BULL. AUSTRAL. MATH. SOC. 42A04, 42A44
VOL. 9 (1973), 187-218.

Helgason's number and lacunarity constants

R.E. Edwards and Kenneth A. Ross

This paper studies the connection between the best possible value of a constant in the compact abelian case of a known inequality due to Helgason and the Λ_2-constants of sets of characters. Various estimates of and expressions for the best possible value are given.

1. Introduction; the numbers M_G and $\underline{\underline{h}}$

1.1. Helgason ([7], p. 245; [8], (36.10)) shows that if G is a CAG (= compact Hausdorff abelian group), then the inequality

(a) $$\|h\|_2 \le M\sup\left\{\|h*f\|_1 : f \in L^1(G),\ \|f^\wedge\|_u \le 1\right\}$$

holds for all $h \in L^2(G)$ with $M = \sqrt{2}$. [Note that the supremum in (a) is unaltered if we write $f \in \underline{\underline{T}}(G)$ in place of $f \in L^1(G)$, where $\underline{\underline{T}}(G)$ denotes the set of complex-valued trigonometric polynomials on G.] Moreover (see 1.3 below), (a) is equivalent to the inequality

(b) $$\|F\|_2 \le M\sup\{\|Ff^\wedge\|_1 : f \in C(G),\ \|f\|_u \le 1\}$$

holding for all $F \in C^{\hat{G}}$, where $\hat{G}$ denotes the character group of G and

Received 30 March 1973. The second author was partly supported by a grant from the National Science Foundation of the USA. He also gratefully acknowledges help and facilities of the Australian National University during his visit from September to December 1972. The authors are grateful to Professor P.A.P. Moran for helpful discussions, and in particular for the indication of the relevance of the central limit theorem to Theorem 2.11.

$C(G)$ the set of continuous complex-valued functions on G. Inequality (b) appears in Theorem (2.1) of [3].

For a given G, we will denote by M_G the smallest number $M \geq 0$ for which (a) (or (b)) is true. Clearly, $M_G \geq 1$ for every CAG G.

In what follows we introduce a certain number $\underline{h}$, defined in terms of Λ_2-constants of large finite sets (see 1.4 and 1.5 below), which we call the *Helgason number*. The reason for the name is that we shall prove the following facts:

(i) $M_G \leq \underline{h}$ for every CAG G (Corollary 1.8);

(ii) $M_G = \underline{h}$ for certain specifiable CAGs G (Corollary 1.12, Theorem 3.6, Corollary 3.8).

Helgason's result is included in the inequalities

(iii) $2\pi^{-\frac{1}{2}} \leq \underline{h} \leq 2^{\frac{1}{2}}$ (Theorem 2.11, Corollary 2.5),

which we shall prove on the way.

We introduce also a somewhat similarly-defined number $\underline{h}_n$ for every positive integer n, showing that

(iv) $\underline{h}_n \leq \underline{h}_{n+1}$ and $\underline{h} = \lim_{n\to\infty} \underline{h}_n$ (Lemma 1.6).

We will also show that

(v) $\underline{h}_2 = \pi\sqrt{2}/4$ (Theorem 2.10), and that

(vi) $\underline{h}_n = \sup\left(\sum_{k=1}^{n} c_k^2\right)^{\frac{1}{2}} \Big/ E_n(c_1, \ldots, c_n) \leq (2-1/n)^{\frac{1}{2}}$,

where

$$E_n(c_1, \ldots, c_n) = (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left|c_1 e^{i\theta_1} + \ldots + c_n e^{i\theta_n}\right| d\theta_1 \ldots d\theta_n$$

and $c_1, \ldots, c_n$ denote nonnegative real numbers, not all zero (Corollaries 2.7 and 2.4).

In Section 3, we show that each $\underline{h}_n$ can be given in terms of sets of

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

characters of T (the circle group) only.

We have been unable to evaluate $\underline{\underline{h}}$; it would be very interesting to know whether or not $\underline{\underline{h}} < \sqrt{2}$.

We start with a simple lemma.

LEMMA 1.2. *If G is a CAG and $g \in \underline{\underline{T}}(G)$, then*

$$\|g\|_1 = \sup\left\{\left|\sum_{\chi\in\hat{G}} g^\wedge(\chi)f^\wedge(\chi)\right| : f \in C(G),\ \|f\|_u \le 1\right\} .$$

Proof. If λ_G denotes normalised Haar measure on G , then

$$\begin{aligned}\|g\|_1 &= \int |g|d\lambda_G = \sup\left\{\left|\int g(x)f\left(x^{-1}\right)d\lambda_G(x)\right| : f \in C(G),\ \|f\|_u \le 1\right\}\\ &= \sup\left\{\left|\sum_{\chi\in\hat{G}} g^\wedge(\chi)\int \chi(x)f\left(x^{-1}\right)d\lambda_G(x)\right| : f \in C(G),\ \|f\|_u \le 1\right\}\\ &= \sup\left\{\left|\sum_{\chi\in\hat{G}} g^\wedge(\chi)f^\wedge(\chi)\right| : f \in C(G),\ \|f\|_u \le 1\right\} .\end{aligned}$$

1.3. Now we verify the equivalence of (a) and (b) in 1.1. The supremum on the right of (a) is

$$\sup\left\{\left\|\sum_{\chi\in\hat{G}} a(\chi)h^\wedge(\chi)\chi\right\|_1 : \operatorname{supp} a \ \text{ finite},\ \|a\|_u \le 1\right\}$$

which, by Lemma 1.2, is equal to

$$\begin{aligned}&\sup\left\{\left|\sum_{\chi\in\hat{G}} a(\chi)h^\wedge(\chi)f^\wedge(\chi)\right| : \operatorname{supp} a \ \text{ finite},\ \|a\|_u \le 1,\ f \in C(G),\ \|f\|_u \le 1\right\}\\ &\qquad = \sup_f \sup_a \left\{\left|\sum_{\chi\in\hat{G}} a(\chi)h^\wedge(\chi)f^\wedge(\chi)\right|\right\}\\ &\qquad = \sup\left\{\|h^\wedge f^\wedge\|_1 : f \in C(G),\ \|f\|_u \le 1\right\} .\end{aligned}$$

Thus (a) is equivalent to (b) for F $(= h^\wedge)$ in $l^2(\hat{G})$; but this is easily seen to be equivalent to (b) for arbitrary $F \in C^{\hat{G}}$.

1.4. If G is a CAG and E is a subset of $\hat{G}$, we write $\underline{\underline{T}}_E(G)$ for the set of $f \in \underline{\underline{T}}(G)$ such that $f^\wedge(\chi) = 0$ for every $\chi \in \hat{G}\backslash E$. We also write

$$\Lambda_G(E) = \sup\left\{\|f\|_2 : f \in \underline{\underline{T}}_E(G),\ \|f\|_1 = 1\right\} \le \infty$$

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

and call $\Lambda_G(E)$ the Λ_2*-constant* of E. It is easy to see that $\Lambda_G(E)$ is a finite assumed maximum whenever E is finite. Moreover,

$$\Lambda_G(E) = \sup\{\Lambda_G(F) : F \text{ finite}, F \subseteq E\} .$$

1.5. Define sets S and S_n (n a positive integer) of nonnegative real numbers as follows.

S is the set of real numbers $\kappa \geq 0$ with the property that, for every positive integer n, there exists a CAG K_n and an n-element subset E_n of $\hat{K}_n$ such that

$$\Lambda_{K_n}(E_n) \leq \kappa .$$

S_n is the set of real numbers $\kappa \geq 0$ with the property that there exists a CAG K and an n-element subset E of $\hat{K}$ such that

$$\Lambda_K(E) \leq \kappa .$$

The proof of Corollary 2.4 below shows incidentally that $2^{\frac{1}{2}} \in S$.

We now define

$$\underline{\underline{h}} = \inf S , \quad \underline{\underline{h}}_n = \inf S_n .$$

It is simple to verify that

$$S_{n+1} \subseteq S_n , \quad S = \bigcap_{n=1}^{\infty} S_n .$$

These observations render the next lemma obvious.

LEMMA 1.6. *We have* $\underline{\underline{h}}_n \leq \underline{\underline{h}}_{n+1}$ *for every positive integer* n, *and*

$$\underline{\underline{h}} = \lim_{n\to\infty} \underline{\underline{h}}_n .$$

THEOREM 1.7. *Let* n *be a positive integer. Then* (b) *of* 1.1 *holds with* $M = \underline{\underline{h}}_n$ *for every CAG* G *and every* $F \in C^{\hat{G}}$ *whose support has cardinal* $\nu(\mathrm{supp}F)$ *at most* n.

