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On iterates of e^z

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Abstract. It is proved that for the map $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^z$, the family $\{f^n\}_{n=0}^{\infty}$ is not normal at any point. In particular, f is topologically transitive.

Let $f: \mathbb{C} \to \mathbb{C}$ be an analytic function. In the theory of iterates of f an important role is played by the set $F(f)$ of points at which $\{f^n\}_{n=0}^{\infty}$ is not a normal family.

The aim of this paper is to prove that, for the map $f(z) = e^z$, the set $F(f)$ is the whole plane. This was conjectured in 1926 by Fatou [**2**]. Some progress has been made by Töpfer [**3**] who proved that $F(f) \supset \overline{\bigcup_{n=0}^{\infty} f^{-n}\mathbb{R}}$, and by Baker [**1**] who proved that any limit function of a subsequence of $(f^n)_{n=0}^{\infty}$ on any component of $\mathbb{C} \backslash F(f)$ has to be one of the constants $f^n(0)$, $n = 0, 1, \ldots, +\infty$. In my proof I do not use the above results.

More information concerning this and similar problems, together with historical references, can be found in [**1**].

In the sequel we assume that $f(z) = e^z$.

LEMMA 1. *Let $z \in \mathbb{C}$. Then* $|\mathrm{Im}\,(f^n(z))| \le |(f^n)'(z)|$.

Proof. We have $f(x+iy) = e^x \cos y + ie^x \sin y$. Since $|\sin y| \le |y|$, we obtain $|\mathrm{Im}\,(f(w))| \le |\mathrm{Im}\, w| \cdot |f(w)|$ for every $w \in \mathbb{C}$. But $f'(w) = f(w)$, and hence if $w \notin \mathbb{R}$ then $|\mathrm{Im}\,(f(w))|/|\mathrm{Im}\, w| \le |f'(w)|$. If $f^n(z) \notin \mathbb{R}$ then, using this inequality for $w = f(z), \ldots, f^{n-1}(z)$, we obtain

$$\frac{|\mathrm{Im}\,(f^n(z))|}{|\mathrm{Im}\,(f(z))|} = \prod_{k=1}^{n-1} \frac{|\mathrm{Im}\,(f(f^k(z)))|}{|\mathrm{Im}\,(f^k(z))|} \le \prod_{k=1}^{n-1} |f'(f^k(z))|.$$

But $|\mathrm{Im}\,(f(z))| \le |f(z)| = |f'(z)|$, and hence

$$|\mathrm{Im}\,(f^n(z))| \le \prod_{k=0}^{n-1} |f'(f^k(z))| = |(f^n)'(z)|.$$

If $f^n(z) \in \mathbb{R}$ then the inequality is obvious. □

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Let $S=\{z\in\mathbb{C}: |\mathrm{Im}\, z|\le\frac{1}{3}\pi\}$.

LEMMA 2. (*a*) *If* $z\in S$ *then* $\mathrm{Re}\,(f(z))\ge\mathrm{Re}\, z+(1-\ln 2)$.
(*b*) *If* $z\in\mathbb{C}\backslash\mathbb{R}$ *then there exists* $n\ge 0$ *such that* $f^n(z)\notin S$.

Proof. For $y\in[-\frac{1}{3}\pi,\frac{1}{3}\pi]$ we have $\cos y\ge\frac{1}{2}$ and hence if $x+iy\in S$ then $\mathrm{Re}\,(f(x+iy))=e^x\cos y\ge\frac{1}{2}e^x$. Let $\phi(x)=\frac{1}{2}e^x-x$. Then $\phi'(x)=\frac{1}{2}e^x-1$ and hence $\inf_{\mathbb{R}}\phi=\phi(\ln 2)=1-\ln 2$. Therefore, if $z\in S$ then $\mathrm{Re}\,(f(z))\ge\mathrm{Re}\, z+(1-\ln 2)$.

