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NORMAL-EQUIVALENT OPERATORS AND OPERATORS WITH DUAL OF SCALAR-TYPE

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Abstract If $T \in L(X)$ is such that T' is a scalar-type prespectral operator, then $\operatorname{Re} T'$ and $\operatorname{Im} T'$ are both dual operators. It is shown that that the possession of a functional calculus for the continuous functions on the spectrum of T is equivalent to T' being scalar-type prespectral of class X, thus answering a question of Berkson and Gillespie.

Keywords: hermitian; normal; scalar type; prespectral; operator; functional calculus

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Introduction

The class of scalar-type spectral operators on a Banach space was introduced by Dunford [**8**] as a natural analogue of the normal operators on Hilbert space. They can be characterized by their possession of a weakly compact functional calculus for continuous functions on the spectrum [**9**, Corollary 1] or [**11**, Theorem]. The more general class of scalar-type prepectral operators of class Γ was introduced by Berkson and Dowson [**2**]. They proved that if $T \in L(X)$ admits a $C(\sigma(T))$ functional calculus, then T' is scalar-type prespectral of class X. The converse implication is immediate if X is reflexive [**6**, Theorem 6.17] or $\sigma(T) \subseteq \mathbb{R}$ [**6**, Theorem 16.15 and the proof of Theorem 16.16]. The question raised by Berkson and Gillespie [**3**, Remark 1] has remained open for some time. The problem amounts to finding a decomposition for T with commuting real and imaginary parts, given that T' has such a decomposition. We show that this can always be done, developing the properties of (strongly) normal (equivalent) operators for this purpose.

1. Normal-equivalent operators

Throughout X will be a Banach space endowed with its norm $\|\cdot\|$. We write X' for its norm dual and $L(X)$, $L(X')$ for the Banach algebra of all bounded linear operators on X and X' respectively. When $T \in L(X)$ we denote its dual (or adjoint) by T'.

Definition 1.1. An operator $T \in L(X)$ is hermitian if

$$\|\exp(\mathrm{i}tT)\| = 1 \qquad (t \in \mathbb{R}).$$

An operator $T \in L(X)$ is hermitian if and only if $T' \in L(X')$ is hermitian.

Definition 1.2. An operator $T \in L(X)$ is normal if $T = R + \mathrm{i}J$, where R and J are commuting hermitian operators.

We shall need the following Fuglede-type result [**7**, Lemma 3], and generalizations of it.

Lemma 1.3. *If $T = R + \mathrm{i}J$, where $RJ = JR$ and $\{R, J\}$ is hermitian, and if $A \in L(X)$ is such that $AT = TA$, then $AR = RA$, $AJ = JA$.*

Remark 1.4. If $T \in L(X)$ is normal, then the operators R and J are determined uniquely by T, and we write

$$T^* = R - \mathrm{i}J.$$

Uniqueness follows from Lemma 1.3.

If $T \in L(X)$ is normal then $T' \in L(X')$ is normal. The converse of this was proved by Behrends in [**1**].

Definition 1.5. An operator $R \in L(X)$ is hermitian-equivalent if and only if there exists an equivalent norm on X with respect to which R is hermitian.

Equivalently, R is hermitian-equivalent if and only if there is an $M(> 1)$ such that

$$\|\exp(\mathrm{i}tR)\| \leqslant M \qquad (t \in \mathbb{R}).$$

If this condition is satisfied, then

$$|x| = \sup\{\|\exp(\mathrm{i}tR)x\| : t \in \mathbb{R}\}$$

defines a norm on X, equivalent to $\|\cdot\|$, with respect to which R is hermitian.

More generally, a set $\Lambda \subseteq L(X)$ is hermitian-equivalent if and only if there is an equivalent norm on X with respect to which every operator in Λ is hermitian. It is known [**6**, Theorem 4.17] that when Λ is a commutative subset of $L(X)$, then Λ is hermitian-equivalent if and only if each operator in the closed real linear span of Λ is hermitian-equivalent; and, most importantly for our study, that any bounded Boolean algebra of projections on X is hermitian-equivalent [**6**, Theorem 5.4].

Lemma 1.6. *An operator $R \in L(X)$ is hermitian-equivalent if and only if $R' \in L(X')$ is hermitian-equivalent.*

Proof. $\sup_{t\in\mathbb{R}} \|\exp(\mathrm{i}tR)\| = \sup_{t\in\mathbb{R}} \|\exp(\mathrm{i}tR')\|$. □

The following result is an immediate consequence of Lemma 1.3.

Lemma 1.7. *If $T = R + \mathrm{i}J$ where $RJ = JR$ and $\{R, J\}$ is hermitian-equivalent, and if $A \in L(X)$ is such that $AT = TA$, then $AR = RA$, $AJ = JA$.*

Definition 1.8. An operator $T \in L(X)$ is normal-equivalent if $T = R + \mathrm{i}J$, where $RJ = JR$ and $\{R, J\}$ is hermitian-equivalent.

