alpha} \cdot y^{\beta} = \mathbb{L}^{\frac{1}{2}\langle\alpha,\beta\rangle_Q} y^{\alpha+\beta}. \tag{1·1}$$

If $\widetilde{Q}$ is the r-framed quiver associated to $(Q, 0)$, one has a decomposition

$$\mathcal{T}_{\widetilde{Q}} = \left(\prod_{d\geq 0} \mathcal{M}_{\mathbb{C}} \cdot y_{\infty}^{d}\right) \oplus \mathcal{T}_Q,$$

where we have set $y_{\infty} = y^{(1,\mathbf{0})}$. A generator $y^{\alpha} \in \mathcal{T}_Q$ will be identified with its image $y^{(0,\alpha)} \in \mathcal{T}_{\widetilde{Q}}$.

1·3. *Quiver representations and their stability*

Let Q be a quiver. A *representation* ρ of Q is the datum of a finite dimensional $\mathbb{C}$-vector space ρ_i for every vertex $i \in Q_0$, and a linear map $\rho(a)\colon \rho_i \to \rho_j$ for every edge $a\colon i \to j$ in Q_1. The *dimension vector* of ρ is $\underline{\dim}\,\rho = (\dim_{\mathbb{C}} \rho_i)_i \in \mathbb{N}^{Q_0}$, where $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

Convention 1·1. Let Q be a quiver, $\widetilde{Q}$ its r-framing. The dimension vector of a representation $\widetilde{\rho}$ of $\widetilde{Q}$ will be denoted by (d, α), where $d = \dim_{\mathbb{C}} \widetilde{\rho}_{\infty} \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{Q_0}$.

The space of all representations of Q with a fixed dimension vector $\alpha \in \mathbb{N}^{Q_0}$ is the affine space

$$\mathrm{R}(Q, \alpha) = \prod_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{t(a)}}, \mathbb{C}^{\alpha_{h(a)}}).$$

The gauge group $\mathrm{GL}_{\alpha} = \prod_{i\in Q_0} \mathrm{GL}_{\alpha_i}$ acts on $\mathrm{R}(Q, \alpha)$ by $(g_i)_i \cdot (\rho(a))_{a\in Q_1} = (g_{h(a)} \circ \rho(a) \circ g_{t(a)}^{-1})_{a\in Q_1}$.

Following [25], we recall the notion of (semi)stability of a representation.

Definition 1·4. A *central charge* is a group homomorphism $\mathrm{Z}\colon \mathbb{Z}^{Q_0} \to \mathbb{C}$ such that the image of $\mathbb{N}^{Q_0} \setminus 0$ lies inside $\mathbb{H}_+ = \{re^{\sqrt{-1}\pi\varphi} | r > 0,\ 0 < \varphi \leq 1\}$. For every $\alpha \in \mathbb{N}^{Q_0} \setminus 0$, we denote by $\varphi(\alpha)$ the real number φ such that $\mathrm{Z}(\alpha) = re^{\sqrt{-1}\pi\varphi}$. It is called the phase of α with respect to Z.

Note that every vector $\zeta \in \mathbb{R}^{Q_0}$ induces a central charge Z_{ζ} if we set $\mathrm{Z}_{\zeta}(\alpha) = -\zeta \cdot \alpha + |\alpha|\sqrt{-1}$, where $|\alpha| = \sum_{i\in Q_0} \alpha_i$. We denote by φ_{ζ} the induced phase function, and we set $\varphi_{\zeta}(\rho) = \varphi_{\zeta}(\underline{\dim}\,\rho)$ for every representation ρ of Q.

Definition 1·5. Fix $\zeta \in \mathbb{R}^{Q_0}$. A representation ρ of Q is called ζ-semistable if

$$\varphi_{\zeta}(\rho') \leq \varphi_{\zeta}(\rho)$$

for every nonzero proper subrepresentation $0 \neq \rho' \subsetneq \rho$. If '$\leq$' can be replaced by '$<$', we say that ρ is ζ-stable. Vectors $\zeta \in \mathbb{R}^{Q_0}$ are referred to as stability parameters.

Definition 1·6. Let $\alpha \in \mathbb{N}^{Q_0}$ be a dimension vector. A stability parameter ζ is called α-generic if for any $0 < \beta < \alpha$ one has $\varphi_\zeta(\beta) \neq \varphi_\zeta(\alpha)$.

The sets of ζ-stable and ζ-semistable representations with given dimension vector α form a chain of open subsets

$$\mathrm{R}^{\zeta\text{-st}}(Q,\alpha) \subset \mathrm{R}^{\zeta\text{-ss}}(Q,\alpha) \subset \mathrm{R}(Q,\alpha).$$

If ζ is α-generic, one has $\mathrm{R}^{\zeta\text{-st}}(Q,\alpha) = \mathrm{R}^{\zeta\text{-ss}}(Q,\alpha)$.

1·4. *Quivers with potential*

Let Q be a quiver, and let $\mathbb{C}Q$ be the path algebra of Q. A finite linear combination of cyclic paths $W \in \mathbb{C}Q$ is called a *potential*. Given a cyclic path w and an arrow $a \in Q_1$, one defines the noncommutative derivative

$$\frac{\partial w}{\partial a} = \sum_{\substack{w=cac' \\ c,c' \text{ paths in } Q}} c'c \ \in \ \mathbb{C}Q.$$

This rule extends to an operator $\partial/\partial a$ acting on every potential. The *Jacobi algebra* $J = J_{Q,W}$ of (Q, W) is the quotient of $\mathbb{C}Q$ by the two-sided ideal generated by $\partial W/\partial a$ for all edges $a \in Q_1$. For every $\alpha \in \mathbb{N}^{Q_0}$, the potential $W = \sum_c a_c c$ determines a regular function

$$f_\alpha : \mathrm{R}(Q,\alpha) \to \mathbb{A}^1, \quad \rho \longmapsto \sum_{c \text{ cycle in } Q} a_c \mathrm{Tr}(\rho(c)).$$

The points in the critical locus $\mathrm{crit} f_\alpha \subset R(Q,\alpha)$ correspond to α-dimensional *J-modules*.

Fix an α-generic stability parameter $\zeta \in \mathbb{R}^{Q_0}$. If $f_{\zeta,\alpha} : \mathrm{R}^{\zeta\text{-st}}(Q,\alpha) \to \mathbb{A}^1$ is the restriction of f_α, then

$$\mathfrak{M}(J,\alpha) = [\mathrm{crit} f_\alpha / \mathrm{G}_\alpha], \quad \mathfrak{M}_\zeta(J,\alpha) = [\mathrm{crit} f_{\zeta,\alpha}/\mathrm{GL}_\alpha]$$

are, by definition, the stacks of α-dimensional J-modules and ζ-stable J-modules.

Definition 1·7. A quiver with potential (Q, W) admits a cut if there is a subset $I \subset Q_1$ such that every cyclic monomial appearing in W contains exactly one edge in I.

From now on we assume (Q, W) admits a cut. This condition ensures that the motive $[\mathfrak{M}(J,\alpha)]_{\mathrm{vir}}$ introduced in the next definition lives in $\mathcal{M}_{\mathbb{C}}$. The 3-loop quiver L_3 with the cubic potential $W = A_3[A_1, A_2]$ has $I = \{A_3\}$ as a cut. We go back to this in Example 1·13.

Definition 1·8 ([25]). We define motivic Donaldson–Thomas invariants

$$\left[\mathfrak{M}(J,\alpha)\right]_{\mathrm{vir}} = \frac{\left[\mathrm{crit} f_\alpha\right]_{\mathrm{vir}}}{[\mathrm{GL}_\alpha]_{\mathrm{vir}}}, \qquad \left[\mathfrak{M}_\zeta(J,\alpha)\right]_{\mathrm{vir}} = \mathbb{L}^{\frac{1}{2}\chi(\alpha,\alpha)} \frac{\left[f_{\zeta,\alpha}^{-1}(0)\right] - \left[f_{\zeta,\alpha}^{-1}(1)\right]}{[\mathrm{GL}_\alpha]}$$

in $\mathcal{M}_{\mathbb{C}}$, where $[\mathrm{GL}_\alpha]_{\mathrm{vir}}$ is taken as in Example 1·1. The generating function

$$A_U = \sum_{\alpha \in \mathbb{N}^{Q_0}} \left[\mathfrak{M}(J,\alpha)\right]_{\mathrm{vir}} \cdot y^\alpha \ \in \ \mathcal{T}_Q \tag{1·2}$$

is called the universal series of (Q, W).

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By [25, lemma 1.9], since we assumed the existence of a cut, we have an identity $\left[\mathfrak{M}_\zeta(J,\alpha)\right]_{\mathrm{vir}} = [\operatorname{crit} f_{\zeta,\alpha}]_{\mathrm{vir}}/[\mathrm{G}_\alpha]_{\mathrm{vir}}$ as long as α has 1 as one of its components.

Definition 1·9. A stability parameter $\zeta \in \mathbb{R}^{Q_0}$ is called generic if $\zeta \cdot \underline{\dim}\,\rho \neq 0$ for every nontrivial ζ-stable J-module ρ.

1·5. *Framed motivic DT invariants*

Let Q be a quiver, $r \geq 1$ be an integer, and consider its r-framing $\widetilde{Q}$ with respect to a vertex $0 \in Q_0$ (Definition 1·3). A representation $\widetilde{\rho}$ of $\widetilde{Q}$ can be uniquely written as a pair (u, ρ), where ρ is a representation of Q and $u = (u_1, \ldots, u_r)$ is an r-tuple of linear maps $u_i\colon \widetilde{\rho}_\infty \to \rho_0$. From now on, we assume our framed representations to satisfy $\dim_{\mathbb{C}} \widetilde{\rho}_\infty = 1$, so that according to Convention 1·1 we can write $\underline{\dim}\,\widetilde{\rho} = (1, \underline{\dim}\,\rho)$. We also view ρ as a subrepresentation of $\widetilde{\rho}$ of dimension $(0, \underline{\dim}\,\rho)$, based at the vertex $0 \in Q_0$.

