The first of the seven chapters deals then with those beginnings and several strands of earlier work on large cardinal properties, as well as Gödel's notion of his universe of constructible sets, L, and Scott's seminal result that V could not be L if there was a σ-complete 2-valued measure on an uncountable cardinal number. This phylogenetic account is then developed in the subsequent text, and forms the basis of chapters on measurability, and its relation to embeddings of inner models of set theory; to model theoretic consequences of that measurability, and indiscernibles for first order structures; and to extensions of such embedding properties in terms of large cardinal axioms (that these days are often interpreted as embeddings of the universe V into some inner model of the ZFC axioms with particular properties). A chapter is devoted to the set theory of the reals. This serves to introduce forcing (although the basics of this technique are assumed of the reader). The keywork here was that of Solovay who showed the consistency of all sets of reals being Lebesgue measurable, if one assumed the consistency of an inaccessible cardinal. Forcing and large cardinal connections with properties of the real continuum are laid out. Descriptive set theoretical representations of sets of real numbers are given here (which will be needed for later work in the Chapter on Determinacy). Some may feel that what is commonly called 'Set Theory of the Reals' and the so-called cardinal invariants are given short shrift here (although this reviewer is not amongst them). The reader will find little on inner model theory nor a general survey of forcing consistency results. They are promised in a subsequent volume.

In the eager rush of mathematics that often stylises 'developmental' texts history is often trampled underfoot and, if mentioned, is all too often at the mercy of the *Weltanschauung* of the author, or hastily disposed of in footnotes or an embarrassed appendix. But here there is a sensitive interpretation of the notions of past mathematicians and we hear how their own views coloured their work and the subject's development. The text is bookended by an Introduction which gives an overview of that evolution and its formative influences, and at the end by an appendix in which philosophical discussion has been coralled. The avowed purpose of the latter is to pre-empt, or perhaps defuse, attempts to 'over-metaphysicize' the discussion on mathematical truth, existence, and such concepts that are thrown into sharp relief in the light of any discussion of set theory as a foundation, and even more so when in the blinding glare of large cardinal hypotheses. Quite rightly, it is judged that the autonomy of set theory *qua* mathematical practice is a justification in itself, and that set theory provides an open-ended framework for the interpretation of mathematical systems rather than (a now rather simple view) a reductionist 'foundation' for all mathematics.

The exposition is intelligent and well-paced; misprints are extremely few; as a source book it is a compendium of references, well indexed, and it will become literally the reference book, a Baedeker, for the enquiring student of the subject. It should therefore be on every University Library's mathematical shelf.

P. D. WELCH

SHELAH, S. *Cardinal arithmetic* (Oxford Logic Guides Vol. 29, Clarendon Press, Oxford, 1994), xxxi+481 pp., 0 19 853785 9, (hardback) £65.

Since its inception cardinal arithmetic, or rather the cardinal exponentiation function, has been problematic. Cantor, the founder of set theory, showed (1874) that $2^{\aleph_0} > \aleph_0$, or more generally that $2^{\aleph_\alpha} > \aleph_\alpha$, but was unable to prove the Continuum Hypothesis, that $2^{\aleph_0} = \aleph_1$. Since 1963 we now know why: Cohen showed by his method of forcing that it was consistent with ZFC (the widely accepted axioms of Zermelo-Fraenkel set theory, with the axiom of Choice) that $2^{\aleph_0}$ could be almost anything (the caveat being due to the only other restriction on $2^{\aleph_\alpha}$ known – due to König (1927) – that $cf(2^{\aleph_\alpha}) \neq cf(\alpha)$). (Here cf, or cofinality, of λ is in fact the size of a smallest family of sets cofinal in the partial order $(\mathcal{P}_{<\lambda}(\lambda), \subseteq)$ where $\mathcal{P}_{<\lambda}(X)$ is the set of subsets of X of cardinality $< \lambda$. We identify $\aleph_\alpha$ with ω_α.)

Expanding on this forcing method, Easton showed that the exponential function $F(\alpha) = 2^{\aleph_\alpha}$ could be almost anything consistent with König and common sense for *regular* $\aleph_\alpha$ (that is those α so that $cf(\aleph_\alpha) = \aleph_\alpha$), but the same could not be shown for *singular* cardinals (i.e. those with $cf(\aleph_\alpha) < \aleph_\alpha$). This was thought merely to require extra effort.