Remark 1.9. The operator $T = R + \mathrm{i}J$ is normal-equivalent if and only if $RJ = JR$ and

$$\|\exp(\mathrm{i}sR + \mathrm{i}tJ)\| \leqslant M$$

for some M and all real s, t.

Lemma 1.10. *If $T \in L(X)$ is normal-equivalent then T can be expressed uniquely in the form $R + \mathrm{i}J$, where $RJ = JR$ and $\{R, J\}$ is hermitian-equivalent.*

Proof. If $T = R + \mathrm{i}J = R_1 + \mathrm{i}J_1$, where $RJ = JR$, $R_1J_1 = J_1R_1$, $\{R, J\}$ and $\{R_1, J_1\}$ are hermitian-equivalent, then by Lemma 1.7 $\{R, J, R_1, J_1\}$ is a commuting hermitian-equivalent set: by [**6**, Theorem 4.17] we can renorm X to make them simultaneously hermitian. Since $R - R_1 = i(J_1 - J)$ we have

$$\sigma(R - R_1) = \sigma(J_1 - J) = \{0\} :$$

by Sinclair's theorem $R = R_1$, $J = J_1$. □

If $T \in L(X)$ is normal-equivalent then $T' \in L(X')$ is normal-equivalent. The converse also holds. We model our proof on that of Behrends [**1**]. It depends on Lemma 1.11, which is essentially due to Behrends [**1**]: for completeness we include a proof.

In the following lemma we shall make use of the canonical projection on the third dual of X. If $i_X : X \to X''$ is the canonical injection, then $P = i_{X'}(i_X)'$ is a projection on X''' whose range is $i_{X'}(X')$ and whose kernel is $(i_X(X))^{\perp}$. We have the following facts about i_X, $i_{X'}$, $(i_X)'$ and P:

1. $(i_X)'i_{X'} = (\text{identity})_{X'}$,
2. $Pi_{X'} = i_{X'}$,
3. $(i_X)'P = (i_X)'$,
4. $\|P\| = 1$,
5. $\langle i_X x, Py'''\rangle = \langle x, (i_X)'Py'''\rangle = \langle x, (i_X)'y'''\rangle = \langle i_X x, y'''\rangle$ for each x in X and y''' in X'''.

Lemma 1.11. *An operator $T \in L(X')$ is of the form S' (for some $S \in L(X)$) if and only if T'' commutes with the projection $P = i_{X'}(i_X)' : X''' \to X'''$.*

Proof. First note that if $S \in L(X)$ then

$$S''i_X = i_X S.$$

If now $T = S'$ for some $S \in L(X)$ then

$$T'i_X = i_X S$$

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so

$$(i_X)'T'' = S'(i_X)' = T(i_X)'$$

and

$$PT'' = i_{X'}(i_X)'T'' = i_{X'}S'(i_X)' = i_{X'}T(i_X)'.$$

Next note that

$$T''i_{X'} = i_{X'}T$$

from which

$$T''P = T''i_{X'}(i_X)' = i_{X'}T(i_X)' :$$

so

$$PT'' = T''P.$$

Conversely, suppose $T''P = PT''$. If $y''' \perp i_X(X)$, that is, $Py''' = 0$, then

$$\begin{aligned}\langle T'i_X x, y'''\rangle &= \langle i_X x, T''y'''\rangle \\ &= \langle i_X x, PT''y'''\rangle \qquad \text{(by 5 above)} \\ &= \langle i_X x, T''Py'''\rangle \\ &= 0,\end{aligned}$$

i.e. $y''' \perp T'i_X(X)$. It follows that $T'i_X(X) \subseteq i_X(X)$ so that

$$S = (i_X)^{-1}T'i_X : X \to X$$

is well-defined: and then $T = S'$. □

We can now prove the following theorem, which generalizes that of Behrends [**1**, Theorem 1].

Theorem 1.12. *If $T' \in L(X')$ is normal-equivalent then $T \in L(X)$ is normal-equivalent.*

Proof. If $T' \in L(X')$ is normal-equivalent then $T' = R + \mathrm{i}J$ where R, J commute and $\|\exp(\mathrm{i}sR + \mathrm{i}tJ)\| \leqslant M$ for some M and all real s, t. Also $T''' = R'' + \mathrm{i}J''$ is normal-equivalent. By Lemma 1.11 we have $T'''P = PT'''$; by Lemma 1.7 we get $R''P = PR''$ and $J''P = PJ''$: hence, by Lemma 1.11, there are $H,\ K \in L(X)$ such that $H' = R, K' = J$. So $T = H + \mathrm{i}K$; now

$$\|\exp(\mathrm{i}sH + \mathrm{i}tK)\| = \|\exp(\mathrm{i}sR + \mathrm{i}tJ)\| \leqslant M$$

for all real s, t, so T is normal-equivalent (Remark 1.9). □

Definition 1.13. An operator $T \in L(X)$ is strongly normal if $T = R + \mathrm{i}J$ where $RJ = JR$ and the set $\{R^m J^n : m, n = 0, 1, 2, \dots\}$ is hermitian.