Definition 1·10. Fix $\zeta \in \mathbb{R}^{Q_0}$. A representation (u, ρ) of $\widetilde{Q}$ (resp. a $\widetilde{J}$-module) with $\dim_{\mathbb{C}} \widetilde{\rho}_\infty = 1$ is said to be ζ-(semi)stable if it is (ζ_∞, ζ)-(semi)stable in the sense of Definition 1·5, where $\zeta_\infty = -\zeta \cdot \underline{\dim}\,\rho$.

We now define motivic DT invariants for moduli stacks of r-framed representations of a given quiver Q. Fix a potential W on Q. Let $\widetilde{Q}$ be the r-framing of Q at a given vertex $0 \in Q_0$, and let $\widetilde{J}$ be the Jacobi algebra $J_{\widetilde{Q},W}$, where W is viewed as a potential on $\widetilde{Q}$ in the obvious way. For a generic stability parameter $\zeta \in \mathbb{R}^{Q_0}$, and an arbitrary dimension vector $\alpha \in \mathbb{N}^{Q_0}$, set

$$\zeta_\infty = -\zeta \cdot \alpha, \quad \widetilde{\zeta} = (\zeta_\infty, \zeta), \quad \widetilde{\alpha} = (1, \alpha).$$

As in Section 1·4, consider the trace map $f_{\widetilde{\alpha}}\colon \mathrm{R}(\widetilde{Q}, \widetilde{\alpha}) \to \mathbb{A}^1$, induced by W, and its restriction to the framed-stable locus $f_{\widetilde{\zeta},\widetilde{\alpha}}\colon \mathrm{R}^{\widetilde{\zeta}\text{-st}}(\widetilde{Q}, \widetilde{\alpha}) \to \mathbb{A}^1$. Define the moduli stacks

$$\mathfrak{M}(\widetilde{J}, \alpha) = \left[\operatorname{crit} f_{\widetilde{\alpha}} \,/ \mathrm{GL}_\alpha\right], \quad \mathfrak{M}_\zeta(\widetilde{J}, \alpha) = \left[\operatorname{crit} f_{\widetilde{\zeta},\widetilde{\alpha}} \,/ \mathrm{GL}_\alpha\right].$$

Note that we are not quotienting by $\mathrm{GL}_{\widetilde{\alpha}} = \mathrm{GL}_\alpha \times \mathbb{C}^\times$, but only by GL_α.

Definition 1·11. (See [25] for $r = 1$) We define r-framed motivic Donaldson–Thomas invariants

$$\left[\mathfrak{M}(\widetilde{J}, \alpha)\right]_{\mathrm{vir}} = \frac{\left[\operatorname{crit} f_{\widetilde{\alpha}}\right]_{\mathrm{vir}}}{[\mathrm{GL}_\alpha]_{\mathrm{vir}}}, \qquad \left[\mathfrak{M}_\zeta(\widetilde{J}, \alpha)\right]_{\mathrm{vir}} = \frac{\left[\operatorname{crit} f_{\widetilde{\zeta},\widetilde{\alpha}}\right]_{\mathrm{vir}}}{[\mathrm{GL}_\alpha]_{\mathrm{vir}}} \tag{1·3}$$

in $\mathcal{M}_{\mathbb{C}}$, and the associated motivic generating functions

$$\widetilde{A}_U = \sum_{\alpha \in \mathbb{N}^{Q_0}} \left[\mathfrak{M}(\widetilde{J}, \alpha)\right]_{\mathrm{vir}} \cdot y^{\widetilde{\alpha}} \in \mathcal{T}_{\widetilde{Q}}$$

$$\mathsf{Z}_\zeta = \sum_{\alpha \in \mathbb{N}^{Q_0}} \left[\mathfrak{M}_\zeta(\widetilde{J}, \alpha)\right]_{\mathrm{vir}} \cdot y^{\widetilde{\alpha}} \in \mathcal{T}_{\widetilde{Q}}.$$

The fact that the r-framed invariants (1·3) live in $\mathcal{M}_{\mathbb{C}}$ (i.e. have no monodromy) follows from [25, Lemma 1·10]. The reason is that the dimension vector $\widetilde{\alpha} = (1, \alpha)$ contains ‘1’ as a component.

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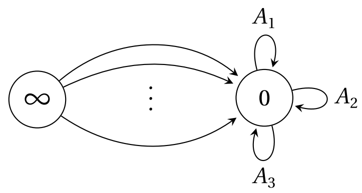

Fig. 1. The r-framed 3-loop quiver $\widetilde{L}_3$.

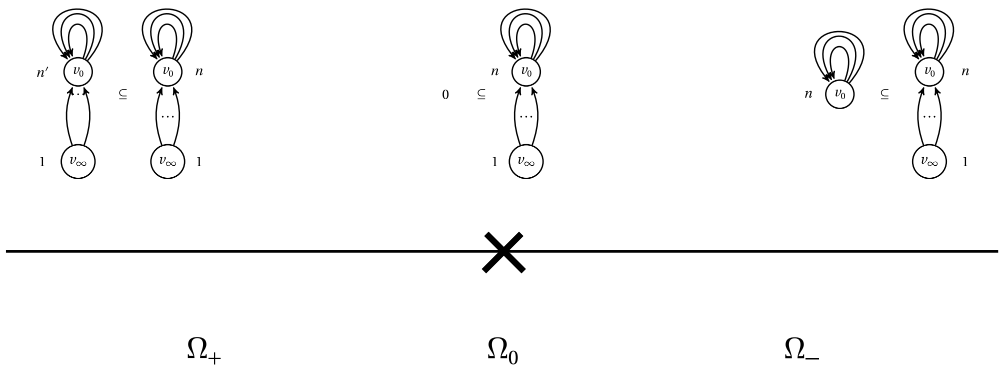

Fig. 2. An illustration of Proposition 2·6.

1·6. *Dimensional reduction*

If I is a cut for (Q, W), one can define a new quiver $Q_I = (Q_0, Q_1 \setminus I)$. Let $J_{W,I}$ be the quotient of $\mathbb{C}Q_I$ by the two-sided ideal generated by the noncommutative derivatives $\partial W/\partial a$ for $a \in I$. Let $\mathrm{R}(J_{W,I}, \alpha) \subset \mathrm{R}(Q_I, \alpha)$ be the space of $J_{W,I}$-modules of dimension vector α. Then one has the following dimensional reduction principle.

PROPOSITION 1·12. ([25, proposition 1·15]) *Suppose I is a cut for (Q, W). Set* $\mathrm{d}_I(\alpha) = \sum_{a\in I} \alpha_{t(a)}\alpha_{h(a)}$. *Then*

$$A_U = \sum_{\alpha\in\mathbb{N}^{Q_0}} \mathbb{L}^{\frac{1}{2}\chi_Q(\alpha,\alpha)+\mathrm{d}_I(\alpha)} \frac{[\mathrm{R}(J_{W,I},\alpha)]}{[\mathrm{GL}_\alpha]} \cdot y^\alpha.$$

Example 1·13. Let $Q = L_3$ be the 3-loop quiver (see Figure 1, and remove the framing vertex to obtain a picture of this quiver) with the potential $W = A_3[A_1, A_2]$. Notice that $J = J_{L_3,W} = \mathbb{C}[x, y, z]$, and $I = \{A_3\}$ is a cut for (L_3, W). The quiver Q_I is the 2-loop quiver and $J_{W,I} = \mathbb{C}[x, y]$. We have $\mathrm{d}_I(n) = n^2$ and $\chi_Q(n, n) = -2n^2$. Therefore Proposition 1·12 yields an identity

$$\sum_{n\geq 0} \big[\mathfrak{M}(J, n)\big]_{\mathrm{vir}} \cdot y^n = \sum_{n\geq 0} \frac{[C_n]}{[\mathrm{GL}_n]} \cdot y^n = \prod_{m\geq 1}\prod_{k\geq 1} \left(1 - \mathbb{L}^{2-k} y^m\right)^{-1}, \tag{1·4}$$

where $R(J_{W,I}, n)$ is identified with the *commuting variety*

$$C_n = \{(A_1, A_2) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^{\oplus 2} | [A_1, A_2] = 0\} \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^{\oplus 2},$$

and the second identity in (1·4) is the Feit–Fine formula [3, 5, 15].

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Remark 1·14. The universal series A_U has been computed for several homogeneous deformations of the potential W of example 1·13 in [7].

2. *Motivic DT invariants of the Quot scheme of points*

2·1. *Stability on the framed 3-loop quiver*

The main character in this section is the framed quiver $\widetilde{L}_3$ of Figure 1, which we equip with the potential $W = A_3[A_1, A_2]$.

Definition 2·1. Let $\widetilde{\rho} = (u, \rho)$ be a representation of $\widetilde{L}_3$ of dimension $(1, n)$. We denote by $\langle u, \rho\rangle \subset \widetilde{\rho}$ the smallest subrepresentation of $\widetilde{\rho}$ containing $u(\rho_\infty)$. More precisely, if $\rho = (A_1, A_2, A_3) \in R(L_3, n)$, then $\langle u, \rho\rangle$ is the subrepresentation of $\widetilde{\rho}$ with $\langle u, \rho\rangle_\infty = \widetilde{\rho}_\infty = \mathbb{C}$ and

$$\langle u, \rho\rangle_0 = Span_{\mathbb{C}}\{A_1^{a_1}A_2^{a_2}A_3^{a_3} \cdot u_\ell(1) | a_i \geq 0, 1 \leq \ell \leq r\} \subset \widetilde{\rho}_0,$$

and with linear maps induced naturally by those defined by $\widetilde{\rho}$.