Proof. Let $\kappa \in S_n$ and let K be a CAG such that there exists an

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

n-element subset $E = \{\zeta_1, \ldots, \zeta_n\}$ of $\hat{K}$ for which $\Lambda_K(E) \le \kappa$. Suppose $\nu(\mathrm{supp}F) = r \le n$ and enumerate $\mathrm{supp}F$ as $\{\chi_1, \ldots, \chi_r\}$. Then, for every $x \in G$, we have

$$\left(\sum_{j=1}^{r} |F(\chi_j)|^2\right)^{\frac{1}{2}} \le \kappa \int_K \left|\sum_{j=1}^{r} F(\chi_j)\chi_j(x)\zeta_j(y)\right| d\lambda_K(y) .$$

Integrating over G and using Fubini's Theorem, this gives

$$\left(\sum_{j=1}^{r} |F(\chi_j)|^2\right)^{\frac{1}{2}} \le \kappa \int_K \left\|\sum_{j=1}^{r} F(\chi_j)\zeta_j(y)\chi_j\right\|_{L^1(G)} d\lambda_K(y) ,$$

which shows that

$$\left(\sum_{j=1}^{r} |F(\chi_j)|^2\right)^{\frac{1}{2}} \le \kappa \left\|\sum_{j=1}^{r} F(\chi_j)\zeta_j(y_0)\chi_j\right\|_{L^1(G)}$$

for some $y_0 \in K$. Using Lemma 1.2, it follows that

$$\left(\sum_{j=1}^{r} |F(\chi_j)|^2\right)^{\frac{1}{2}} \le \kappa\sup\left\{\left|\sum_{j=1}^{r} F(\chi_j)\zeta_j(y_0)f^\wedge(\chi_j)\right| : f \in C(G),\ \|f\|_u \le 1\right\}$$
$$\le \kappa\sup\left\{\sum_{j=1}^{r} |F(\chi_j)f^\wedge(\chi_j)| : f \in C(G),\ \|f\|_u \le 1\right\} .$$

Since this is true for every $\kappa \in S_n$, it remains true with $\underline{\underline{h}}_n$ in place of κ. Thus, (b) of 1.1 is true with $M = \underline{\underline{h}}_n$ for the stated functions F.

COROLLARY 1.8. *We have* $M_G \le \underline{\underline{h}}$ *for every CAG* G.

Proof. By Lemma 1.6 and Theorem 1.7, (b) of 1.1 holds with $M = \underline{\underline{h}}$ for every $f \in C^{\hat{G}}$ having a finite support. But then (b) holds with $M = \underline{\underline{h}}$ for every $F \in C^{\hat{G}}$, and so $M_G \le \underline{\underline{h}}$.

REMARK. From Theorem 1.7 it follows that, if G is of finite order n, then $M_G \le \underline{\underline{h}}_n$ which, by Corollary 2.4, is at most $(2-1/n)^{\frac{1}{2}} < 2^{\frac{1}{2}}$. Thus Helgason's inequality (that is, 1.1 (a) with $M = 2^{\frac{1}{2}}$) is not best possible when only groups of given finite order n are considered. In addition it can be shown that, if G is the subgroup $\{-1, 1\}$ of T, then $M_G = 1$ whereas (by Theorem 2.10) $\underline{\underline{h}}_2 = \pi\sqrt{2}/4 > 1$.

THEOREM 1.9. *Suppose that G is a CAG, that E is a Sidon subset of $\hat{G}$, and that $S_G(E)$ is the Sidon constant of E, that is, the smallest nonnegative real number κ for which*

$$\|f^\wedge\|_1 \le \kappa\|f\|_u$$

for every $f \in \underline{\underline{T}}_E(G)$. Then

$$\Lambda_G(E) \le M_G S_G(E) .$$

Proof. Let $f \in \underline{\underline{T}}_E(G)$. Using (b) of 1.1 with $M = M_G$, we have

$$\|f\|_2 = \|f^\wedge\|_2 \le M_G \sup\left\{\sum_{\chi\in\hat{G}} |f^\wedge(\chi)g^\wedge(\chi)| : g \in C(G),\ \|g\|_u \le 1\right\}$$
$$= M_G \sup\left\{\left|\sum_{\chi\in E} f^\wedge(\chi)\omega(\chi)g^\wedge(\chi)\right| : g \in C(G),\ \|g\|_u \le 1,\ \omega \in \Omega\right\} ,$$

where $\Omega = T^{\hat{G}}$. Writing κ for $S_G(E)$, a known property of Sidon sets ([8], (37.2)) asserts that every $\omega \in \Omega$ agrees on E with $\mu_\omega^\wedge$ for some $\mu_\omega \in M(G)$ satisfying $\|\mu_\omega\| \le \kappa$. It follows that

$$\|f\|_2 \le M_G \sup\left\{\left|\sum_{\chi\in\hat{G}} f^\wedge(\chi)k^\wedge(\chi)\right| : k \in C(G),\ \|k\|_u \le \kappa\right\}$$
$$= M_G \kappa \sup\left\{\left|\sum_{\chi\in\hat{G}} f^\wedge(\chi)k^\wedge(\chi)\right| : k \in C(G),\ \|k\|_u \le 1\right\}$$
$$= M_G \kappa \|f\|_1 ,$$

the last step by Lemma 1.2. Thus $\Lambda_G(E) \le M_G \kappa$.

COROLLARY 1.10. *Let G be a CAG. Then*

(1) $$M_G \le \underline{\underline{h}} \le M_G \liminf_{n\to\infty} \inf\{S_G(E) : E \subseteq \hat{G},\ \nu(E) = n\} .$$

(The infimum of the empty set is understood to be ∞.)

Proof. The first inequality in (1) is just Corollary 1.8. For the rest, let t_n denote the infimum appearing in (1), which we may assume to be finite. If $E \subseteq \hat{G}$ and $\nu(E) = n$, then $\Lambda_G(E) \in S_n$ and so $\Lambda_G(E) \geq \underline{\underline{h}}_n$. By Theorem 1.9 we therefore have

$$\underline{\underline{h}}_n \leq \Lambda_G(E) \leq M_G S_G(E) .$$

From this it follows that

(2) $$\underline{\underline{h}}_n \leq M_G t_n .$$

The second inequality in (1) follows from (2) and Lemma 1.6.

1.11. If G is a CAG, a subset E of $\hat{G}$ will be termed *strongly independent* if, whenever $\chi_1, \ldots, \chi_n$ denote distinct elements of E and $m_1, \ldots, m_n$ denote integers, the relation

$$\chi_1^{m_1} \ldots \chi_n^{m_n} = 1$$

implies that $m_1 = \ldots = m_n = 0$. For example, if I is any set and $G = T^I$, then the set of projections

$$\pi_{i_0} : (x_i)_{i \in I} \mapsto x_{i_0}$$

with $i_0 \in I$ is a strongly independent subset of $\hat{G}$.

We list several properties of strongly independent sets which will be useful in the sequel.

(i) If G is a CAG and E a subset of $\hat{G}$, then E is strongly independent if and only if the mapping $\phi : x \mapsto (\chi(x))_{\chi \in E}$ maps G onto T^E , where T denotes the circle group.

Proof. The image $H = \phi(G)$ is a closed subgroup of T^E. If the character group of T be identified with Z (the additive group of integers) in the usual fashion, the annihilator A in $(T^E)^\wedge$ of H is precisely the set of Z-valued functions $\chi \mapsto m(\chi)$ on E having finite supports and such that

$$\prod_{\chi \in E} \chi^{m(\chi)} = 1 .$$

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

The strong independence of E is equivalent to the assertion that A is the trivial subgroup of $P^*_{\chi \in E} Z$. Since H is the annihilator in T^E of A , this occurs if and only if $H = T^E$.

(ii) If G *is a CAG and* E *a strongly independent subset of* $\hat{G}$ *, then* $S_G(E) = 1$.

Proof. This follows at once from *(i)* and the definition of $S_G(E)$ in 1.9.

(iii) Suppose that G *is a CAG and that* E *is a strongly independent subset of* $\hat{G}$ *. If* $\chi_1, \ldots, \chi_n$ *are distinct elements of* E *and* $c_1, \ldots, c_n$ *are complex numbers, then*

$$\int_G \left|\sum_{k=1}^{n} c_k \chi_k\right| d\lambda_G = \int_G \left|\sum_{k=1}^{n} |c_k| \chi_k\right| d\lambda_G .$$

Proof. For $k \in \{1, 2, \ldots, n\}$, choose $\omega_k \in T$ such that $c_k = |c_k| \omega_k$. By *(i)*, there exists $a \in G$ such that $\chi_k(a) = \omega_k$ for $k \in \{1, 2, \ldots, n\}$. Then $\sum_{k=1}^{n} c_k \chi_k$ is the a-translate of $\sum_{k=1}^{n} |c_k| \chi_k$, and the stated equality follows from translation-invariance of λ_G .