Remark 1.14. If $T \in L(X)$ is strongly normal, $T = R + \mathrm{i}J$ as above, then the set $\{g_1(R,J) + \mathrm{i}g_2(R,J) : g_1, g_2 \in C_{\mathbb{R}}(\sigma(T))\}$ is a commutative C^*-algebra under the operator norm and the natural involution $(g_1(R,J) + \mathrm{i}g_2(R,J))^* = g_1(R,J) - \mathrm{i}g_2(R,J)$, where $C_{\mathbb{R}}(\sigma(T)))$ is the Banach algebra of continuous real-valued functions in two variables on $\sigma(T)$ [**4**, § 38].

Definition 1.15. An operator $T \in L(X)$ is strongly normal-equivalent if $T = R + \mathrm{i}J$ where $RJ = JR$ and the set $\{R^m J^n : m, n = 0, 1, 2, \dots\}$ is hermitian-equivalent.

Remark 1.16. If $T \in L(X)$ is strongly normal-equivalent then $T' \in L(X')$ is strongly normal-equivalent.

The next result is a refinement of Theorem 1.12.

Theorem 1.17. *If $T' \in L(X')$ is strongly normal-equivalent then $T \in L(X)$ is strongly normal-equivalent.*

Proof. Suppose that there exist operators R and J such that $T' = R + \mathrm{i}J$ and there is an equivalent norm $|\cdot|$ on X' with respect to which the set

$$\{R^m J^n : m, n = 0, 1, 2, \dots\}$$

is hermitian. Since T' is normal-equivalent, by Theorem 1.12 there exist H, K and such that $T = H + \mathrm{i}K$ where $HK = KH$ and H, K are hermitian-equivalent. The set $\{R^m J^n : m, n = 0, 1, 2, \dots\}$ is hermitian-equivalent. So there is an $M(\geqslant 1)$ such that

$$\|\exp(\mathrm{i}tR^m J^n)\| \leqslant M \qquad (t \in \mathbb{R},\ m, n = 0, 1, 2 \dots)$$

and we have

$$\|\exp(\mathrm{i}tH^m K^n)\| = \|\exp(\mathrm{i}tR^m J^n)\| \leqslant M \qquad (t \in \mathbb{R},\ m, n = 0, 1, 2, \dots).$$

If we define

$$|||x||| = \sup\{\|\exp(\mathrm{i}tH^m K^n)x\| : t \in \mathbb{R},\ m, n = 0, 1, 2, \dots\}$$

then $|||\cdot|||$ is a norm on X, equivalent to the original norm, and for each $t \in \mathbb{R}$ we have

$$|||\exp(\mathrm{i}tH^m K^n)||| = 1 \qquad (m, n = 0, 1, 2, \dots).$$

Therefore with this norm the set $\{H^m K^n : m, n = 0, 1, 2, \dots\}$ is hermitian: hence T is strongly normal-equivalent. □

Note that if T is strongly normal-equivalent then the closed linear span of $\{R^m J^n : m, n = 0, 1, 2, \dots\}$ is an hermitian-equivalent set [**6**, Theorem 4.17]: equivalently,

$$\{f(R,J) : f \in C_{\mathbb{R}}(\sigma(T))\}$$

is hermitian-equivalent. We may therefore introduce yet another norm, γ, on X, with respect to which T will be strongly normal:

$$\gamma(x) = \sup\{\|\exp(\mathrm{i}f(R,J))x\| : f \in C_{\mathbb{R}}(\sigma(T))\}.$$

Then $\gamma(x) \geqslant |||x|||$: so $|||x'||| \leqslant \gamma(x')$ for $x' \in X'$.

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Questions 1.18.

(a) Do γ and $|||\cdot|||$ coincide?

(b) Does the norm $|\cdot|$ (on X') coincide with either the dual of of γ or the dual of $|||\cdot|||$?

(c) Is $|\cdot|$ (on X') automatically a dual norm? That is, does there exist an equivalent norm η on X such that $|x'| = \sup\{|\langle x, x'\rangle| : \eta(x) = 1\}$?