From now on we identify the space of stability parameters for L_3 with $\mathbb{R}$.

LEMMA 2·2. *Let $\zeta \in \mathbb{R}$ be a stability parameter, and let $\widetilde{\rho} = (u, \rho)$ be a representation of $\widetilde{L}_3$ of dimension (1, n). Set $\widetilde{\zeta} = (-n\zeta, \zeta)$. Then*:

(i) *if $\zeta < 0$, $\widetilde{\rho}$ is ζ-semistable if and only if it is ζ-stable if and only if $\widetilde{\rho} = \langle u, \rho\rangle$*;

(ii) *if $\zeta = 0$, $\widetilde{\rho}$ is ζ-semistable*;

(iii) *if $\zeta > 0$, $\widetilde{\rho}$ is ζ-semistable if and only if it is ζ-stable if and only if $n = 0$.*

Proof. For the case $\zeta < 0$ we refer to [1, proposition 2·4]. Consider the case $\zeta > 0$. If we had $n = \dim_{\mathbb{C}} \widetilde{\rho}_0 > 0$, then $\rho \subset \widetilde{\rho}$ would be destabilising, for $Z_{\widetilde{\zeta}}(0, n) = -n\zeta + n\sqrt{-1}$ implies $\varphi_{\widetilde{\zeta}}(\rho) > 1/2 = \varphi_{\widetilde{\zeta}}(\widetilde{\rho})$, since $Z_{\widetilde{\zeta}}(1, n) = (n+1)\sqrt{-1}$ has vanishing real part. On the other hand, if $n = 0$ then $\widetilde{\rho}$ is simple and hence ζ-stable. In the case $\zeta = 0$ there is nothing to prove, as all representations have phase $1/2$.

Consider the following regions of the space of stability parameters $\mathbb{R}$:

- $\Omega_+ = \{\zeta \in \mathbb{R} | \zeta < 0\}$,
- $\Omega_0 = \{\zeta \in \mathbb{R} | \zeta = 0\}$,
- $\Omega_- = \{\zeta \in \mathbb{R} | \zeta > 0\}$.

By Lemma 2·2 the space of stability parameters on $\widetilde{L}_3$ admits a particularly simple wall-and-chamber decomposition $\mathbb{R} = \Omega_+ \amalg \Omega_0 \amalg \Omega_-$ with one wall (the origin) and two chambers.

2·2. *The virtual motive of the Quot scheme of points*

Let $\widetilde{L}_3$ be the r-framed 3-loop quiver (Figure 1), and fix the potential $W = A_3[A_1, A_2]$. Fix a stability parameter $\zeta^+ \in \Omega_+ = \mathbb{R}_{<0}$. Fixing $n \geq 0$ and setting $\widetilde{\zeta}^+ = (-n\zeta, \zeta)$, $\widetilde{n} = (1, n)$, the quotient stack

$$\mathfrak{M}_{\zeta^+}(\widetilde{L}_3, n) = \left[\mathrm{R}^{\widetilde{\zeta}^+\text{-st}}(\widetilde{L}_3, \widetilde{n}) \,/\, \mathrm{GL}_n\right]$$

is a smooth quasiprojective *variety* of dimension $2n^2 + rn$, called the *noncommutative Quot scheme* in [1]. The regular function $f_n \colon \mathrm{R}(\widetilde{L}_3, \widetilde{n}) \to \mathbb{A}^1$ given by taking the trace of

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W descends to a regular function on $\mathfrak{M}_{\zeta^+}(\widetilde{L}_3, n)$, still denoted f_n. We have the following description of the Quot scheme of length n quotients of $\mathscr{O}_{\mathbb{A}^3}^{\oplus r}$.

PROPOSITION 2·3. ([1, theorem 2·6]) *There is an identity of closed subschemes*

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n) = \mathrm{crit}(f_n) \subset \mathfrak{M}_{\zeta^+}(\widetilde{L}_3, n).$$

Thanks to Proposition 2·3, we can form the virtual motives of the Quot scheme, as in Section 1·1·1, and define their generating function

$$\mathsf{DT}_r^{\mathrm{points}}(\mathbb{A}^3, q) = \sum_{n \geq 0} \left[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)\right]_{\mathrm{vir}} \cdot q^n \in \mathcal{M}_{\mathbb{C}}[\![q]\!]. \tag{2·1}$$

Remark 2·4. The main result of [3] is the formula

$$\mathsf{DT}_1^{\mathrm{points}}(\mathbb{A}^3, q) = \prod_{m \geq 1} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{2+k-\frac{m}{2}} q^m\right)^{-1}.$$

The series $\mathsf{DT}_1^{\mathrm{points}}(Y, q)$ studied in [3] for an arbitrary smooth 3-fold Y also appeared in [11] as the wall-crossing factor in the motivic DT/PT correspondence based at a fixed smooth curve $C \subset Y$ in Y. This correspondence refined its enumerative counterpart [32, 33]. The same phenomenon occurred in [25, 26] in the context of framed motivic DT invariants. See [4, 35] for a generalisation $\mathsf{DT}_r^{\mathrm{points}}(Y, q)$ of (2·1) to an arbitrary smooth 3-fold Y. See [34] for a plethystic formula expressing the naive motives $[\mathrm{Quot}_Y(F, n)] \in K_0(\mathrm{Var}_{\mathbb{C}})$ in terms of the motives of the punctual Quot schemes.

The main identity of Theorem 0·1

$$\mathsf{DT}_r^{\mathrm{points}}(\mathbb{A}^3, q) = \prod_{m \geq 1} \prod_{k=0}^{rm-1} \left(1 - \mathbb{L}^{2+k-\frac{rm}{2}} q^m\right)^{-1} \tag{2·2}$$

will be proved in the next subsection. An argument following closely the technique used in the $r = 1$ case by Behrend–Bryan–Szendrői [3], was given in the PhD theses of the first and third authors [6, 31]. In the following subsection, we will instead exploit an r-framed version of motivic wall-crossing. This technique, inspired by [25, 26, 28], will be also applied to small crepant resolutions of affine toric Calabi–Yau 3-folds in [8].

Granting (2·2) for the moment, we prove in the next lemma the second assertion in Theorem 0·1, i.e. Formula (0·2).

LEMMA 2·5. *There is an identity*

$$\prod_{m \geq 1} \prod_{k=0}^{rm-1} \left(1 - \mathbb{L}^{2+k-\frac{rm}{2}} q^m\right)^{-1} = \prod_{i=1}^{r} \mathsf{DT}_1^{\mathrm{points}}\left(\mathbb{A}^3, q\mathbb{L}^{\frac{-r-1}{2}+i}\right). \tag{2·3}$$

Proof. The sought after identity follows from a simple manipulation:

$$\prod_{i=1}^{r} \mathsf{DT}_1^{\mathrm{points}}(\mathbb{A}^3, q\mathbb{L}^{\frac{-r-1}{2}+i}) = \prod_{i=1}^{r} \prod_{m \geq 1} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{2+k-\frac{m}{2}} \mathbb{L}^{\frac{-r-1}{2}m+im} q^m\right)^{-1}$$

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$$= \prod_{m\geq 1}\prod_{k=0}^{m-1}\prod_{i=1}^{r}\left(1-\mathbb{L}^{2+k+(i-1)m-\frac{rm}{2}}q^m\right)^{-1}$$
$$= \prod_{m\geq 1}\prod_{k=0}^{rm-1}\left(1-\mathbb{L}^{2+k-\frac{rm}{2}}q^m\right)^{-1},$$

as claimed.

2·3. *Calculation via wall-crossing*

In this subsection we prove Theorem 0·1. Consider the universal generating function

$$\widetilde{A}_U = \sum_{n\geq 0}\left[\mathfrak{M}(\widetilde{J},n)\right]_{\mathrm{vir}}\cdot y^{(1,n)} \in \mathcal{T}_{\widetilde{L}_3}$$

as an element of the motivic quantum torus. To (generic) stability parameters $\zeta^{\pm}\in\Omega_{\pm}$ we associate elements (cf. Definition 1·11)

$$\mathsf{Z}_{\zeta^{\pm}} = \sum_{n\geq 0}\left[\mathfrak{M}_{\zeta^{\pm}}(\widetilde{J},n)\right]_{\mathrm{vir}}\cdot y^{(1,n)} \in \mathcal{T}_{\widetilde{L}_3}.$$

By Lemma 2·2 (iii), we have an identity

$$\mathsf{Z}_{\zeta^-} = y_\infty = y^{(1,0)}, \tag{2·4}$$

whereas the series Z_{ζ^+} is, essentially, a "shift" of the generating function $\mathsf{DT}_r^{\mathrm{points}}(\mathbb{A}^3, y^{(0,1)})$. More precisely, by Proposition 2·3 we have an identification

$$\mathfrak{M}_{\zeta^+}(\widetilde{J},n) = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)\subset \mathfrak{M}_{\zeta^+}(\widetilde{L}_3,n)$$

of critical loci sitting inside the noncommutative Quot scheme; in particular, the associated virtual motives are the same. The shift is intended in the following sense: since $\langle(0,n),(1,0)\rangle_{\widetilde{L}_3} = rn$, the product rule (1·1) yields the identity

$$y^{(1,n)} = \mathbb{L}^{-\frac{rn}{2}}y^{(0,n)}\cdot y_\infty \in \mathcal{T}_{\widetilde{L}_3}. \tag{2·5}$$

Since we can express $y^{(0,n)}$ as the n-fold product of $y^{(0,1)}$ with itself, we obtain

$$\mathsf{Z}_{\zeta^+} = \mathsf{DT}_r^{\mathrm{points}}\left(\mathbb{A}^3, \mathbb{L}^{-\frac{r}{2}}y^{(0,1)}\right)\cdot y_\infty. \tag{2·6}$$