COROLLARY 1.12. *Let* G *be a CAG with the property that, for every positive integer* n *,* $\hat{G}$ *contains an* n*-element strongly independent set. Then* $M_G \in S$ *and* $\underline{\underline{h}} = M_G$.

Proof. For each positive integer n , let I_n be an n-element strongly independent subset of $\hat{G}$. By 1.11 *(ii)*, we have $S_G(I_n) = 1$ and so, by Theorem 1.9, $\Lambda_G(I_n) \le M_G$. Since this is the case for every positive integer n , it follows that $M_G \in S$. This entails that $\underline{\underline{h}} \le M_G$ and the rest ensues from Corollary 1.8.

REMARK. From Corollaries 1.8 and 1.12 it follows that $\underline{\underline{h}}$ is the maximum of the numbers M_G when G ranges over the class of CAGs.

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

1.13. We insert here some remarks about the effect of continuous group homomorphisms.

Let G and K be CAGs and suppose that ϕ is a continuous homomorphism of G onto K. Write ϕ^* for the dual isomorphism of $\hat{K}$ into $\hat{G}$ defined by $\phi^*(\zeta) = \zeta \circ \phi$ for $\zeta \in \hat{K}$, and let Φ denote the mapping $f \mapsto f \circ \phi$ of $C(K)$ into $C(G)$. In what follows, E denotes a subset of $\hat{K}$ and $F = \phi^*(E) \subseteq \hat{G}$. It is plain that

(1) Φ preserves uniform norms

and that

(2) Φ maps $C_E(K)$ onto $C_F(G)$.

$\big(C_E(K)$ denotes the set of $g \in C(K)$ such that $g\hat{}(\zeta) = 0$ for $\zeta \in \hat{K}\backslash E$, and $C_F(G)$ is defined analogously.$\big)$

By considering the functional $f \mapsto \int_G (\Phi f)d\lambda_G$ and invoking the uniqueness of normalised Haar measure on K, we infer that

(3) $$\int_G (f\circ\phi)d\lambda_G = \int_K fd\lambda_K$$

for every $f \in C(K)$.

From (3) we may infer first that

(4) ϕ preserves L^p-norms $(0 < p < \infty)$

and second that, if $\chi \in \hat{G}$ and $f \in C(K)$, then

(5) $(f\circ\phi)\hat{}(\chi) = f\hat{}\big(\phi^{*-1}(\chi)\big)$ if $\chi \in \phi^*(\hat{K})$ and 0 otherwise.

In particular,

(6) $$\|(f\circ\phi)\hat{}\|_1 = \|f\hat{}\|_1 .$$

In view of (4) and (2), it follows that the Λ_2-constant of F is equal to the Λ_2-constant of E. Similarly, from (1), (2) and (6) it appears that the Sidon constant of F is equal to the Sidon constant of E.

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

From (3) it follows also that

(7) $$(f\circ\phi) * (g\circ\phi) = (f*g)\circ\phi$$

for f and g in $C(K)$. If ϕ is an isomorphism (which occurs if and only if ϕ^* maps $\hat{K}$ onto $\hat{G}$, that is, if and only if ϕ maps $C(K)$ onto $C(G)$), we infer from (7) and reference to 1.1 (a) that $M_G = M_K$.

We end this section by recording another property of the number M_G for a given G.

LEMMA 1.14. *Suppose that G is a CAG, that $1 \le p \le 2$, and that $q = 2p/(2-p)$. For $F \in C^{\hat{G}}$ we have*

$$\|F\|_q = \sup\{\|F\phi\|_2 : \|\phi\|_{p'} = 1\}.$$

Proof. We have

$$\sup\left\{\|F\phi\|_2^2 : \|\phi\|_{p'} = 1\right\} = \sup\left\{\|F^2\phi^2\|_1 : \|\phi\|_{p'} = 1\right\}.$$

Now $\|\phi\|_{p'} = 1$ if and only if $\|\phi^2\|_{\frac{1}{2}p'} = 1$; and every nonnegative ψ satisfying $\|\psi\|_{\frac{1}{2}p'} = 1$ has the form ϕ^2 for some ϕ satisfying $\|\phi\|_{p'} = 1$. So the above supremum equals

$$\sup\left\{\|F^2\psi\|_1 : \|\psi\|_{\frac{1}{2}p'} = 1\right\} = \|F^2\|_{(\frac{1}{2}p')'}.$$

Since $(\frac{1}{2}p')' = p/(2-p) = \frac{1}{2}q$, the supremum equals

$$\|F^2\|_{\frac{1}{2}q} = \|F\|_q^2.$$

THEOREM 1.15. *Let G be a CAG, $1 \le p \le 2$ and $q = 2p/(2-p)$. Then*

$$\|F\|_q \le M_G \sup\{\|Ff^\wedge\|_p : f \in C(G),\ \|f\|_u \le 1\}$$

for every $F \in C^{\hat{G}}$. If $F \in C^{\hat{G}}$ and $Ff^\wedge \in l^p(\hat{G})$ for every $f \in C(G)$, then $F \in l^q(\hat{G})$. (*Cf.* [3], Corollary (2.3).)

Proof. By Lemma 1.14 and (b) of 1.1, we have

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

$$\begin{aligned}\|F\|_q &= \sup\{\|F\phi\|_2 : \|\phi\|_{p'} = 1\} \\ &\le M_G \sup_{\|\phi\|_{p'}=1} \sup_{\|f\|_u \le 1} \|F\phi f^\wedge\|_1 \\ &= M_G \sup_f \sup_\phi \|Ff^\wedge \phi\|_1 \\ &= M_G \sup_f \|Ff^\wedge\|_p .\end{aligned}$$

The rest follows from the closed graph theorem.

2. Estimates for $\underline{h}_n$ and $\underline{h}$

THEOREM 2.1. *Let n be a positive integer, K any CAG and I any n-element strongly independent subset of $\hat{K}$. Let G be any CAG and E a subset of $\hat{G}$ having at least n elements. Then*

$$\Lambda_K(I) \le \Lambda_G(E) .$$

Proof. Enumerate I as $\{\zeta_1, \ldots, \zeta_n\}$ and choose n distinct elements $\chi_1, \ldots, \chi_n$ of E. Any $f \in \underline{\underline{T}}_I(K)$ can be written

$$f = \sum_{k=1}^{n} c_k \zeta_k ,$$

the c_k being complex numbers. For $y \in K$ let

$$f_y : x \mapsto \sum_{k=1}^{n} c_k \zeta_k(y) \chi_k(x) ,$$

so that $f_y \in \underline{\underline{T}}_E(G)$. Then

$$\|f\|_2 = \left(\sum_{k=1}^{n} |c_k|^2\right)^{\frac{1}{2}} = \|f_y\|_2 \le \Lambda_G(E) \|f_y\|_1$$

$$= \Lambda_G(E) \int_G \left|\sum_{k=1}^{n} c_k \zeta_k(y) \chi_k(x)\right| d\lambda_G(x) ,$$

and so also (using Fubini's Theorem)

$$\|f\|_2 \le \Lambda_G(E) \int_G \left\{\int_K \left|\sum_{k=1}^{n} c_k \zeta_k(y) \chi_k(x)\right| d\lambda_K(y)\right\} d\lambda_G(x) .$$

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

By 1.11 *(iii)*, the inner integral is equal to

$$\int_K \left|\sum_{k=1}^{n} c_k \zeta_k(y)\right| d\lambda_K(y) = \|f\|_1 ,$$

which is independent of $x \in G$. Thus

$$\|f\|_2 \le \Lambda_G(E)\|f\|_1 ,$$

showing that $\Lambda_K(I) \le \Lambda_G(E)$.

COROLLARY 2.2. *Let K and I be as in Theorem 2.1. Then* $\underline{\underline{h}}_n = \min\{\Lambda_G(E) : G \text{ a CAG}, E \subseteq \hat{G}, \nu(E) = n\} = \Lambda_K(I)$.