2. Scalar-type operators

A family $\Gamma \subseteq X'$ is called *total* if and only if $x \in X$ and $\langle x, y'\rangle = 0$, for all $y' \in \Gamma$, together imply that $x = 0$. Let Σ be a σ-algebra of subsets of an arbitrary set Ω. and let Γ be a total subset of X'. A spectral measure of class $(\Sigma,\ \Gamma)$ on X is a uniformly bounded Boolean algebra homomorphism from Σ into the Boolean algebra of projections on X such that for all $x \in X$ and $y' \in \Gamma$, $\langle E(\cdot)x, y'\rangle$ is countably additive on Σ. See [**6**] for a fuller account.

In the following definition Σ_p denotes the σ-algebra of Borel subsets of the complex plane.

Definition 2.1. An operator S in $L(X)$ is called a prespectral operator of class Γ if there is a spectral measure $E(\cdot)$ of class (Σ_p, Γ) on X such that for all $\delta \in \Sigma_p$

1. $SE(\delta) = E(\delta)S \quad (\delta \in \Sigma_p)$
2. $\sigma(S \mid E(\delta)X) \subseteq \overline{\delta} \quad (\delta \in \Sigma_p)$.

The spectral measure $E(\cdot)$ is called a resolution of the identity of class Γ for S. If in addition, $S = \int_{\sigma(S)} \lambda E(\mathrm{d}\lambda)$, then S is said to be a scalar-type operator of class Γ.

Definition 2.2. An operator $S \in L(X)$ is a spectral operator if there is a spectral measure $E(\cdot)$ defined on Σ_p with values in $L(X)$ such that

1. $E(\cdot)$ is countably additive on Σ_p in the strong operator topology,
2. $SE(\tau) = E(\tau)S \quad (\tau \in \Sigma_p)$,
3. $\sigma(S \mid E(\tau)X) \subseteq \overline{\tau} \quad (\tau \in \Sigma_p)$.

Remark 2.3. The operator $S \in L(X)$ is spectral if and only if it is prespectral of class X' [**6**, Theorem 6.5].

The next result extends that of Berkson and Gillespie [**3**, Theorem 8] and answers the question of [**3**, Remark 1 on Theorem 9] affirmatively.

Theorem 2.4. *Let $S \in L(X)$. Then the following conditions are equivalent:*

(1) *$S' \in L(X')$ is a scalar-type of class X,*

(2) *$S \in L(X)$ is strongly normal-equivalent,*

(3) *there exist a compact subset* Ω *of* $\mathbb{C}$ *and a norm continuous representation* $\Theta : C(\Omega) \mapsto X$ *such that* $\Theta(z \mapsto z) = S$, $\Theta(z \mapsto 1) = I$.

Proof. $1 \Rightarrow 2$. Suppose that $S' \in L(X')$ is scalar-type of class X with spectral measure $E(\cdot)$. There is a norm $|\cdot|$ on X', equivalent to the original norm $\|\cdot\|$, for which the values of $E(\cdot)$ are simultaneously hermitian [**6**, Theorem 5.4]. Then, putting $R = \int_{\sigma(S)} \operatorname{Re}\lambda E(\mathrm{d}\lambda)$ and $J = \int_{\sigma(S)} \operatorname{Im}\lambda E(\mathrm{d}\lambda)$, we see that $S' = R + \mathrm{i}J$, $RJ = JR$ and $\{R^m J^n : m, n = 0, 1, 2, 3, \dots\}$ is $|\cdot|$-hermitian [**6**, proof of Theorem 5.40]: so S' is strongly normal-equivalent. Hence, by Theorem 1.17, S is strongly normal-equivalent.

$2 \Rightarrow 3$. If $|\cdot|$ is a norm equivalent to the original norm on X such that $S = H + \mathrm{i}K$, where $HK = KH$ and

$$\{H^m K^n : m, n = 0, 1, 2, 3, \dots\}$$

is $|\cdot|$-hermitian, then, using Sinclair's theorem as in the proof of [**6**, Theorem 5.41], we have

$$|p(H, K)| \leqslant 2 \sup\{|p(\operatorname{Re}\lambda, \operatorname{Im}\lambda)| : \lambda \in \sigma(S)\}$$

for all polynomials $p(x, y)$ with complex coefficients. The Stone–Weierstrass theorem ensures the existence of the functional calculus Θ as claimed.

$3 \Rightarrow 1$. This is immediate from [**6**, Theorem 5.21]. □

The following results are immediate corollaries of Theorem 2.4, and Theorems 3.1 and 3.2 of [**5**]; see also [**10**].

Corollary 2.5. *Let X be a Banach space which does not contain a subspace isomorphic to c_0. Then $S \in L(X)$ is scalar-type spectral if and only if S satisfies any (and hence all) of the condition in Theorem 2.4.*

The converse of Corollary 2.5 is true in any Banach space (see [**6**, Theorem 5.22]).

Corollary 2.6. *Let X be a Banach space which contains a subspace isomorphic to c_0. Then there exists an operator which satisfies the three condition of Theorem 2.5, but which is not scalar-type spectral.*

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