The last generating function we need to analyse is

$$A_U = \sum_{n\geq 0}\left[\mathfrak{M}(J,n)\right]_{\mathrm{vir}}\cdot y^{(0,n)} \in \mathcal{T}_{L_3}\subset \mathcal{T}_{\widetilde{L}_3},$$

whose n-th coefficient is the virtual motive of the stack of 0-dimensional $\mathbb{C}[x,y,z]$-modules of length n. This was already computed in Example 1·13:

$$A_U(y) = \prod_{m\geq 1}\prod_{k\geq 1}\left(1-\mathbb{L}^{2-k}y^m\right)^{-1}. \tag{2·7}$$

PROPOSITION 2·6. *In $\mathcal{T}_{\widetilde{L}_3}$, there are identities*

$$\mathsf{Z}_{\zeta^+}\cdot A_U = \widetilde{A}_U = A_U\cdot \mathsf{Z}_{\zeta^-}. \tag{2·8}$$

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Proof. Let $\widetilde{\rho} = (u, \rho)$ be a $\widetilde{J}$-module. Consider $\zeta^+ \in \Omega_+$, and let $\langle u, \rho \rangle \subset \widetilde{\rho}$ be the submodule introduced in Definition 2·1. We have that $\langle u, \rho \rangle$ is ζ^+-stable by Lemma 2·2(i) and the quotient $\widetilde{\rho}/\langle u, \rho \rangle$ is supported at the vertex 0. From this we deduce the decomposition $\widetilde{A}_U = \mathsf{Z}_{\zeta^+} \cdot A_U$.

Consider now $\zeta^- \in \Omega_-$. The quotient of $\widetilde{\rho}$ by the submodule ρ based at the vertex 0 is the simple module supported at the framing vertex ∞. By Lemma 2·2 (iii) this is the unique ζ^--stable module for the current choice of ζ^-, so we obtain the decomposition $\widetilde{A}_U = A_U \cdot \mathsf{Z}_{\zeta^-}$.

Remark 2·7. In the previous proof, the decompositions in the motivic quantum torus follow directly from the functoriality of the Harder–Narasimhan filtration, combined with the existence of a well-defined motivic integration map, for which we refer to Mozgovoy's work [27, 28], formalising an original construction by Kontsevich–Soibelman [20].

Note that we have an identity

$$y_\infty \cdot y^{(0,n)} = \mathbb{L}^{-\frac{rn}{2}} \cdot y^{(1,n)} = \mathbb{L}^{-rn} \cdot y^{(0,n)} \cdot y_\infty,$$

where we have used (2·5) for the second equality. By Formula (2·6), the left-hand term of Formula (2·8) can then be rewritten as

$$\begin{aligned}\mathsf{DT}_r^{\text{points}}\left(\mathbb{A}^3, \mathbb{L}^{-\frac{r}{2}} y^{(0,1)}\right) &\cdot \sum_{n\geq 0} \left[\mathfrak{M}(J,n)\right]_{\text{vir}} \cdot y_\infty \cdot y^{(0,n)} \\ &= \mathsf{DT}_r^{\text{points}}\left(\mathbb{A}^3, \mathbb{L}^{-\frac{r}{2}} y^{(0,1)}\right) \cdot \sum_{n\geq 0} \left[\mathfrak{M}(J,n)\right]_{\text{vir}} \mathbb{L}^{-rn} \cdot y^{(0,n)} \cdot y_\infty.\end{aligned}$$

Therefore, by Equation (2·4), identifying the two expressions for $\widetilde{A}_U$ in Equation (2·8) yields

$$\mathsf{DT}_r^{\text{points}}\left(\mathbb{A}^3, \mathbb{L}^{-\frac{r}{2}} y^{(0,1)}\right) \cdot A_U\left(\mathbb{L}^{-r} y^{(0,1)}\right) \cdot y_\infty = A_U\left(y^{(0,1)}\right) \cdot y_\infty,$$

which is equivalent to

$$\mathsf{DT}_r^{\text{points}}\left(\mathbb{A}^3, \mathbb{L}^{-\frac{r}{2}} y^{(0,1)}\right) = \frac{A_U\left(y^{(0,1)}\right)}{A_U\left(\mathbb{L}^{-r} y^{(0,1)}\right)}.$$

Setting $q = \mathbb{L}^{-\frac{r}{2}} y^{(0,1)}$, and using Equation (2·7), a simple substitution yields

$$\begin{aligned}\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3, q) &= \frac{A_U\left(\mathbb{L}^{\frac{r}{2}} q\right)}{A_U\left(\mathbb{L}^{-\frac{r}{2}} q\right)} \\ &= \prod_{m\geq 1} \prod_{j\geq 0} \frac{\left(1 - \mathbb{L}^{1-j+\frac{rm}{2}} q^m\right)^{-1}}{\left(1 - \mathbb{L}^{1-j-\frac{rm}{2}} q^m\right)^{-1}} \\ &= \prod_{m\geq 1} \prod_{j=0}^{rm-1} \left(1 - \mathbb{L}^{1-j+\frac{rm}{2}} q^m\right)^{-1}\end{aligned}$$

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$$= \prod_{m\geq 1}\prod_{k=0}^{rm-1}\left(1-\mathbb{L}^{2+k-\frac{rm}{2}}q^m\right)^{-1}.$$

Formula (0·1) is proved. By Lemma 2·5, Theorem 0·1 is proved.

Remark 2·8. The generating function $\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3, (-1)^r q)$ admits the plethystic expression

$$\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3, (-1)^r q) = Exp\left(\frac{(-1)^r q\mathbb{L}^{\frac{3}{2}}}{\left(1-(-\mathbb{L}^{-\frac{1}{2}})^r q\right)\left(1-(-\mathbb{L}^{\frac{1}{2}})^r q\right)}\frac{\mathbb{L}^{-\frac{r}{2}}-\mathbb{L}^{\frac{r}{2}}}{\mathbb{L}^{-\frac{1}{2}}-\mathbb{L}^{\frac{1}{2}}}\right).$$

This is exploited in [35, section 4] to define $[\text{Quot}_Y(F,n)]_{\text{vir}}$ for all 3-folds Y and locally free sheaves F over Y. The shift by the sign $(-1)^r$ is needed in order to write the series as a plethystic exponential (and cannot be naively removed from both sides). This is the case for K-theoretic DT invariants as well [14, theorem A].

Remark 2·9. The Euler number specialisation $\mathbb{L}^{1/2} \to -1$ applied to Formula (0·1) yields a formula for the generating function of virtual Euler characteristics of Quot schemes,

$$\sum_{n\geq 0}\chi_{\text{vir}}(\text{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n))\cdot q^n = \mathsf{M}((-1)^r q)^r,$$

where $\mathsf{M}(q) = \prod_{m\geq 1}(1-q^m)^{-m}$ is the MacMahon function, the generating function of plane partitions, and where $\chi_{\text{vir}}(-) = \chi(-,\nu)$ is the Euler characteristic weighted by Behrend's microlocal function [2]. The above identity can be seen as the Calabi–Yau specialisation (i.e. the specialisation $s_1+s_2+s_3=0$) of the generating function of cohomological Donaldson–Thomas invariants of $\mathbb{A}^3$,

$$\sum_{n\geq 0}\mathsf{DT}_r^{\text{coh}}(\mathbb{A}^3, n)\cdot q^n = \mathsf{M}((-1)^r q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3}} \in \mathbb{Q}(\!(s_1,s_2,s_3)\!)[\![q]\!],$$

obtained in [14, theorem B] as a higher rank version of [22, theorem 1], where s_1, s_2 and s_3 are the equivariant parameters of the torus $\mathbb{T} = \mathbb{G}_m^3$ acting on the Quot scheme.

3. *The normal limit law and asymptotics*

In Section 3·1, we introduce a family of random variables on the space of r-coloured plane partitions, and we describe the asymptotics of the members of the family after suitable normalisation in Proposition 3·1. Theorem 0·2, the main theorem of the section, is deduced from Proposition 3·1 in Section 3·2. Finally, Section 3·3 is entirely devoted to the proof of Proposition 3·1.

3·1. *Random variables on r-coloured plane partitions*

We introduce a multivariate function

$$F(u,v,w,z) = \prod_{l=1}^{r}\prod_{m\geq 1}\prod_{k=1}^{m}\left(1-wu^k v^{ml} z^m\right)^{-1}. \tag{3·1}$$

The coefficient of z^n is a polynomial on the three variables u, v and w, which we denote by $Q_n(u,v,w)$, whose coefficients are nonnegative integers. When $u=v=w=1$, we obtain

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1			
2	2		
2	2	1	
3	3	2	
5	4	3	1

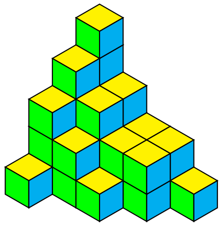

Fig. 3. A plane partition π of size $|\pi| = 31$, $\Delta(\pi) = 9$, $\Delta_+(\pi) = 12$, and $\Delta_-(\pi) = 10$.

the well known MacMahon function raised to the power r, which is the generating function for the r-tuples of plane partitions. Hence, $Q_n(1, 1, 1)$ is the number of r-tuples of plane partitions of total size n, i.e. the number of r-coloured plane partitions $(\pi_1, \pi_2, \ldots, \pi_r)$ such that $\sum_{j=1}^{r} |\pi_j| = n$, where $|\pi_j|$ denotes the sum of the entries of the plane partition π_j (or the number of boxes, cf. Figure 3). The polynomial $Q_n(u, v, w)$, when divided by $Q_n(1, 1, 1)$, represents the joint probability generating function of some random variables X_n, Y_n, and Z_n on the space of r-coloured plane partitions of size n, where each r-tuple is equally likely. More precisely, we have

$$\frac{Q_n(u, v, w)}{Q_n(1, 1, 1)} = \mathbb{E}\left(u^{X_n} v^{Y_n} w^{Z_n}\right). \tag{3·2}$$

To describe these random variables, we need to define certain parameters of plane partitions. For a plane partition π, let $\Delta(\pi)$ denote the sum of the diagonal parts of π, $\Delta_+(\pi)$ denote the sum of the upper diagonal parts, and $\Delta_-(\pi)$ denote the sum of the lower diagonal parts. See Figure 3 for an example of a plane partition showing the values of these parameters.