Proof. Let

$$c = \inf\{\Lambda_G(E) : G \text{ a CAG}, E \subseteq \hat{G}, \nu(E) = n\} .$$

The definitions in 1.5 show that $c = \underline{\underline{h}}_n$. On the other hand, Theorem 2.1 shows that c is an assumed minimum equal to $\Lambda_K(I)$.

REMARK 2.3. Corollary 2.2 shows that $\underline{\underline{h}}_n$ can be computed in terms of Λ_2-constants of n-element strongly independent sets of characters. Although there are no nontrivial independent subsets of $\hat{T}$, Theorem 3.5 below shows that $\underline{\underline{h}}_n$ can nevertheless be given in terms of Λ_2-constants of n-element subsets of $\hat{T}$.

COROLLARY 2.4. *We have* $\underline{\underline{h}}_n \le (2-1/n)^{\frac{1}{2}}$.

Proof. In view of Corollary 2.2, it suffices to show that

$$\Lambda_K(P) \le (2-1/n)^{\frac{1}{2}} ,$$

where $K = T^n$ and $P = \{\pi_1, \ldots, \pi_n\}$ is the set of all projections of K. There exists $f \in \underline{\underline{T}}_P(K)$ such that $\|f\|_1 = 1$ and

$$\Lambda_K(P) = \|f\|_2 . \tag{1}$$

Write

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

$$f = \sum_{k=1}^{n} c_k \pi_k ,$$

where the c_k are certain complex numbers. Then

$$\int_K |f|^4 d\lambda_K = \sum_{j,k,l,m=1}^{n} c_j c_k \bar{c}_l \bar{c}_m \int_K \pi_j \pi_k \bar{\pi}_l \bar{\pi}_m d\lambda_K ,$$

the integrals remaining being equal to 1 or 0 according as the integrand is or is not the character 1 of K. It follows that

$$(2) \qquad \int_K |f|^4 d\lambda_K = \sum_{k=1}^{n} |c_k|^4 + 2 \sum_{j,k=1, j\neq k}^{n} |c_j|^2 |c_k|^2 .$$

On the other hand,

$$(3) \qquad \left(\int_K |f|^2 d\lambda_K\right)^2 = \left(\sum_{k=1}^{n} |c_k|^2\right)^2$$

$$= \sum_{k=1}^{n} |c_k|^4 + \sum_{j,k=1, j\neq k}^{n} |c_j|^2 |c_k|^2 .$$

Write $|c_k|^2 = A_{k-1}$ for $k \in \{1, 2, \ldots, n\}$. We claim that

$$(4) \qquad \sum_{r=0}^{n-1} A_r^2 + 2 \sum_{r,s=0, r\neq s}^{n-1} A_r A_s \leq (2-1/n)\left(\sum_{r=0}^{n-1} A_r^2 + \sum_{r,s=0, r\neq s}^{n-1} A_r A_s\right) ,$$

that is, that

$$\sum_{r,s=0, r\neq s}^{n-1} A_r A_s \leq (n-1) \sum_{r=0}^{n-1} A_r^2 .$$

In fact, define $\rho : Z \to \{0, 1, \ldots, n-1\}$ by

$$t = qn + \rho(t) ,$$

where $q \in Z$. Then

$$\sum_{r,s=0, r\neq s}^{n-1} A_r A_s = \sum_{r=0}^{n-1} \sum_{s=0, s\neq r}^{n-1} A_r A_s$$

which, since $m \mapsto \rho(r+m)$ maps $\{1, 2, \ldots, n-1\}$ one-to-one onto $\{0, 1, \ldots, n-1\}\backslash\{r\}$, equals

$$\sum_{r=0}^{n-1}\sum_{m=1}^{n-1} A_r A_{\rho(r+m)} = \sum_{m=1}^{n-1}\sum_{r=0}^{n-1} A_r A_{\rho(r+m)}$$
$$\leq \sum_{m=1}^{n-1}\left(\sum_{r=0}^{n-1} A_r^2\right)^{\frac{1}{2}}\left(\sum_{r=0}^{n-1} A_{\rho(r+m)}^2\right)^{\frac{1}{2}} .$$

Since $r \mapsto \rho(r+m)$ maps $\{0, 1, \ldots, n-1\}$ one-to-one onto itself, this equals

$$\sum_{m=1}^{n-1}\left(\sum_{r=0}^{n-1} A_r^2\right) = (n-1)\sum_{r=0}^{n-1} A_r^2 ,$$

which verifies (4). Collecting (2), (3) and (4), we see that

$$\|f\|_4^4 \leq (2-1/n)\|f\|_2^4 ,$$

and hence

(5) $$\|f\|_4 \leq (2-1/n)^{1/4}\|f\|_2 .$$

From (5) and Hölder's inequality it follows that

(6) $$\|f\|_2 \leq (2-1/n)^{\frac{1}{2}}\|f\|_1 = (2-1/n)^{\frac{1}{2}} ,$$

and the proof is completed by reference to (1).

COROLLARY 2.5. *We have* $\underline{\underline{h}} \leq \sqrt{2}$.

Proof. Lemma 1.6 and Corollary 2.4.

Corollaries 1.8 and 2.5 provide an alternative proof of Helgason's version of 1.1 (a).

COROLLARY 2.6. *Let* K *be a CAG such that* $\hat{K}$ *contains an infinite strongly independent set* I . *Then*

$$\underline{\underline{h}} = \min\{\Lambda_G(E) : G \text{ a CAG}, E \subseteq \hat{G}, E \text{ infinite}\} = \Lambda_K(I) .$$

In particular,

$$\underline{\underline{h}} = \Lambda_{T^\infty}(\{\pi_1, \pi_2, \ldots\}) ,$$

where $T^\infty = T^N$ *with* $N = \{1, 2, \ldots\}$ *and* π_n *is the* n-*th projection of* T^∞

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

Proof. Let G be a CAG and E an infinite subset of $\hat{G}$. Let F be any finite subset of I. By Theorem 2.1 and Corollary 2.2, we have

$$(1) \qquad \Lambda_G(E) \geq \Lambda_K(F) = \underline{\underline{h}}_n \quad \text{where} \quad n = \nu(F) .$$

Hence $\Lambda_G(E) \geq \underline{\underline{h}}_n$ for all n and so, by Lemma 1.6, $\Lambda_G(E) \geq \underline{\underline{h}}$. Using (1) and Lemma 1.6, we also have

$$\begin{aligned}\Lambda_K(I) &= \sup\{\Lambda_K(F) : F \subseteq I,\ F \text{ finite}\} \\ &= \sup\{\underline{\underline{h}}_n : n = 1, 2, \ldots\} = \underline{\underline{h}} ,\end{aligned}$$

and this completes the proof.

COROLLARY 2.7. *If n is a positive integer, then*

$$\underline{\underline{h}}_n = \sup\left(\sum_{k=1}^{n} c_k^2\right)^{\frac{1}{2}} \Big/ E_n(c_1, \ldots, c_n) ,$$

where

$$E_n(c_1, \ldots, c_n) = (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left|c_1 e^{i\theta_1} + \ldots + c_n e^{i\theta_n}\right| d\theta_1 \ldots d\theta_n$$

and the supremum is taken over all nonnegative numbers $c_1, \ldots, c_n$, not all zero.

Proof. Applying Corollary 2.2 with $K = T^n$ and I the set of all projections of T^n, we see that

$$\underline{\underline{h}}_n = \sup\left(\sum_{k=1}^{n} |c_k|^2\right)^{\frac{1}{2}} \Big/ \left\|\sum_{k=1}^{n} c_k \pi_k\right\|_1 ,$$

the c_k being complex and not all zero. By 1.11 *(iii)*,

$$\left\|\sum_{k=1}^{n} c_k \pi_k\right\|_1 = \left\|\sum_{k=1}^{n} |c_k| \pi_k\right\|_1 ,$$

and so we may assume all the c_k to be real and nonnegative. Finally, since $\lambda_{T^n} = \lambda_T \otimes \ldots \otimes \lambda_T$ (n factors),

$$\left\| \sum_{k=1}^{n} c_k \pi_k \right\|_1 = E_n(c_1, \ldots, c_n) .$$

REMARK 2.8. Corollary 2.7 indicates connections between the numbers $\underline{\underline{h}}_n$ and the so-called Pearson random walk ([*10*], pp. 419-421; [9], pp. 496-500; [*1*], pp. 10-13), wherein the walker begins at the origin and walks in the plane for a distance c_1 at random angle θ_1, then proceeds for a distance c_2 at a random angle θ_2, and so on. The integral $E_n(c_1, \ldots, c_n)$ plainly denotes the expected distance of the walker from the origin after completing the first n steps. A search of the literature indicates that the numbers $E_n(c_1, \ldots, c_n)$ have not yet been computed or estimated by machine.

