EXISTENCE OF CERTAIN ANALYTIC HOMEOMORPHISMS

Z. A. Melzak

(received December 5, 1958)

1. This note has its origin in the following problem: do there exist non-trivial increasing continuous functions on $[0, 1]$ to $[0, 1]$, which map the following sets in $[0, 1]$ onto themselves: the rational, the algebraic and the transcendental numbers? One such function is obviously $f(x) = x$; more generally, $f(x) = (c + 1)x/(cx + 1)$, with c rational and non-negative, satisfies the conditions. Let G denote the space of order-preserving homeomorphisms of $[0, 1]$ onto $[0, 1]$, in the uniform metric. It follows from Theorem 1 below that the set S of all such functions is dense in G. S is clearly a subgroup of G and one may ask what are its group-theoretic properties. We shall not consider these questions.

2. In 1925 Franklin proved the following theorem [1] :if X and Y are two countable sets, both dense in (0, 1), then there exists an analytic function f in G, such that $f(X) = Y$. By a change in Franklin's method the above theorem will be generalized as follows.

THEOREM 1. Let $\{X_i\}$ and $\{Y_i\}$, $i=1,2,\ldots$, be two sequences of countable sets, each set being dense in (0, 1). Let the sets of each sequence be pairwise disjoint. Then there exists an analytic function f in G such that $f(X_i) = Y_i$, $i=1,2,\ldots$.

The proof proceeds by the method of successive approximations. Let

$$\chi(\,(i,j)\,) = (i+j-2)\,(i+j-1)/2 + j$$

be the usual 1 : 1 correspondence between the set of ordered pairs of positive integers and the set of positive integers themselves; let its inverse be

$$\chi^{-1}(n) = (\,\varphi(n),\ \omega(n)\,).$$

https://doi.org/10.4153/CMB-1959-010-8 Published online by Cambridge University Press

In the sequel only the function $\varphi(n)$ will be used. At the n-th stage of the approximation process a function $f_n(x)$ in G is obtained which sends certain two points in $X_{\varphi(n)}$ onto two points in $Y_{\varphi(n)}$:

$$f_n(x_{\varphi(n),\psi(n)}) = y_{\varphi(n),\alpha(n)} ,$$

$$f_n(x_{\varphi(n),\beta(n)}) = y_{\varphi(n),\gamma(n)} .$$

Here $x_{\varphi(n),\psi(n)}$ and $y_{\varphi(n),\gamma(n)}$ are the first two as yet unused points of $X_{\varphi(n)}$ and $Y_{\varphi(n)}$ respectively. In addition f_n preserves all the correspondences established at the earlier stages and the limit $f_n(x)$, as $n \to \infty$, is an analytic function in G.

The first few approximations will now be set up. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive constants such that

$$\sum_{j=1}^{\infty} \varepsilon_j = h < 1.$$

Let

$$x_{\varphi(1),\psi(1)} = x_{1,1},$$

$$g_1(x) = x + a_1 x(x-1),$$

$$h_1(y) = y + b_1 y(y-1)(y-y_{\varphi(1),\alpha(1)}),$$

where a_1, b_1 and $y_{\varphi(1),\alpha(1)}$ are so chosen that

$$|a_1| < \varepsilon_1/2! \ , \ |b_1| < \varepsilon_1/3! \ ,$$

$$y_{\varphi(1),\alpha(1)} = g_1(x_{\varphi(1),\psi(1)}) ,$$

$$g_1(x_{\varphi(1),\beta(1)}) = h_1(y_{\varphi(1),\gamma(1)}) ,$$

for some $x_{\varphi(1),\beta(1)}$; here $y_{\varphi(1),\gamma(1)}$ is defined to be $y_{1,1}$ if $\alpha(1) > 1$ and $y_{1,2}$ if $\alpha(1) = 1$. Since X_1 and Y_1 are dense in (0, 1) the required numbers can be found.

Consider the equation

$$h_1(y) = g_1(x);$$

https://doi.org/10.4153/CMB-1959-010-8 Published online by Cambridge University Press

this determines a function f_1 in G:

$$y = f_1(x),$$

$$f_1(x_{\varphi(1),\psi(1)}) = y_{\varphi(1),\alpha(1)},$$

$$f_1(x_{\varphi(1),\beta(1)}) = y_{\varphi(1),\gamma(1)}.$$

The function $f_1(x)$ is the first approximation to f. To find $f_2(x)$ introduce first two auxiliary functions $g_2(x)$ and $h_2(x)$ and four numbers

$$x_{\varphi(2),\psi(2)},\quad y_{\varphi(2),\alpha(2)},$$

$$x_{\varphi(2),\beta(2)},\quad y_{\varphi(2),\gamma(2)},$$

such that

$$g_2(x) = g_1(x)+a_2x(x-1)(x-x_{\varphi(1),\psi(1)})(x-x_{\varphi(1),\beta(1)}),$$

$$h_2(y)=h_1(y)+b_2y(y-1)(y-y_{\varphi(1),\gamma(1)})(y-y_{\varphi(1),\alpha(1)})(y-y_{\varphi(2),\alpha(2)}),$$