The parameter $\Delta(\pi)$ is also known as the trace of π and it has been studied in the literature, see for instance [19] and the references therein. In particular, one has

$$\prod_{m\geq 1} (1 - wz^m)^{-m} = \sum_{\pi} w^{\Delta(\pi)} z^{|\pi|}.$$

Similarly, we can find in [24] that

$$\prod_{m\geq 1} \prod_{k=1}^{m} (1 - q^{2k-m} z^m)^{-1} = \sum_{\pi} q^{\Delta(\pi)+\Delta_+(\pi)-\Delta_-(\pi)} z^{|\pi|}.$$

We can easily deduce from these two identities that for an r-coloured plane partition $\overline{\pi} = (\pi_1, \pi_2, \ldots, \pi_r)$ of total size n, we have

$$\begin{aligned} X_n(\overline{\pi}) &= \frac{n}{2} + \frac{1}{2}\left(\sum_{l=1}^{r} (\Delta(\pi_l) + \Delta_+(\pi_l) - \Delta_-(\pi_l))\right) \\ &= \sum_{l=1}^{r} (\Delta(\pi_l) + \Delta_+(\pi_l)), \\ Y_n(\overline{\pi}) &= \sum_{l=1}^{r} l|\pi_l|, \quad \text{and} \end{aligned}$$

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$$Z_n\,(\overline{\pi}) = \sum_{l=1}^{r} \Delta(\pi_l).$$

When $r = 1$, Kamenov and Mutafchiev [19] proved that the distribution of the trace of a random plane partition of size n, when suitably normalised, is asymptotically normal. Morrison [24] also established asymptotic normality for any random variable of the form $\delta\Delta(\pi) + \Delta_+(\pi) - \Delta_-(\pi)$, where π is a random plane partition of size n and δ is a fixed real number. We show that for any fixed integer $r \geq 1$, any nontrivial linear combination of the variables X_n, Y_n and Z_n, when suitably normalised, converges weakly to a normal distribution. It is worth noting that the random variable Y_n is non-constant only when $r > 1$.

PROPOSITION 3·1. *For any fixed real vector* $(\alpha, \beta, \gamma) \neq (0, 0, 0)$*, there exist sequences of real numbers* μ_n *and* $\sigma_n \geq 0$ *such that the normalised random variable*

$$\frac{\alpha X_n + \beta Y_n + \gamma Z_n - \mu_n}{\sigma_n}$$

converges weakly to the standard normal distribution. Moreover, μ_n *and* σ_n *satisfy the following asymptotic formulas as* $n \to \infty$:

$$\mu_n = \left(\frac{1}{2}\alpha + \frac{r+1}{2}\beta\right) n + \frac{r^{1/3}\zeta(2)(\alpha + 2\gamma)}{2^{5/3}(\zeta(3))^{2/3}}\, n^{2/3} + \mathcal{O}(n^{1/3}), \ \ \textit{and}$$

$$\sigma_n^2 \sim \begin{cases} \dfrac{\alpha^2 + (r^2-1)\beta^2}{2^{7/3}(r\zeta(3))^{1/3}}\, n^{4/3} & \textit{if } (\alpha, \beta) \neq (0, 0), \\[2ex] \dfrac{r^{1/3}\gamma^2}{3(2\zeta(3))^{2/3}} n^{2/3} \log n & \textit{otherwise.} \end{cases}$$

Looking at the asymptotic behaviour of the random variables X_n, Y_n and Z_n, when divided by $n^{2/3}$, we observe from the above result that the random variable $n^{-2/3}Z_n$ degenerates as $n \to \infty$. Furthermore, by the Camér–Wold device [9], the random variables $n^{-2/3}X_n$, $n^{-2/3}Y_n$ converge jointly to a bivariate normal distribution with a diagonal covariance matrix. The appropriate normalisation of Z_n is $n^{-1/3}(\log n)^{-1/2}Z_n$ which is, when centred, asymptotically normal. This asymptotic normality and the asymptotic formulas for μ_n and σ_n^2 agree with the main result in [19] when $r = 1$ and $(\alpha, \beta, \gamma) = (0, 0, 1)$.

Convention 3·1. We shall use the Vinogradov notation $\ll$ interchangeably with the $\mathcal{O}$-notation. For instance, by $f(n) \ll g(n)$ (or $g(n) \gg f(n)$) as $n \to \infty$, we mean that there exists a positive constant C such that $|f(n)| \leq Cg(n)$ for sufficiently large n.

Theorem 0·2 now follows immediately from Proposition 3·1 as we will see next.

3·2. *Deducing Theorem* 0·2

Granting Proposition 3·1, we can now finish the proof of Theorem 0·2, our second main result.

THEOREM 3·2. *As* $n \to \infty$*, the normalised random variable* $n^{-2/3}S_{n,r}$ *converges in distribution to* $\mathcal{N}(\mu, \sigma^2)$ *with*

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$$\mu = \frac{r^{1/3}\pi^2}{2^{5/3}(\zeta(3))^{2/3}} \quad \textit{and} \quad \sigma^2 = \frac{r^{5/3}}{(2\zeta(3))^{1/3}},$$

where $\zeta(s)$ is Riemann's zeta function.

Proof. With the change of variable $T = \mathbb{L}^{1/2}$, the combination of the equations (0·1) and (0·2) in Theorem 0·1 yields

$$\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3, q) = \prod_{l=1}^{r}\prod_{m\geq 1}\prod_{k=0}^{m-1}\left(1 - T^{4+2k-m}\left(qT^{-r-1+2l}\right)^m\right)^{-1}.$$

The product on the right-hand side can be expressed in terms of our auxiliary function $F(u, v, w, z)$ defined at the beginning of this section. First we write it as follows

$$\prod_{l=1}^{r}\prod_{m\geq 1}\prod_{k=0}^{m-1}\left(1 - T^{4+m-2(m-k)}\left(qT^{r+1-2(r-l+1)}\right)^m\right)^{-1}.$$

If k goes from 0 to $m-1$, then $m-k$ goes from m to 1. Similarly, if l goes from 1 to r, then $r-l+1$ goes from r to 1. Therefore, we have

$$\begin{aligned}
\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3, q) &= \prod_{l=1}^{r}\prod_{m\geq 1}\prod_{k=1}^{m}\left(1 - T^{4+m-2k}(T^{r+1-2l}q)^m\right)^{-1} \\
&= \prod_{l=1}^{r}\prod_{m\geq 1}\prod_{k=1}^{m}\left(1 - T^{4-2k-2ml}(T^{r+2}q)^m\right)^{-1} \\
&= F(T^{-2}, T^{-2}, T^4, T^{r+2}q).
\end{aligned}$$

This implies that

$$\mathbb{E}(T^{S_{n,r}}) = \frac{M_{n,r}(T)}{M_{n,r}(1)} = \mathbb{E}\left(T^{4Z_n - 2X_n - 2Y_n + (r+2)n}\right), \tag{3·3}$$

where $M_{n,r}(T)$ is, as defined in (0·3), the coefficient of q^n in $\mathsf{DT}_r^{\text{points}}(\mathbb{A}^3, q)$. The second equality in (3·3) makes use of Equation (3·2). Thus, $S_{n,r}$ has the same distribution as $4Z_n - 2X_n - 2Y_n + (r+2)n$ — a shifted linear combination of the variables X_n, Y_n, and Z_n. Now, applying Proposition 3·1 with $(\alpha, \beta, \gamma) = (-2, -2, 4)$, we deduce that the normalised random variable $n^{-2/3}(4Z_n - 2X_n - 2Y_n + (r+2)n)$ converges weakly to the normal distribution $\mathcal{N}(\mu, \sigma^2)$ with

$$\mu = \frac{r^{1/3}\pi^2}{2^{5/3}(\zeta(3))^{2/3}} \quad \text{and} \quad \sigma^2 = \frac{r^{5/3}}{(2\zeta(3))^{1/3}},$$

which proves Theorem 0·2.

3·3. *Proof of Proposition 3·1*

Morrison used the method of moments to prove his result in [24]. However, due to the appearance of the second variable Y_n and the complication that comes with it, we decided to use a different approach. We follow the method that Hwang used in [18] to prove limit theorems for the number of parts in the so-called restricted partitions (these are one dimensional partitions with some restrictions on the parts). The first part of the proof is based on the

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saddle-point method to get an asymptotic formula for $Q_n(u, v, w)$ as $n \to \infty$, and the second is a perturbation technique to deduce the central limit theorem.