LEMMA 2.9. *(i) Let G be a CAG and let χ_1 and χ_2 be elements of $\hat{G}$ such that $\phi = \chi_1\chi_2^{-1}$ is of infinite order. Then*

$$\Lambda_G(\{\chi_1, \chi_2\}) = \pi\sqrt{2}/4 .$$

(ii) If G is a connected CAG, then $\Lambda_G(E) = \pi\sqrt{2}/4$ for every two-element subset E of $\hat{G}$.

Proof. *(i)* Let $E = \{\chi_1, \chi_2\}$. We need to show that the maximum of $\|g\|_2/\|g\|_1$, for $g = c_1\chi_1 + c_2\chi_2$ subject to $(c_1, c_2) \neq (0, 0)$, is $\pi\sqrt{2}/4$. In doing this we may plainly assume that $|c_1| \leq |c_2| = 1$ and also that $c_2 = 1$. Let $r = |c_1|$ and select $w \in T$ so that $wr = c_1$. Then we have

$$\|g\|_2 = (1+r^2)^{\frac{1}{2}} .$$

The character ϕ is of infinite order if and only if $\{\phi\}$ is strongly independent, and (by 1.11 *(i)*) this is so if and only if $\phi(G) = T$. Also we have $g = (f\circ\phi)\chi_2$ where $f(z) = c_1z + 1$ for $z \in T$. Hence by 1.13 (3), we have

$$\|g\|_1 = \|f\circ\phi\|_1 = \int_T |f(z)|d\lambda_T(z) = \int_T |c_1 z+1|d\lambda_T(z)$$
$$= \int_T |rwz+1|d\lambda_T(z)$$
$$= \int_T |rz+1|d\lambda_T(z) \quad \text{(by invariance of } \lambda_T\text{)}$$
$$= (2\pi)^{-1}\int_{-\pi}^{\pi} |re^{i\theta}+1|d\theta$$
$$= \pi^{-1}\int_0^{\pi} (1+r^2+2r\cos\theta)^{\frac{1}{2}}d\theta\ .$$

Thus we have to show that the maximum of

$$(1+r^2)^{\frac{1}{2}}\Big/\pi^{-1}\int_0^{\pi} (1+r^2+2r\cos\theta)^{\frac{1}{2}}d\theta\ ,$$

subject to $0 \le r \le 1$, is $\pi\sqrt{2}/4$, that is, that the minimum of

$$(1+r^2)^{-\frac{1}{2}}\int_0^{\pi} (1+r^2+2r\cos\theta)^{\frac{1}{2}}d\theta\ ,$$

subject to $0 \le r \le 1$, is $2\sqrt{2}$. On putting $a = (1+r^2)^{-1}2r$, it comes to the same thing to show that the minimum of

$$I(a) = \int_0^{\pi} (1+a\cos\theta)^{\frac{1}{2}}d\theta\ ,$$

subject to $0 \le a \le 1$, is $2\sqrt{2}$. Now

$$I(1) = \int_0^{\pi} (1+\cos\theta)^{\frac{1}{2}}d\theta = \sqrt{2}\int_0^{\pi} \cos\tfrac{1}{2}\theta d\theta = 2\sqrt{2}\int_0^{\frac{1}{2}\pi} \cos\alpha d\alpha = 2\sqrt{2}\ ,$$

and so it will suffice to show that $I'(a) \le 0$ for $0 < a < 1$. But

$$I'(a) = \tfrac{1}{2}\int_0^{\pi} \cos\theta(1+a\cos\theta)^{-\frac{1}{2}}d\theta = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} + \tfrac{1}{2}\int_{\frac{1}{2}\pi}^{\pi}$$
$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos\theta(1+a\cos\theta)^{-\frac{1}{2}}d\theta - \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos\phi(1-a\cos\phi)^{-\frac{1}{2}}d\phi$$
$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos\theta\left[(1+a\cos\theta)^{-\frac{1}{2}}-(1-a\cos\theta)^{-\frac{1}{2}}\right]d\theta\ ,$$

which is nonpositive since the integrand is nonpositive throughout the range of integration.

(ii) This statement follows from *(i)* because, if G is connected, $\phi(G)$ is a closed connected subgroup of T and so coincides with T if and only if it has at least two elements, that is, if and only if ϕ is not the constant character 1 .

THEOREM 2.10. *We have* $\underline{\underline{h}}_2 = \pi\sqrt{2}/4$.

Proof. By Lemma 2.9, we have

$$\Lambda_{T^2}(P) = \pi\sqrt{2}/4$$

where $P = \{\pi_1, \pi_2\}$ is the set of projections of T^2 . Now apply Corollary 2.2.

REMARK. It is evident from Lemma 1.6 and Theorem 2.10 that

$$\underline{\underline{h}} \geq \underline{\underline{h}}_2 = \sqrt{\pi}2/4 = 1.1107 \ldots .$$

Here is a slight improvement on this estimate.

THEOREM 2.11. *We have*

$$\underline{\underline{h}} \geq 2\pi^{-\frac{1}{2}} = 1.1284 \ldots .$$

Proof. Our aim is to apply the two-dimensional central limit theorem; see, for example, [4], Section VIII.4, Theorem 2. The underlying probability space will be (S, m) , where $S = T^N$, $N = \{1, 2, \ldots\}$ and m is normalised Haar measure on S . As before, if $k \in N$, π_k denotes the k-th projection of T^N . Let

$$X_k = \left(\mathrm{Re}\,\pi_k, \mathrm{Im}\,\pi_k\right) = \left(X_k^{(1)}, X_k^{(2)}\right) .$$

Then $X_1, X_2, \ldots$ are mutually independent two-dimensional real random variables with a common distribution. Moreover, $E\left(X_k^{(\alpha)}\right) = 0$ for all $k \in N$ and $\alpha \in \{1, 2\}$ and the common covariance matrix $\left(E\left(X_k^{(\alpha)} X_k^{(\beta)}\right)\right)_{\alpha,\beta=1,2}$ is equal to

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

where $\sigma_1^2 = \sigma_2^2 = \frac{1}{2}$ and $\rho = 0$. Consequently the central limit theorem asserts that the distributions ν_n of the random variables

$$S_n = n^{-\frac{1}{2}}(X_1 + \ldots + X_n)$$

converge (weakly) to the distribution

$$\nu = g\lambda_{R^2} ,$$

where λ_{R^2} denotes Lebesgue measure on R^2 and

$$g(x_1, x_2) = \pi^{-1}\exp\left(-\left(x_1^2+x_2^2\right)\right) .$$

We now show that

(1) $$\int_{R^2} |x|^2 d\nu_n(x) = 1 \quad \text{for all } n ,$$

where $|x| = |(x_1, x_2)| = \left(x_1^2+x_2^2\right)^{\frac{1}{2}}$ for $x \in R^2$. In fact, by definition of ν_n we have

$$\int_{R^2} |x|^2 d\nu_n(x) = \int_S |S_n|^2 dm = n^{-1} \int_S |X_1 + \ldots + X_n|^2 dm$$

$$= n^{-1} \int_{T^N} \left(\left(\sum_{k=1}^{n} \mathrm{Re}\,\pi_k\right)^2 + \left(\sum_{k=1}^{n} \mathrm{Im}\,\pi_k\right)^2\right) dm .$$

We also find that

(2) $$\int_{T^N} \left(\sum_{k=1}^{n} \mathrm{Re}\,\pi_k\right)^2 dm = \sum_{k=1}^{n} \int_{T^N} (\mathrm{Re}\,\pi_k)^2 dm$$

$$= \sum_{k=1}^{n} (2\pi)^{-1} \int_{-\pi}^{\pi} \cos^2\theta d\theta = \tfrac{1}{2}n ,$$

and similarly

(3) $$\int_{T^N} \left(\sum_{k=1}^{n} \operatorname{Im}\pi_k\right)^2 dm = \tfrac{1}{2}n .$$

Equalities (2) and (3) lead directly to (1).