$$|a_2| < \varepsilon_2/4!,\quad |b_2| < \varepsilon_2/5!,$$

$$x_{\varphi(2),\psi(2)} = x_{\varphi(2),1},$$

$$h_1(y_{\varphi(2),\alpha(2)}) = g_2(x_{\varphi(2),\psi(2)}),$$

$$y_{\varphi(2),\gamma(2)} = y_{\varphi(2),1} \text{ if } \alpha(2) > 1,$$

$$= y_{\varphi(2),2} \text{ if } \alpha(2) = 1,$$

$$g_2(x_{\varphi(2),\beta(2)}) = h_2(y_{\varphi(2),\gamma(2)}).$$

By the hypotheses on the sets $X_{\varphi(2)}$ and $Y_{\varphi(2)}$ the required constants can always be found. Now consider the equation

$$h_2(y) = g_2(x);$$

this determines y as a function of x :

https://doi.org/10.4153/CMB-1959-010-8 Published online by Cambridge University Press

$$y = f_2(x), \qquad f_2 \text{ in } G,$$

$$f_2(x_{\varphi(1),\psi(1)}) = y_{\varphi(1),\alpha(1)},$$

$$f_2(x_{\varphi(1),\beta(1)}) = y_{\varphi(1),\gamma(1)},$$

$$f_2(x_{\varphi(2),\psi(2)}) = y_{\varphi(2),\alpha(2)},$$

$$f_2(x_{\varphi(2),\beta(2)}) = y_{\varphi(2),\gamma(2)}.$$

The general method of procedure is now clear - at the (n+1)-th stage one determines the two auxiliary functions $g_{n+1}(x)$ and $h_{n+1}(x)$ by recursion, and two new pairs of points in $X_{\varphi(n+1)}$ and $Y_{\varphi(n+1)}$ are made to correspond under $f_{n+1}(x)$. The equation $h_n(y) = g_n(x)$, which determines $f_n(x)$, becomes in the limit

$$y + \sum_{j=1}^{\infty} b_j\, y(y-1)(y-y_{\varphi(j),\alpha(j)}) \prod_{k=1}^{j-1} (y-y_{\varphi(k),\gamma(k)})(y-y_{\varphi(k),\alpha(k)})$$

$$= x + \sum_{j=1}^{\infty} a_j\, x(x-1) \prod_{k=1}^{j-1} (x-x_{\varphi(k),\psi(k)})(x-x_{\varphi(k),\beta(k)}),$$

which may be written as $h(y) = g(x)$. Here h and g are increasing analytic functions in G and consequently y is thereby determined as an analytic function: $y = f(x)$, f in G.

By construction, each $x_{i,j}$ is mapped onto some $y_{i,k}$ and conversely, each $y_{j,m}$ is an image of some $x_{j,p}$. Since $f(x)$ is 1 : 1 this completes the proof.

3. COROLLARY 1. Any function $F(x)$ in G of class $C^{(n)}$, whose derivative is bounded away from zero, can be uniformly approximated by analytic functions $f(x)$ in G, such that $f(X_i) = Y_i$; the first n derivatives of $F(x)$ are uniformly approximated by those of $f(x)$.

COROLLARY 2. Any continuous function $F(x)$ in G can be uniformly approximated by analytic functions $f(x)$ in G, such that $f(X_i) = Y_i$.

The proofs of these corollaries follow exactly the proofs of Theorems II and III in [1].

COROLLARY 3. A continuous function $F(x)$ in G can be uniformly approximated by analytic functions $f(x)$ in G, which map the following sets onto themselves: the rational numbers in $[0,1]$, the transcendental numbers in $[0,1]$, and the irrational algebraic numbers in $[0,1]$ of degree $n \geqslant 2$. Alternatively, $f(x)$ can be so chosen that it maps the set of transcendental numbers in $[0,1]$ onto itself and sends each algebraic number x in $[0,1]$ into $K(x)$, the field obtained by adjoining x to the field of rational numbers.

Proof. In each case it suffices to set up the sets X_i and Y_i. In the first case let $X_i = Y_i$ = the set of algebraic numbers in $[0,1]$ of degree i, $i = 1, 2, 3, \ldots$; the other conditions of the theorem are easily verified. In the second case let $x \sim y$ be an equivalence relation defined as follows: $x \sim y$ if and only if $K(x) = K(y)$. This equivalence relation introduces then a partition of the algebraic numbers of $[0,1]$ into disjoint residue classes; there are countably many of these and each is a countable dense set on $[0,1]$; now let $X_i = Y_i$ = the i-th residue class.

It is easily shown that under the hypotheses of Theorem 1 the set of all functions f with required properties is uncountable because to any sequences $\{X_i\}$, $\{Y_i\}$, one can adjoin (in uncountably many ways) two new sets X_o and Y_o. When proper disjointness and density conditions are satisfied, it is possible then to have $f(X_o) = Y_o$ as well as $f(X_i) = Y_i$, $i = 1, 2, \ldots$.

REFERENCES

1. P. Franklin, Analytic transformations of linear everywhere dense point sets, Trans. Amer. Math. Soc. 27(1925), 91-100.

McGill University

https://doi.org/10.4153/CMB-1959-010-8 Published online by Cambridge University Press