3·3·1. *Saddle-point method*

The goal here is to obtain an asymptotic formula for $Q_n(u, v, w)$ as $n \to \infty$, where (u, v, w) is allowed to vary in a fixed real neighbourhood of $(1, 1, 1)$. To simplify our notation, define $\Phi(t) = -\log(1 - e^{-t})$, and for real numbers a, b and c, we let

$$f(x, y) = \sum_{l=1}^{r} \sum_{m=1}^{\infty} \sum_{k=1}^{m} \Phi(xm + y(c + ak + mbl)).$$

The function f depends on (a, b, c) but we drop this dependence for now to ease notation. Also, for a positive number ρ, we make the substitution $z = e^{-\tau}$, where $\tau = \rho + it$. Hence,

$$f(\tau, \rho) = -\sum_{l=1}^{r} \sum_{m=1}^{\infty} \sum_{k=1}^{m} \log\left(1 - e^{-\rho(c+ak+mbl)-\tau m}\right) = \log F(e^{-a\rho}, e^{-b\rho}, e^{-c\rho}, e^{-\tau}).$$

One can easily verify that if (u, v, w) is bounded (which is the case throughout this section), then there exists a fixed positive real number R such that the product in (3·1) converges absolutely whenever $|z| < R$. Hence, $F(u, v, w, z)$, as function of z, is analytic in a complex neighbourhood of 0. By Cauchy's integral formula, we have

$$Q_n(e^{-a\rho}, e^{-b\rho}, e^{-c\rho}) = \frac{e^{n\rho}}{2\pi} \int_{-\pi}^{\pi} \exp\left(f(\rho + it, \rho) + nti\right) dt. \tag{3·4}$$

We now use the saddle-point method to estimate the above integral. We choose ρ to be the positive solution of the equation

$$n = -f_x(\rho, \rho) = \sum_{l=1}^{r} \sum_{m=1}^{\infty} \sum_{k=1}^{m} \frac{me^{-\rho(m+(c+ak+mbl))}}{1 - e^{-\rho(m+(c+ak+mbl))}}, \tag{3·5}$$

where f_x denotes the partial derivative of f with respect to x. Similar notations will be used for other partial derivatives. Note that there is a unique positive solution $\rho = \rho(n, a, b, c)$ of Equation (3·5) since the function defined by the series is strictly decreasing as a function of ρ, provided that a, b and c are small enough (it suffices for instance to assume that $|c| + |a| + r|b| < 1$). Furthermore, we observe that $\rho \to 0$ as $n \to \infty$. The following lemma reveals the asymptotic dependence between n and ρ.

LEMMA 3·3. *Let ϵ be a number in the interval $[0, 1/2]$ and ρ be the solution of Equation (3·5). Then we have*

$$\frac{2r\zeta(3)}{(1+\epsilon)^3} \rho^{-3} + \mathcal{O}(\rho^{-2}) \le n \le \frac{2r\zeta(3)}{(1-\epsilon)^3} \rho^{-3} + \mathcal{O}(\rho^{-2}), \tag{3·6}$$

as $n \to \infty$, uniformly for $|c| + |a| + r|b| \le \epsilon$, where the implied constants in the $\mathcal{O}$-terms are independent of ϵ.

Proof. Recall from (3·5) that

$$n = -f_x(\rho,\rho) = -\sum_{l=1}^{r}\sum_{m=1}^{\infty}\sum_{k=1}^{m} m\Phi'\left(\rho m\left(1+\frac{c+ak+mbl}{m}\right)\right).$$

Under the assumption that $|c|+|a|+r|b|\leq\epsilon$, for any $m\geq 1$, $1\leq k\leq m$ and $1\leq l\leq r$, we have

$$|c+ak+mbl|\leq(|c|+|a|+r|b|)\,m\leq\epsilon m.$$

Moreover, the function $\Phi'(x)$ is an increasing function. Therefore,

$$\Phi'((1-\epsilon)\rho m)\leq\Phi'\left(\rho m\left(1+\frac{c+ak+mbl}{m}\right)\right)\leq\Phi'((1+\epsilon)\rho m).$$

Multiplying by m and summing over $m\geq 1$, $1\leq k\leq m$ and $1\leq l\leq r$, we obtain

$$f_x((1-\epsilon)\rho,0)\leq f_x(\rho,\rho)\leq f_x((1+\epsilon)\rho,0).$$

We can obtain asymptotic estimates of the lower and upper bounds as $\rho\to 0^+$. This can be done via Mellin transform. The reader can consult [17] for a comprehensive survey on the Mellin transform method. The Mellin transform of $f_x((1-\epsilon)t,0)$ is

$$\int_0^{\infty} f_x((1-\epsilon)t,0)t^{s-1}dt = -r(1-\epsilon)^{-s}\zeta(s-2)\zeta(s)\Gamma(s),$$

which has simple poles at $s=3$ and $s=1$. The other singularities are precisely at the negative odd integers. Thus, we have

$$f_x((1-\epsilon)\rho,0) = -\frac{2r\zeta(3)}{(1-\epsilon)^3}\,\rho^{-3} + \frac{r}{12(1-\epsilon)}\,\rho^{-1} - \frac{r}{2\pi i}\int_{-i\infty}^{i\infty}\zeta(s-2)\zeta(s)\Gamma(s)((1-\epsilon)\rho)^{-s}ds. \qquad (3\cdot 7)$$

Since $|\zeta(it-2)\zeta(it)\Gamma(it)|$ decays exponentially fast as $t\to\pm\infty$, the absolute value of the integral on the right-hand side is bounded by an absolute constant. The same argument works for the estimate of the upper bound $f_x((1+\epsilon)\rho,0)$. This completes the proof of the lemma.

Next, we split the integral on the right-hand side of (3·4) into two parts as follows: let

$$\mathcal{I}_1 = \frac{e^{n\rho}}{2\pi}\int_{-\rho^C}^{\rho^C}\exp\left(f(\rho+it,\rho)+nti\right)dt,$$

where C is an absolute constant in the interval $(5/3,2)$, and let $\mathcal{I}_2 = Q_n(e^{-a\rho},e^{-b\rho},e^{-c\rho})-\mathcal{I}_1$.

Estimate of $\mathcal{I}_1$

For the rest of this section, we work under the condition of Lemma 3·3, that is $|c|+|a|+r|b|\leq\epsilon$ and $\epsilon\in[0,1/2]$. For $-\rho^C\leq t\leq\rho^C$, Equation (3·5) and a Taylor approximation of

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$f(\rho+it,\rho)$ give

$$f(\rho+it,\rho)+nit=f(\rho,\rho)-f_{xx}(\rho,\rho)\frac{t^2}{2}+\mathcal{O}\left(\rho^{3C}\max_{-\rho^{\rho}\leq\theta\leq\rho^{C}}\left|f_{xxx}(\rho+i\theta,\rho)\right|\right),$$

where the implied constant in the error term is absolute. To estimate the error term, observe that

$$f_{xxx}(\tau,\rho)=-\sum_{l=1}^{r}\sum_{m=1}^{\infty}\sum_{k=1}^{m}\frac{m^3e^{-\tau m-\rho(c+ak+mbl)}(1+e^{-\tau m-\rho(c+ak+mbl)})}{(1-e^{-\tau m-\rho(c+ak+mbl)})^3}.$$

For any real number θ and $\tau=\rho+i\theta$ we have

$$|1+e^{-\tau m-\rho(c+ak+mbl)}|\leq 1+e^{-\rho m-\rho(c+ak+mbl)},$$
$$|1-e^{-\tau m-\rho(c+ak+mbl)}|\geq 1-e^{-\rho m-\rho(c+ak+mbl)}.$$

Hence $|f_{xxx}(\rho+i\theta,\rho)|$ is bounded above by $|f_{xxx}(\rho,\rho)|$. We can estimate $|f_{xxx}(\rho,\rho)|$ as we did for $f_x(\rho,\rho)$ in the proof of Lemma 3·3. We obtain

$$|f_{xxx}(\rho+i\theta,\rho)|\leq|f_{xxx}(\rho,\rho)|=\mathcal{O}(\rho^{-5}),$$

where the implied constant depends only on r. Therefore, for $|t|\leq\rho^C$ we have

$$f(\rho+it,\rho)+nit=f(\rho,\rho)-f_{xx}(\rho,\rho)\frac{t^2}{2}+\mathcal{O}(\rho^{3C-5}).$$

Since we chose $C>5/3$, we have $\rho^{3C-5}=o(1)$. Thus

$$\mathcal{I}_1=\frac{e^{f(\rho,\rho)+n\rho}}{2\pi}\int_{-\rho^C}^{\rho^C}e^{-f_{xx}(\rho,\rho)t^2/2}\,dt\left(1+\mathcal{O}(\rho^{3C-5})\right).$$

It remains to estimate the integral on the right-hand side as $\rho\to 0^+$. Note that $f_{xx}(\rho,\rho)>0$ and $f_{xx}(\rho,\rho)\gg\rho^{-4}$ (again via Mellin transform as in Lemma 3·3), so we have

$$\begin{aligned}\int_{-\rho^C}^{\rho^C}e^{-f_{xx}(\rho,\rho)t^2/2}\,dt&=\int_{-\infty}^{\infty}e^{-f_{xx}(\rho,\rho)t^2/2}\,dt-2\int_{\rho^C}^{\infty}e^{-f_{xx}(\rho,\rho)t^2/2}\,dt\\&=\sqrt{\frac{2\pi}{f_{xx}(\rho,\rho)}}+\mathcal{O}\left(\int_{\rho^C}^{\infty}e^{-\rho^Cf_{xx}(\rho,\rho)t/2}\,dt\right)\\&=\sqrt{\frac{2\pi}{f_{xx}(\rho,\rho)}}+\mathcal{O}\left(\rho^{4-c}e^{-A\rho^{2C-4}}\right),\end{aligned}$$

where $A>0$ and the hidden constants in the error terms above depend only on r. Thus, since we chose $C<2$, the term $\rho^{4-c}e^{-A\rho^{2C-4}}$ tends to zero faster than any power of ρ as $\rho\to 0^+$. Hence, we obtain an estimate for $\mathcal{I}_1$

$$\mathcal{I}_1=\frac{e^{f(\rho,\rho)+n\rho}}{\sqrt{2\pi f_{xx}(\rho,\rho)}}\left(1+\mathcal{O}(\rho^{3C-5})\right)\ \text{as}\ \ \rho\to 0^+. \tag{3·8}$$

This estimate holds uniformly for $|c|+|a|+r|b|\leq\epsilon$ and $\epsilon\in[0,1/2]$.