From (1) and Lemma 2.12 proved below (with $F(x) = |x|^2 + 1$ and $f(x) = |x|$), we obtain

$$\lim_{n\to\infty} \int_{R^2} |x| d\nu_n(x) = \int_{R^2} |x| d\nu(x) ,$$

and hence

(4) $$\lim_{n\to\infty} \int_{T^N} n^{-\frac{1}{2}} |X_1 + \ldots + X_n| dm = \int_{R^2} |x| g(x) d\lambda_{R^2}(x) .$$

In the notation introduced in Corollary 2.7, the left hand side of (4) is equal to

$$\lim_{n\to\infty} n^{-\frac{1}{2}} E_n(1, \ldots, 1) ,$$

while the right hand side of (4) is equal to

$$\pi^{-1} \int_0^{2\pi} \int_0^{\infty} re^{-r^2} r dr d\theta = 2 \int_0^{\infty} r^2 e^{-r^2} dr = \int_0^{\infty} e^{-s} s^{\frac{1}{2}} ds$$
$$= \Gamma(3/2) = \tfrac{1}{2}\pi^{\frac{1}{2}} ,$$

so that

(5) $$\lim_{n\to\infty} n^{-\frac{1}{2}} E_n(1, \ldots, 1) = \tfrac{1}{2}\pi^{\frac{1}{2}} .$$

Hence, by Lemma 1.6 and Corollary 2.7,

$$\underline{\underline{h}} = \sup_n \underline{\underline{h}}_n \geq \lim_{n\to\infty} n^{\frac{1}{2}} / E_n(1, \ldots, 1) = 2\pi^{-\frac{1}{2}} .$$

REMARK. It seems quite possible that the supremum appearing in Corollary 2.7 is attained when all the c_k are equal, that is, that

$$\underline{\underline{h}}_n = n^{\frac{1}{2}} / E_n(1, \ldots, 1) .$$

If this is so, 2.11 (5) and Lemma 1.6 imply that $\underline{\underline{h}} = 2\pi^{-\frac{1}{2}}$. Note that

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

Corollary 2.7 and examination of the proof of Lemma 2.9 confirm that $\underline{\underline{h}}_2 = 2^{\frac{1}{2}}/E_2(1, 1)$.

LEMMA 2.12. *Let* μ *and* μ_n $(n = 1, 2, \ldots)$ *be positive Radon measures on* R^m . *Let* F *be a positive continuous function and* f *a complex-valued continuous function on* R^m . *Suppose that*

(i) $\mu_n \to \mu$ *weakly in the dual of* $C_{00}(R^m)$;

(ii) $M = \sup_n \int F d\mu_n < \infty$;

(iii) $\lim_{|x|\to\infty} |f(x)|/F(x) = 0$.

Then

(iv) $\sup_n \int |f| d\mu_n < \infty$;

(v) $\int F d\mu \le M$, $\int |f| d\mu < \infty$;

(vi) $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$.

Proof. By *(iii)*, there is a nonnegative number C such that

(1) $$|f| \le CF$$

and hence *(iv)* follows from *(ii)*. For the rest of the proof we may assume without loss of generality that f is real-valued and nonnegative. Let $(f_k)_{k=1}^{\infty}$ be an increasing sequence of functions in $C_{00}(R^m)$ such that

(2) $$0 \le f_k \le 1 , \quad f_k(x) = 1 \quad \text{for} \quad |x| \le k .$$

By *(i)*, (2) and *(ii)* we have

$$\int f_k F d\mu = \lim_{n\to\infty} \int f_k F d\mu_n \le \liminf_{n\to\infty} \int F d\mu_n \le M$$

for every k and so monotone convergence shows that

(3) $$\int F d\mu \le M .$$

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

Now (1) and (3) entail $\int f d\mu < \infty$. Thus *(v)* is true. Next, if we define

$$\varepsilon_k = \sup\{f(x)/F(x) : |x| \geqq k\} ,$$

(iii) shows that

(4) $$\lim_{k\to\infty} \varepsilon_k = 0$$

and (2) shows that $(1-f_k)f \leq \varepsilon_k F$. Thus

$$f_k f \leq f = f_k f + (1-f_k)f \leq f_k f + \varepsilon_k F ,$$

and *(ii)* implies that

$$\int f_k f d\mu_n \leq \int f d\mu_n \leq \int f_k f d\mu_n + \varepsilon_k \int F d\mu_n$$
$$\leq \int f_k f d\mu_n + \varepsilon_k M .$$

Letting $n \to \infty$, it follows from *(i)* that

$$\int f_k f d\mu \leq \liminf_{n\to\infty} \int f d\mu_n \leq \limsup_{n\to\infty} \int f d\mu_n \leq \int f_k f d\mu + \varepsilon_k M .$$

Now we let $k \to \infty$ and use (4) and monotone convergence to conclude that

$$\int f d\mu \leq \liminf_{n\to\infty} \int f d\mu_n \leq \limsup_{n\to\infty} \int f d\mu_n \leq \int f d\mu ,$$

which completes the proof.

2.13. We consider briefly the "change-of-arguments" operators T_ω introduced in [3]. This will lead to a slight improvement of Helgason's inequality 1.1 (a) and an alternative characterisation of $\underline{\underline{h}}$.

Let G be a CAG and write Ω for $T^{\hat{G}}$. (The present Ω is denoted by Ω^* in [3].) For $\chi \in \hat{G}$, π_χ denotes the χ-th projection on Ω , so that $\pi_\chi(\omega) = \omega(\chi)$ for every $\omega \in \Omega$.

For $\omega \in \Omega$, T_ω denotes the unitary endomorphism of $L^2(G)$ defined by

$$T_\omega f = \sum_{\chi\in\hat{G}} \omega(\chi) f^\wedge(\chi)\chi .$$

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

THEOREM 2.14. *Let G be a CAG and let the notation be as in 2.13.*

(i) We have

$$\|T_{\omega_0}f\|_1 \le \|f\|_2 \le \underline{\underline{h}} \int_\Omega \|T_\omega f\|_1 d\lambda_\Omega(\omega) \tag{1}$$

for $\omega_0 \in \Omega$ and $f \in L^2(G)$.

(ii) If E is an infinite subset of $\hat{G}$ and k a real number such that

$$\|f\|_2 \le k \int_\Omega \|T_\omega f\|_1 d\lambda_\Omega(\omega) \tag{2}$$

for every $f \in \underline{\underline{T}}_E(G)$, then $k \ge \underline{\underline{h}}$.

Proof. *(i)* The first inequality is trivial, since

$$\|T_{\omega_0}f\|_1 \le \|T_{\omega_0}f\|_2 = \|f\|_2 .$$

For the rest, it is sufficient to deal with the case in which $f \in \underline{\underline{T}}(G)$, for then a simple approximation argument extends the inequality to a general element of $L^2(G)$. We then have, by Fubini's Theorem,

$$\begin{aligned}\int_\Omega \|T_\omega f\|_1 d\lambda_\Omega(\omega) &= \int_\Omega \left\{\int_G \left|\sum_{\chi\in\hat{G}} \omega(\chi)f^\wedge(\chi)\chi(x)\right| d\lambda_G(x)\right\} d\lambda_\Omega(\omega) \\ &= \int_G \left\{\int_\Omega \left|\sum_{\chi\in\hat{G}} f^\wedge(\chi)\chi(x)\pi_\chi(\omega)\right| d\lambda_\Omega(\omega)\right\} d\lambda_G(x) .\end{aligned}$$

By Corollary 2.2 or Corollary 2.6, the Λ_2-constant of the set of all projections π_χ of Ω is at most $\underline{\underline{h}}$, so that the last-written inner integral is not less than

$$\underline{\underline{h}}^{-1}\left(\sum_{\chi\in\hat{G}} |f^\wedge(\chi)|^2\right)^{\frac{1}{2}} = \underline{\underline{h}}^{-1}\|f\|_2 ,$$

and the second inequality in (1) follows.

(ii) By Corollary 2.6, applied with Ω in place of G and $E_1 = \{\pi_\chi : \chi \in E\}$ in place of I, it suffices to show that

$$\Lambda_\Omega(E_1) \le k .$$

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

This in turn will follow, if it be shown that

(3) $$\Lambda_\Omega\left(\left\{\pi_{\chi_1}, \ldots, \pi_{\chi_n}\right\}\right) \le k$$

for arbitrary distinct $\chi_1, \ldots, \chi_n \in E$. To this end, let

$$F = \sum_{j=1}^{n} c_j \pi_{\chi_j}$$

and

$$f = \sum_{j=1}^{n} c_j \chi_j ,$$

where the c_j are complex numbers. By (2) we have

$$\|F\|_2 = \left(\sum_{j=1}^{n} |c_j|^2\right)^{\frac{1}{2}} = \|f\|_2 \le k \int_\Omega \|T_\omega f\|_1 d\lambda_\Omega(\omega) .$$

Using Fubini's Theorem, this gives

$$\|F\|_2 \le k \int_G \left\{\int_\Omega \left|\sum_{j=1}^{n} c_j \chi_j(x) \pi_{\chi_j}(\omega)\right| d\lambda_\Omega(\omega)\right\} d\lambda_G(x) .$$

By 1.11 *(iii)*, the inner integral here is independent of $x \in G$ and equal to

$$\int_\Omega \left|\sum_{j=1}^{n} c_j \pi_{\chi_j}(\omega)\right| d\lambda_\Omega(\omega) = \|F\|_1 .$$

Thus, $\|F\|_2 \le k\|F\|_1$, which verifies (3) and completes the proof.