However Silver surprised the set-theoretical community by showing, for example, that if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ held for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$ (and similarly for any cardinal $\aleph_\lambda$ with $cf(\lambda)$ uncountable). The case of $cf(\lambda) = \omega$ remained open. Following up on this work, Galvin and Hajnal showed, again for example, that $\aleph_{\omega_1}$ a strong limit cardinal (meaning that $\alpha < \omega_1 \Rightarrow 2^{\aleph_\alpha} < \aleph_{\omega_1}$) gave a bound on $2^{\aleph_{\omega_1}} \leq \aleph_{(2^{\aleph_1})^+}$.

However the difference between countable and uncountable cofinalities was thrown into stark relief by Jensen's result that, for example, $2^{\aleph_n} = \aleph_{n+1}$ and $2^{\aleph_\omega} > \aleph_{\omega+1}$ would require a non-trivial embedding of Gödel's universe L of constructible sets to itself. (This was the model that Gödel used to show *inter alia* that for all α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ – the Generalised Continuum Hypothesis – was consistent.) This was a hint that 'large cardinal'-like properties would be needed to 'violate' the GCH first at a singular cardinal in any forcing argument. Indeed there followed a series of forcing results using large cardinals along just these lines, all producing *relative consistency* results that assumed the consistency of some strong axiom with the rest of ZFC.

Shelah's possible cofinality, or pcf, theory provides *absolute* theorems of ZFC. His approach has been to say that the right way to measure the size of $\aleph_\omega^{\aleph_0}$ is by $cf((\mathcal{P}_{<\omega_\infty}(\aleph_\omega), \subseteq))$. The usual formula for $\aleph_\omega^{\aleph_0}$ is then obtained by multiplying this cofinality by $2^{\aleph_0}$. The latter we know is unrestricted by ZFC and Shelah regards this as 'interference' or 'noise' in the true calculation. However, and remarkably, he has shown that the cofinality has a definite value. He studies the set of regular cardinals in the interval $[\aleph_{\omega_1}, cf((\mathcal{P}_{<\omega_\infty}(\aleph_\omega), \subseteq))]$ as *true cofinalities* of reduced products $\Pi(a)_\lambda / J_{<\lambda}$ of sets $(a)_\lambda \subseteq \{\aleph_n : n < \omega\}$ modulo an ideal J_λ on ω. Sloganising, pcf theory is the theory of reduced products of small sets of regular cardinals.

The applications are extraordinarily fruitful. He can prove a result directly generalising Galvin and Hajnal's, that if $2^{\aleph_n} < \aleph_\omega$ (for all $n < \omega$) then $2^{\aleph_\omega} < \aleph_{(2_0^{\aleph})^+}$. Weakening the assumption: if $2^{\aleph_0} < \aleph_\omega$ then $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$. In Chapter IX of the book is a section called 'Why the HELL is it 4?'. (It might be thought that 4 was a contingent artefact of the proof, but Shelah believes otherwise.)

Chapters II to IV deal with Jónsson algebras (an algebra $\mathcal{A} = (A, (f_i)_{i<\omega})$, where each f_i is a finitary function, is called *Jónsson*, if $\mathcal{A}$ has no proper sub-algebra of the same cardinality). A cardinal κ is called *Jónsson*, if there are no Jónsson algebras of size κ. It is easy to show that λ not Jónsson implies that the next cardinal λ^+ is not Jónsson. Thus each $\aleph_n$ is not Jónsson. A long standing question is whether $\aleph_\omega$ can be Jónsson. An application of pcf theory is that $\aleph_{\omega+1}$ cannot be Jónsson. This phenomenon persists higher up for other singular cardinals. There are other applications, too numerous to detail, on chain conditions on Boolean algebras, entangled linear orders, $L_{\infty,\lambda}$-equivalent non-isomorphic models, and colouring relations.

The author claims that Cantor could read this book (there is no ultimate reliance on forcing and little of metamathematical methods). The book is amusingly self-referential, e.g., [Sh 789] in the Bibliography or Contents page turns out to be simply Chapter N. Readers used to the Shelah style will find themselves in familiar territory. Rather than a gently sloping development of a theory, its landscape is alpine with peaks emerging stunningly above the clouds. The reader (and Cantor) must be prepared for some stiff climbing. The mathematics here will remain an important summit of the subject and the Editors have the good fortune of having obtained a landmark volume for the Logic Guide Series.

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