Estimate of $\mathcal{I}_2$

We will prove that $|\mathcal{I}_2|$ is much smaller than $|\mathcal{I}_1|$. To this end, we assume that $\rho^C < t \leq \pi$, where C is as before. We have

$$\begin{aligned}\mathrm{Re}(f(\rho+it,\rho))-f(\rho,\rho) &= -\sum_{l=1}^{r}\sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{j=1}^{\infty} j^{-1}e^{-j\rho(m+(c+ak+mbl))}(1-\cos(mt))\\ &\leq -\sum_{l=1}^{r}\sum_{m=1}^{\infty}\sum_{k=1}^{m} e^{-\rho(m+(c+ak+mbl))}\,(1-\cos(mt))\\ &\leq -r\sum_{m=1}^{\infty} me^{-\rho(1+\epsilon)m}(1-\cos(mt)).\end{aligned}$$

Moreover,

$$\sum_{m=1}^{\infty} me^{-\rho(1+\epsilon)m}(1-\cos(mt)) = \frac{e^{\rho(1+\epsilon)}}{(e^{\rho(1+\epsilon)}-1)^2} - \mathrm{Re}\left(\frac{e^{\rho(1+\epsilon)+it}}{(e^{\rho(1+\epsilon)+it}-1)^2}\right).$$

A lower estimate of the same term can be found in the proof of [21, lemma 5]. By the same argument as the one given in loc. cit., but with $|t| \geq \rho^C$, we get

$$\sum_{m=1}^{\infty} me^{-\rho(1+\epsilon)m}(1-\cos(mt)) \gg (\rho(1+\epsilon))^{2C-4} \quad \text{as} \quad \rho \to 0^+,$$

where the implied constant is independent of ϵ.

Noting that $2C-4<0$, we deduce that $\exp(\mathrm{Re}(f(\rho+it,\rho))-f(\rho,\rho))$ tends to zero faster than any power of ρ as $\rho \to 0^+$. Thus, by (3·8), we find

$$\frac{|\mathcal{I}_2|}{|\mathcal{I}_1|} \ll \sqrt{f_{xx}(\rho,\rho)}\int_{\rho^C}^{\pi} \exp\left(\mathrm{Re}(f(\rho+it,\rho))-f(\rho,\rho)\right) dt,$$

which tends to zero faster than any power of ρ as $\rho \to 0^+$. Recalling that $Q_n(e^{-a\rho}, e^{-b\rho}, e^{-c\rho}) = \mathcal{I}_1 + \mathcal{I}_2$, we finally obtain

$$Q_n(e^{-a\rho}, e^{-b\rho}, e^{-c\rho}) = \frac{e^{f(\rho,\rho)+n\rho}}{\sqrt{2\pi f_{xx}(\rho,\rho)}}\left(1+\mathcal{O}(\rho^{3C-5})\right) \quad \text{as} \quad n \to \infty, \tag{3·9}$$

uniformly for $|c|+|a|+r|b| \leq \epsilon$ and $\epsilon \in [0, 1/2]$, where ρ is the solution of Equation (3·5).

3·3·2. *Perturbation*

Here we set $u = e^{\eta\alpha}$, $v = e^{\eta\beta}$, and $w = e^{\eta\gamma}$ where (α, β, γ) is fixed and η can vary in a small open interval containing zero. Hence, Equation (3·2) becomes

$$\frac{Q_n(u,v,w)}{Q_n(1,1,1)} = \mathbb{E}\left(e^{\eta(\alpha X_n+\beta Y_n+\gamma Z_n)}\right). \tag{3·10}$$

The right-hand side is the moment generating function of the random variable $\alpha X_n + \beta Y_n + \gamma Z_n$. From now on, let ρ_0 be the unique positive number such that $n = -f_x(\rho_0, 0)$. Then, by Lemma 3·3 (with $\epsilon = 0$) we have $n \sim 2r\zeta(3)\rho_0^{-3}$. Moreover, if we write $u = e^{-a\rho}$, $v = e^{-b\rho}$, and $w = e^{-\gamma\rho}$ for $\rho > 0$ as before, then we have $a = -\alpha\eta\rho^{-1}$, $b = -\beta\eta\rho^{-1}$, and

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$c=-\gamma\eta\rho^{-1}$. This implies that

$$|c|+|a|+r|b|=(|\gamma|+|\alpha|+r|\beta|)\eta\rho^{-1}.$$

Now, if we choose η and ρ in such a way that $\eta\rho^{-1}=o(1)$ and $\rho\to 0$ as $n\to\infty$, then by Lemma 3·3 (with $\epsilon\to 0$), we get

$$-f_x(\rho,\rho)\sim 2r\zeta(3)\rho^{-3}.$$

Observe that it is possible to choose such η and $\rho>0$ that satisfy $\eta=o(\rho_0)$ and $\rho=o(\rho_0)$. In this case, the above asymptotic formula implies that $-f_x(\rho,\rho)>-f_x(\rho_0,0)=n$ for large enough n. Similarly, we can also choose η and $\rho>0$ such that $\eta=o(\rho_0)$, $\rho\to 0$, $\eta\rho^{-1}=o(1)$, and $\rho/\rho_0\to\infty$. This time, the asymptotic formula gives $-f_x(\rho,\rho)<n$. Hence, for $\eta=o(\rho_0)$ and n large enough, the equation $n=-f_x(\rho,\rho)$ has a unique solution, which we denote by $\rho(\eta)$. Furthermore, it satisfies $\rho(\eta)\to 0$ and $\eta\rho(\eta)^{-1}=o(1)$ as $n\to\infty$. Therefore, we also have

$$n\sim 2r\zeta(3)\rho(\eta)^{-3}.$$

The latter and the asymptotic estimate $n\sim 2r\zeta(3)\rho_0^{-3}$ yield $\rho(\eta)\sim\rho_0$ as $n\to\infty$ whenever $\eta=o(\rho_0)$.

From this point onward, we assume that $\eta=o(\rho_0)$. Since $\rho(\eta)$ and ρ_0 are asymptotically equivalent as $n\to\infty$, so are $f_{xx}(\rho(\eta),\rho(\eta))$ and $f_{xx}(\rho_0,0)$. Thus, by (3·9), we have

$$\frac{Q_n(u,v,w)}{Q_n(1,1,1)}\sim\exp\Big(f(\rho(\eta),\rho(\eta))-f(\rho_0,0)+n(\rho(\eta)-\rho_0)\Big). \tag{3·11}$$

We want to obtain a precise asymptotic estimate of the exponent of the right-hand side that holds uniformly for $\eta=o(\rho_0)$. We will use a Taylor approximation of the function $f(\rho(\eta),\rho(\eta))$ when η is near 0 (noting that the parameters a, b, and c are themselves functions of η). So to highlight the variable η, we define

$$\begin{aligned}g(x,y)&=\log F(e^{y\alpha},e^{y\beta},e^{y\gamma},e^{-x})\\&=-\sum_{l=1}^{r}\sum_{m=1}^{\infty}\sum_{k=1}^{m}\log\left(1-e^{y(\gamma+\alpha k+m\beta l)-xm}\right)\\&=\sum_{l=1}^{r}\sum_{m=1}^{\infty}\sum_{k=1}^{m}\Phi(xm-y(\gamma+\alpha k+m\beta l)).\end{aligned}$$

This is essentially the same as the function $f(x,y)$ (if (a,b,c) is replaced by $(-\alpha,-\beta,-\gamma)$). However, in our case we have $a=-\alpha\eta\rho(\eta)^{-1}$, $b=-\beta\eta\rho(\eta)^{-1}$, and $c=-\gamma\eta\rho(\eta)^{-1}$. Hence, the meanings of the first and second variables will be different. For instance, we have the equation

$$g(\rho(\eta),\eta)=f(\rho(\eta),\rho(\eta))\ \text{ and }\ g(\rho_0,0)=f(\rho_0,0). \tag{3·12}$$

Similarly, the saddle-point equation $n=-f_x(\rho(\eta),\rho(\eta))$ becomes $n=-g_x(\rho(\eta),\eta)$. First, we apply implicit differentiation (with respect to η) to the latter equation, then by the mean value theorem, we get

$$\rho(\eta)-\rho_0=-\eta\,\frac{g_{xy}(\rho(\theta),\theta)}{g_{xx}(\rho(\theta),\theta)},$$

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for some real number θ between 0 and η. We can estimate $g_{xy}(\rho(\theta),\theta)$ and $g_{xx}(\rho(\theta),\theta)$ via Mellin transform in the same way as in the proof of Lemma 3·3, but we need the Mellin transforms of the functions $g_{xx}(t,0)$ and $g_{xy}(t,0)$. The Mellin transform of $g_{xx}(t,0)$ is $r\zeta(s-3)\zeta(s-1)\Gamma(s)$. To determine the Mellin transform of $g_{x,y}(t,0)$, first we write

$$\begin{aligned} g_{x,y}(t,0) &= -\sum_{l=1}^{r}\sum_{m=1}^{\infty}\sum_{k=1}^{m} m(\gamma+\alpha k+m\beta l)\Phi''(tm) \\ &= -\sum_{m=1}^{\infty} m\left(rm\gamma + r\alpha\frac{m^2+m}{2} + m^2\beta\frac{r^2+r}{2}\right)\Phi''(tm) \\ &= -\frac{r}{2}\sum_{m=1}^{\infty}\left((\alpha+(r+1)\beta)m^3+(\alpha+2\gamma)m^2\right)\Phi''(tm). \end{aligned}$$

Since the Mellin transform of $\Phi''(t)$ is $\zeta(s-1)\Gamma(s)$, the Mellin transform of $g_{x,y}(t,0)$ is

$$-\frac{r}{2}\left((\alpha+(r+1)\beta)\zeta(s-3)+(\alpha+2\gamma)\zeta(s-2)\right)\zeta(s-1)\Gamma(s).$$