COROLLARY 2.15. *The notation is as in* 2.13. *Suppose also that* $E \subseteq \hat{G}$ *and let*

$$\kappa = \sup\{\|T_\omega f\|_1 : f \in \underline{\underline{T}}_E(G), \|f\|_1 = 1, \omega \in \Omega\} .$$

Then

$$\kappa \le \Lambda_G(E) \le \underline{\underline{h}}\kappa .$$

In particular, E *is a* Λ_2*-set if and only if* $\kappa < \infty$.

Proof. The inequality $\kappa \le \Lambda_G(E)$ follows from the first inequality

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

in 2.14 (1), since $\|f\|_2 \le \Lambda_G(E)\|f\|_1$ for every $f \in \underline{\underline{T}}_E(G)$. The inequality $\Lambda_G(E) \le \underline{\underline{h}}\kappa$ follows from the second inequality in 2.14 (1).

3. $\underline{\underline{h}}$ in terms of subsets of $\hat{T}$

In this section we show that each of the numbers $\underline{\underline{h}}_n$ can be given in terms of n-element subsets of $\hat{T}$. In view of Lemma 1.6, $\underline{\underline{h}}$ can therefore be given in terms of finite subsets of $\hat{T}$.

NOTATION 3.1. Here we consider the (compact) circle group T; n will denote a fixed positive integer. For integers $k \ge 2$, we write $E_{n,k}$ for the set of characters $z \mapsto z^{k^{j-1}}$ of T corresponding to $j \in \{1, 2, \ldots, n\}$. In Theorem 3.5, we will prove that $\underline{\underline{h}}_n = \lim_{k\to\infty} \Lambda_T(E_{n,k})$.

For each k, ϕ_k will denote the mapping of T into T^n defined by

$$\phi_k(z) = \left(z, z^k, z^{k^2}, \ldots, z^{k^{n-1}}\right);$$

and H_k will denote the image $\phi_k(T)$ of T. It is evident that ϕ_k is a topological isomorphism of T onto H_k.

DEFINITION 3.2. Let $\mathcal{H}$ denote the set of all closed subgroups of the compact group G. We endow $\mathcal{H}$ with the topology for which an open basis consists of sets of the form

$$\mathcal{U}(K; U_1, \ldots, U_m) = \{H \in \mathcal{H} : H \cap K = \emptyset \text{ and } H \cap U_j \ne \emptyset \text{ for all } j\};$$

here K is a compact subset of G and $U_1, \ldots, U_m$ are nonvoid open subsets of G. A net $(H_\gamma)_{\gamma\in\Gamma}$ in $\mathcal{H}$ is said to converge in the sense of Hausdorff to H_0 in $\mathcal{H}$ provided it converges to H_0 in this topology; in this case we write

$$\lim_\gamma H_\gamma = H_0 \text{ [Hausdorff]}.$$

Since G belongs to $\mathcal{U}(K; U_1, \ldots, U_m)$ if and only if $K = \emptyset$, it follows that

(i) $\lim_{\gamma} H_\gamma = G$ [Hausdorff]

if and only if

(ii) whenever $U_1, \ldots, U_m$ are given nonvoid open subsets of G, there exists a $\gamma_0 \in \Gamma$ such that $\gamma > \gamma_0$ implies $H_\gamma \cap U_j \neq \emptyset$ for all $j \in \{1, 2, \ldots, m\}$.

We need the following lemma due to Fell (see appendix to [6]) and to Bourbaki [2]; see also [5].

LEMMA 3.3. *If (H_γ) is a net of closed subgroups of a compact group G, and if*

(i) $\lim_{\gamma} H_\gamma = G$ [Hausdorff],

then for all F in $C(G)$ we have

(ii) $\int_G F d\lambda_G = \lim_{\gamma} \int_{H_\gamma} F d\lambda_{H_\gamma}$

where λ_G and λ_{H_γ} denote normalised Haar measure on G and H_γ, respectively.

LEMMA 3.4. *Let $(H_k)_{k=2}^{\infty}$ denote the sequence of closed subgroups of T^n defined in 3.1. Then*

$$\lim_{k\to\infty} H_k = T^n \text{ [Hausdorff].}$$

Proof. We establish some local terminology for this proof. By a k^r-sector of T we shall mean a subset of T of the form

$$\{\exp(2\pi i\theta) : jk^{-r} \leq \theta < (j+1)k^{-r}\} ;$$

here r denotes a nonnegative integer and j any integer. A subset E of T^n will be termed k-dense if for every choice of n k-sectors $S_1, \ldots, S_n$ of T, the set

$$E \cap (S_1 \times S_2 \times \ldots \times S_n)$$

is nonvoid. We first prove that

(1) each H_k is k-dense in T^n .

We begin with an observation. If r is a nonnegative integer, if R is a k^r-sector of T , and if S is a k-sector of T , then there is some k^{r+1}-sector $R' \subseteq R$ such that $z \mapsto z^{k^r}$ maps R' into S . In fact, we can write

$$R = \{\exp(2\pi i\theta) : mk^{-r} \leq \theta < (m+1)k^{-r}\}$$

and

$$S = \{\exp(2\pi i\theta) : jk^{-1} \leq \theta < (j+1)k^{-1}\} ,$$

where $m \in Z$ and $j \in \{0, 1, \ldots, k-1\}$, and then set

$$R' = \{\exp(2\pi i\theta) : (mk+j)k^{-r-1} \leq \theta < (mk+j+1)k^{-r-1}\} .$$

Now let $S_1, \ldots, S_n$ be given k-sectors of T . The preceding observation allows us to choose by recurrence k^r-sectors R_r for $r \in \{1, 2, \ldots, n\}$ such that $R_n \subseteq R_{n-1} \subseteq \ldots \subseteq R_1$ and $z \mapsto z^{k^{r-1}}$ maps R_r into S_r for $r \in \{1, 2, \ldots, n\}$. Select any z from R_n . Then $z^{k^{r-1}}$ belongs to S_r for $r \in \{1, 2, \ldots, n\}$ and so $\phi_k(z)$ lies in $S_1 \times S_2 \times \ldots \times S_n$; thus

$$H_k \cap (S_1 \times S_2 \times \ldots \times S_n) \neq \emptyset .$$

This proves (1).

To complete the proof of the lemma, we verify 3.2 (ii) in the present setting. So consider nonvoid open subsets $U_1, \ldots, U_m$ of T^n . A simple argument shows that for each $j \in \{1, 2, \ldots, m\}$, there is an integer k_j such that $E \cap U_j \neq \emptyset$ whenever E is a subset of T^n that is k-dense

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

for some $k \geq k_j$. Thus if $k \geq \max(k_1, k_2, \ldots, k_m)$, then (1) shows that $H_k \cap U_j \neq \emptyset$ for all $j \in \{1, 2, \ldots, m\}$. This verifies 3.2 (ii) and so $\lim_{k\to\infty} H_k = T^n$ in the sense of Hausdorff.

THEOREM 3.5. *For the sequence* $(E_{n,k})_{k=2}^{\infty}$ *of n-element subsets of* $\hat{T}$ *defined in 3.1, we have*

$$\underline{\underline{h}}_n = \lim_{k\to\infty} \Lambda_T(E_{n,k}) .$$

Proof. Since n is fixed throughout the argument, we will write E_k in place of $E_{n,k}$. The definition of $\underline{\underline{h}}_n$ in 1.5 shows that $\Lambda_T(E_k) \geq \underline{\underline{h}}_n$ for all $k \geq 2$ and so

$$\liminf_{k\to\infty} \Lambda_T(E_k) \geq \underline{\underline{h}}_n .$$

It therefore suffices to prove that

$$\limsup_{k\to\infty} \Lambda_T(E_k) \leq \underline{\underline{h}}_n . \tag{1}$$

Assume that (1) fails. Then there is a subsequence (k_r) of integers and a number $\kappa > \underline{\underline{h}}_n$ so that $\Lambda_T\left(E_{k_r}\right) > \kappa$ for all r . Then for each r we have

$$\left(\sum_{j=1}^{n} \left|c_j^{(r)}\right|^2\right)^{\frac{1}{2}} \geq \kappa \int_T \left|\sum_{j=1}^{n} c_j^{(r)} z^{(k_r)^{j-1}}\right| d\lambda_T(z) \tag{2}$$

for suitable complex numbers $c_j^{(r)}$, $j \in \{1, 2, \ldots, n\}$. We may clearly suppose that

$$\left(\sum_{j=1}^{n} \left|c_j^{(r)}\right|^2\right)^{\frac{1}{2}} = 1 \quad \text{for all } r . \tag{3}$$

Let ϕ_k and H_k be as in 3.1. Since ϕ_k is a continuous homomorphism of T onto H_k , 1.13 (3) shows that

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

(4) $$\int_{H_k} f d\lambda_{H_k} = \int_T (f \circ \phi_k) d\lambda_T$$

for all functions f continuous on H_k. Let $\pi_1, \ldots, \pi_n$ denote the projections of T^n. We apply (4) to the right hand side of (2), taking $k = k_r$ and $f = F_r$, where

$$F_r = \left| \sum_{j=1}^{n} c_j^{(r)} \pi_j \right| ,$$

and so obtain

(5) $$\left(\sum_{j=1}^{n} \left| c_j^{(r)} \right|^2 \right)^{\frac{1}{2}} \geq \kappa \int_{H_{k_r}} F_r d\lambda_r ;$$

here we have written λ_r for normalised Haar measure on H_{k_r}. In view of (3), we may suppose (by passing to further subsequences of (k_r) if necessary) that the limits $\lim_{r\to\infty} c_j^{(r)}$ exist. Let

(6) $$c_j = \lim_{r\to\infty} c_j^{(r)} \quad \text{for} \quad j \in \{1, 2, \ldots, n\} ,$$

and define

$$F = \left| \sum_{j=1}^{n} c_j \pi_j \right| .$$

By Lemma 3.4, we have $\lim_{k\to\infty} H_k = T^n$ in the sense of Hausdorff, and so Lemma 3.3 applies to show that

(7) $$\int_{T^n} F d\lambda_{T^n} = \lim_{r\to\infty} \int_{H_{k_r}} F d\lambda_r .$$

From (6) and (3) it follows that

(8) $$\lim_{r\to\infty} \sum_{j=1}^{n} \left| c_j^{(r)} \right|^2 = \sum_{j=1}^{n} |c_j|^2 = 1 .$$

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

From (6) it also follows that F_r converges uniformly to F and so

(9) $$\lim_{r\to\infty} \left| \int_{H_{k_r}} F_r d\lambda_r - \int_{H_{k_r}} F d\lambda_r \right| = 0 .$$

Relations (7), (8) and (9) together with (5) yield

$$\left(\sum_{j=1}^{n} |c_j|^2 \right)^{\frac{1}{2}} \geq \kappa \int_{T^n} F d\lambda_{T^n} ,$$

that is,

(10) $$\left(\sum_{j=1}^{n} |c_j|^2 \right)^{\frac{1}{2}} \geq \kappa \int_{T^n} \left| \sum_{j=1}^{n} c_j \pi_j \right| d\lambda_{T^n} .$$

Since (8) shows that both sides of (10) are nonzero, we conclude that

$$\Lambda_{T^n}(\{\pi_1, \ldots, \pi_n\}) \geq \kappa > \underline{\underline{h}}_n ,$$

which contradicts Corollary 2.2.

We end by using the sets $E_{n,k}$ to establish the following interesting extension of Corollary 1.12.

THEOREM 3.6. *We have* $M_T = \underline{\underline{h}}$.

Proof. In view of Corollary 1.10, it is enough to show that the Sidon constant of $E_{n,k}$ is at most $\sec(2\pi/k)$ for $k \geq 5$. To achieve this we will show that, if $a_1, \ldots, a_n$ are arbitrary complex numbers, then

(1) $$\cos(2\pi/k) \sum_{j=1}^{n} |a_j| \leq \sup\left\{ \left| \sum_{j=1}^{n} a_j z^{k^{j-1}} \right| : z \in T \right\}$$
$$= \sup\left\{ \left| \sum_{j=1}^{n} \omega_j a_j \right| : \omega = (\omega_j) \in H_k \right\} ,$$

where H_k is as in 3.1. We will use 3.4 (1) and the terminology introduced thereabouts. For each j , $a_j = |a_j|\exp(2\pi i\theta_j)$, where θ_j belongs to the interval $\left[m_j k^{-1}, (m_j+1)k^{-1}\right]$ for some integer

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press

$m_j \in \{0, 1, \ldots, k-1\}$. Let S_j denote the k-sector

$$\left\{\exp(2\pi i\theta) : (-m_j-1)k^{-1} \le \theta < -m_j k^{-1}\right\} .$$

By 3.4 (1), some ω in H_k has the property that $\omega_j \in S_j$ for all $j \in \{1, 2, \ldots, n\}$. Then each $\omega_j \exp(2\pi i\theta_j)$ belongs to

$$\{\exp(2\pi i\theta) : -k^{-1} \le \theta < k^{-1}\}$$

and so $\operatorname{Re}(\omega_j a_j) \ge \cos(2\pi/k)|a_j|$. It follows that

$$\cos(2\pi/k) \sum_{j=1}^{n} |a_j| \le \operatorname{Re}\left(\sum_{j=1}^{n} \omega_j a_j\right) \le \left|\sum_{j=1}^{n} \omega_j a_j\right|$$

and hence (1) holds.

REMARK 3.7. It is clear from 3.6 (1) that the Sidon constant of the infinite set of characters $z \mapsto z^{k^{j-1}}$ of T corresponding to $j \in \{1, 2, 3, \ldots\}$ is at most $\sec(2\pi/k)$ when $k \ge 5$.

COROLLARY 3.8. *Let G be a CAG such that $\hat{G}$ contains an element χ_0 of infinite order. Let n and k be positive integers and*

$$F_{n,k} = \left\{\chi_0^{k^{j-1}} : j \in \{1, 2, \ldots, n\}\right\} .$$

Then

(i) $\underline{\underline{h}}_n = \lim_{k\to\infty} \Lambda_G(F_{n,k})$;

(ii) $\underline{\underline{h}} = M_G$.

Proof. We apply the substance of 1.13 with $K = T$, $\phi = \chi_0$ and $E = E_{n,k}$; since χ_0 is of infinite order, $\{\chi_0\}$ is strongly independent and $\phi(G) = T$ by 1.11 (i). Then $F_{n,k} = \phi^*(E_{n,k})$ and so $S_G(F_{n,k}) = S_T(E_{n,k})$ and $\Lambda_G(F_{n,k}) = \Lambda_T(E_{n,k})$. Statement *(i)* accordingly follows from Theorem 3.5, while *(ii)* follows from Corollary 1.10 and the fact (established in the proof of Theorem 3.6) that $S_T(E_{n,k})$ is at most $\sec(2\pi/k)$ for large k .

References

[1] Michael N. Barber and B.W. Ninham, *Random and restricted walks, theory and applications* (Gordon and Breach, New York, London, Paris, 1970).

[2] N. Bourbaki, *Éléments de mathématique*, Fascicule XXIX. Livre VI: Intégration. Chapitre 8: Convolution et représentations (Actualités Scientifiques et Industrielles, No. 1306. Hermann, Paris, 1963).

[3] R.E. Edwards, "Changing signs of Fourier coefficients", *Pacific J. Math.* 15 (1965), 463-475.

[4] William Feller, *An introduction to probability theory*, Vol. 2, 2nd ed. (John Wiley & Sons, New York, London, Sydney, Toronto, 1971).

[5] Jürgen Flachsmeyer und Heiner Zieschang, "Über die schwache Konvergenz der Haarschen Masse von Untergruppen", *Math. Ann.* 156 (1964), 1-8.

[6] James Glimm, "Families of induced representations", *Pacific J. Math.* 12 (1962), 885-911.

[7] S. Helgason, "Multipliers of Banach algebras", *Ann. of Math.* (2) 64 (1956), 240-254.

[8] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis*, Vol. II (Die Grundlehren der mathematischen Wissenschaften, Band 152. Springer-Verlag, Berlin, Heidelberg, New York, 1970).

[9] P.A.P. Moran, *An introduction to probability theory* (Clarendon Press, Oxford, 1968).

[10] G.N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed. (Cambridge University Press, Cambridge, 1944; reprinted 1958).

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT;

Department of Mathematics,
University of Oregon,
Eugene,
Oregon, USA.