We deduce the following asymptotic formulas as $\rho\to 0^+$:

$$g_{x,x}(\rho,0) = 6r\zeta(3)\rho^{-4} + \mathcal{O}(\rho^{-2}), \tag{3·13}$$

$$g_{x,y}(\rho,0) = -3r\zeta(3)(\alpha+(r+1)\beta)\rho^{-4} - r\zeta(2)(\alpha+2\gamma)\rho^{-3} + \mathcal{O}(\rho^{-2}). \tag{3·14}$$

The fact that $\rho(\theta)\sim\rho_0$ (since $|\theta|\le|\eta| = o(\rho_0)$) and the argument in the proof of Lemma 3·3 (with $\epsilon\to 0$) imply

$$\rho(\eta)-\rho_0 \sim -\eta\,\frac{g_{xy}(\rho_0,0)}{g_{xx}(\rho_0,0)} = \mathcal{O}(|\eta|) = o(\rho_0). \tag{3.15}$$

Similarly, we have the following estimate for the third partial derivatives

$$\frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l} g(x,y)\Big|_{(x,y)=(\rho(\theta),\theta)} = \mathcal{O}\left(\rho_0^{-5}\right),$$

uniformly for $\theta = o(\rho_0)$, and for any nonnegative integers k and l such that $k+l=3$. This shows that if $\eta = o(\rho_0)$, then we have the Taylor approximation

$$\begin{aligned} &g(\rho(\eta),\eta) = g(\rho_0,0) + g_y(\rho_0,0)\eta + g_x(\rho_0,0)(\rho(\eta)-\rho_0) \\ &+\frac{1}{2}\left(g_{yy}(\rho_0,0)\,\eta^2 + 2g_{xy}(\rho_0,0)\,\eta(\rho(\eta)-\rho_0) + g_{xx}(\rho_0,0)\,(\rho(\eta)-\rho_0)^2\right) + \mathcal{O}(|\eta|^3\rho_0^{-5}), \end{aligned}$$

where the implied constant in the error term is independent of n. Using (3·15) and saddle-point equation $n = -g_x(\rho_0,0)$, we deduce that

$$\begin{aligned} &g(\rho(\eta),\eta) - g(\rho_0,0) + n(\rho(\eta)-\rho_0) = \\ &\qquad g_y(\rho_0,0)\eta + \left(\frac{g_{yy}(\rho_0,0)g_{xx}(\rho_0,0) - (g_{xy}(\rho_0,0))^2}{g_{xx}(\rho_0,0)}\right)\frac{\eta^2}{2} + \mathcal{O}(|\eta|^3\rho_0^{-5}). \end{aligned}$$

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Let us define

$$\sigma_n^2 = \frac{g_{yy}(\rho_0,0)g_{xx}(\rho_0,0)-(g_{xy}(\rho_0,0))^2}{g_{xx}(\rho_0,0)} \quad \text{and} \quad \mu_n = g_y(\rho_0,0), \tag{3·16}$$

and we choose $\eta = t/\sigma_n$ where t is a fixed real number. Then our estimate (3·11) becomes

$$\frac{Q_n(u,v,w)}{Q_n(1,1,1)} \sim \exp\left(\frac{\mu_n}{\sigma_n}\,t + \frac{t^2}{2} + \mathcal{O}(\sigma_n^{-3}\rho_0^{-5})\right), \tag{3·17}$$

as $n \to \infty$ (this is valid as long as $\eta = t/\sigma_n = o(\rho_0)$, but we do not even know at this stage whether σ_n^2 is positive). Hence, let us estimate σ_n^2.

Assume first that $(\alpha,\beta) \neq (0,0)$. Once again, by the Mellin transform technique, we have

$$g_{yy}(\rho_0,0) \sim r\left(2\alpha^2 + 3(r+1)\alpha\beta + (r+1)(2r+1)\beta^2\right)\zeta(3)\rho_0^{-4}. \tag{3·18}$$

Putting the estimates (3·13), (3·14) and (3·18) into the formula for σ_n^2 in (3·16), we have

$$\sigma_n^2 \sim \left(\frac{r\zeta(3)}{2}\,\alpha^2 + \frac{r(r^2-1)\zeta(3)}{2}\,\beta^2\right)\rho_0^{-4}. \tag{3·19}$$

Hence, $\eta = \mathcal{O}(\rho_0^2)$ and the estimate (3·17) becomes

$$\frac{Q_n(u,v,w)}{Q_n(1,1,1)} \sim \exp\left(\frac{\mu_n}{\sigma_n}\,t + \frac{t^2}{2} + \mathcal{O}(\rho_0)\right).$$

Therefore, if $(\alpha,\beta) \neq (0,0)$ and t is any fixed real number, then the latter identity together with (3·10) yield

$$e^{-\mu_n t\sigma_n^{-1}}\,\mathbb{E}\left(e^{t\,\sigma_n^{-1}(\alpha X_n + \beta Y_n + \gamma Z_n)}\right) \sim e^{t^2/2},$$

as $n \to \infty$. This means, by Curtiss' theorem [10], that

$$\frac{\alpha X_n + \beta Y_n + \gamma Z_n - \mu_n}{\sigma_n} \xrightarrow{\text{d}} \to \mathcal{N}(0,1) \ \text{ as } \ n \to \infty.$$

To obtain the asymptotic formulas for σ_n^2 and μ_n, recall from the saddle-point equation and (3·7) (with $\epsilon = 0$) that $n = 2r\zeta(3)\rho_0^{-3} - 1/12r\rho_0^{-1} + \mathcal{O}(1)$. Inverting this yields

$$\rho_0^{-1} = \frac{n^{1/3}}{(2r\zeta(3))^{1/3}} + \mathcal{O}(n^{-1/3}). \tag{3·20}$$

So we first estimate σ_n^2 and μ_n in terms of ρ_0, then use the above to get the asymptotic formulas in terms of n. Such an estimate for σ_n^2 is already given in (3·19). The estimate of μ_n can be obtained easily from the Mellin transform of $g_y(t,0)$, which is

$$\frac{r}{2}\left((\alpha + (r+1)\beta)\,\zeta(s-2) + (\alpha + 2\gamma)\,\zeta(s-1)\right)\zeta(s)\Gamma(s).$$

By a straightforward calculation, we have

$$\sigma_n^2 \sim \frac{\alpha^2 + (r^2-1)\beta^2}{2^{7/3}(r\zeta(3))^{1/3}}\,n^{4/3} \ \text{ and } \ \mu_n = \left(\frac{1}{2}\alpha + \frac{r+1}{2}\beta\right)n + \frac{r^{1/3}\zeta(2)(\alpha+2\gamma)}{2^{5/3}(\zeta(3))^{2/3}}\,n^{2/3} + \mathcal{O}(n^{1/3}).$$

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If we now assume that $(\alpha,\beta)=(0,0)$ but $\gamma\neq 0$, then the Mellin transform of $g_{yy}(t,0)$ is $r\gamma^2\zeta(s-1)^2\Gamma(s)$ whose dominant singularity is a double pole at $s=2$. This leads to the asymptotic formula $g_{yy}(\rho_0,0)=r\gamma^2\rho_0^{-2}\log\left(\rho_0^{-1}\right)+\mathcal{O}(\rho_0^{-2})$. Hence, the formula in (3·16) gives

$$\sigma_n^2=r\gamma^2\rho_0^{-2}\log\left(\rho_0^{-1}\right)+\mathcal{O}(\rho_0^{-2}). \tag{3·21}$$

Thus, for a fixed real number t, we have

$$\eta=\frac{t}{\sigma_n}=\mathcal{O}(\rho_0|\log\rho_0|^{-1/2}).$$

So we still have our desired condition that $\eta=o(\rho_0)$. Moreover, applying (3·13) and (3·14) with $(\alpha,\beta)=(0,0)$, the estimate (3·15) becomes

$$\rho(\eta)-\rho_0\sim-\eta\,\frac{g_{xy}(\rho_0,0)}{g_{xx}(\rho_0,0)}=\mathcal{O}(|\eta|\rho_0)=\mathcal{O}(\rho_0^2|\log\rho_0|^{-1/2}).$$

On the other hand, for any $\theta=o(\rho_0)$, we have the following:

$$\begin{aligned} g_{xxx}(\rho(\theta),\theta) &= \mathcal{O}(\rho_0^{-5}), & g_{xxy}(\rho(\theta),\theta) &= \mathcal{O}(\rho_0^{-4}),\\ g_{xyy}(\rho(\theta),\theta) &= \mathcal{O}(\rho_0^{-3}|\log\rho_0|), & g_{yyy}(\rho(\theta),\theta) &= \mathcal{O}(\rho_0^{-3}). \end{aligned}$$

Therefore, (3·17) becomes

$$\frac{Q_n(u,v,w)}{Q_n(1,1,1)}\sim\exp\left(\frac{\mu_n}{\sigma_n}\,t+\frac{t^2}{2}+\mathcal{O}\left((\log n)^{-3/2}\right)\right).$$

Just as in the previous case, this is enough to prove the central limit theorem. The asymptotic formula for the variance in terms of n can be obtained from (3·21) using (3·20). The proof of Proposition 3·1 is complete.

Acknowledgments. The very existence of this work owes a debt to Balázs Szendrői, who brought the authors together. We thank him for his support and for generously sharing his ideas throughout the years. We also thank the referees for their useful comments and suggested improvements on the text. A.C. thanks the African Institute for Mathematical Sciences (AIMS), South Africa, for support during the first part of this collaboration. D.R. is supported by the Division for Research Development (DRD) of Stellenbosch University and the National Research Foundation (NRF) of South Africa, Grant number: 129413. A.R. is supported by Dipartimenti di Eccellenza and thanks SISSA for the excellent working conditions.

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