Suppose now that $z\in\mathbb{C}\backslash\mathbb{R}$ and $f^k(z)\in S$ for all $k\ge 0$. Since $1-\ln 2>0$, by induction we obtain $\mathrm{Re}\,(f^k(z))\to\infty$ as $k\to\infty$. For all $y\in[-\frac{1}{2}\pi,\frac{1}{2}\pi]$ we have $|\sin y|\ge(2/\pi)|y|$. Thus

$$|\mathrm{Im}\,(f^{k+1}(z))|\ge e^{\mathrm{Re}(f^k(z))}\cdot(2/\pi)|\mathrm{Im}\,(f^k(z))|\quad\text{for all } k\ge 0.$$

But for k sufficiently large, $e^{\mathrm{Re}(f^k(z))}\cdot 2/\pi>2$, and hence $|\mathrm{Im}\,(f^k(z))|\to\infty$ as $k\to\infty$ – a contradiction. □

LEMMA 3. *Let* $B\subset\mathbb{C}$ *be a disk with a centre b and a radius r. Let* $n\ge 0$ *be such that* $f^n|_B$ *is a homeomorphism. Then* $f^n(B)$ *contains a disk with a centre* $f^n(b)$ *and a radius* $r\cdot\inf_B|(f^n)'|$.

Proof. Let γ be the shortest curve joining $f^n(b)$ with the boundary of $f^n(B)$. Then the curve $(f^n|_B)^{-1}(\gamma)$ joins b with the boundary of B. Thus its length is at least r. Since f^n is holomorphic, it is conformal, i.e. stretching in all directions is equal to the absolute value of the derivative. Hence, the length of γ is at least $r\cdot\inf_B|(f^n)'|$. □

LEMMA 4. *Let V be a non-empty open connected set. Then only finitely many of its images can be disjoint from S.*

Proof. Suppose that there exists an increasing sequence $(n_j)_{j=1}^{\infty}$ such that $f^{n_j}(V)$ is disjoint from S for every j. Then, by lemma 1, $\inf_V|(f^{n_j})'|\ge(\frac{1}{3}\pi)^j$ for all j. If all maps $f^{n_j}|_V$ are homeomorphisms then, by lemma 3, there exists k such that $f^k(V)$ contains a disk of radius π. Then there exists an integer m such that $f^k(V)$ intersects the line $\mathbb{R}+2\pi im$. If some $f^{n_j}|_V$ is not a homeomorphism then, since V is connected, there exist integers $k\in[0,n_j-1]$ and m such that $f^k(V)$ intersects the line $\mathbb{R}+2\pi im$. In both cases, $f^{k+1}(V)$ intersects the real axis. Therefore all sets $f^{n_j}(V)$ for $n_j\ge k+1$ intersect the real axis – a contradiction. □

LEMMA 5. *Let V be a non-empty open connected set such that infinitely many of its images are contained in the half-plane* $H=\{z:\mathrm{Re}\, z>4\}$. *Then some image of V intersects the real axis.*

Proof. Suppose that no image of V intersects the real axis. Then no image of V intersects the boundary of the set $W=\{z:|\mathrm{Im}\, z|\le 2\pi$ and $|\mathrm{Im}\,(f(z))|\le 2\pi\}$. If a connected set A is disjoint from W then either A or $f(A)$ is disjoint from S. Therefore, in view of lemma 4, only a finite number of images of V can be disjoint from W. Hence, almost all images of V are contained in W. If

$$|\mathrm{Im}\, z|=\tfrac{1}{3}\pi\quad\text{and}\quad\mathrm{Re}\, z\ge 4$$

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then

$$|\text{Im}\,(f(z))| = e^{\text{Re}\,z} \sin|\text{Im}\,z| \geq e^4 \sin\tfrac{1}{3}\pi > 2^4 \cdot \tfrac{1}{2} = 2 \cdot 4 > 2\pi.$$

Therefore the boundary of S is disjoint from $W \cap H$. Thus every connected subset of $W \cap H$ is either contained in S or is disjoint from S.

Infinitely many images of V are contained in $W \cap H$. From lemma 2 (a) it follows that $f(S \cap H) \subset H$. Therefore, in view of lemma 2 (b), infinitely many images of V are contained in $H \backslash S$. This contradicts lemma 4. □

LEMMA 6. *Let V be a non-empty open connected set. Then some image of V intersects the real axis.*

Proof. Suppose that no image of V intersects the real axis. By Montel's theorem, $\{f^n|_V\}_{n=0}^{\infty}$ is a normal family of functions. By lemma 5, almost all images of V intersect the disk $D = f(\mathbb{C}\backslash H) = \{z : |z| \leq e^4\}$. Let f_0 be a limit of some subsequence of the sequence $(f^n|_V)_{n=0}^{\infty}$. Then $f_0(V)$ intersects D.

Take a point z belonging to this intersection. If $z \in \mathbb{R}$ then there exists $k \geq 0$ such that $f^k(z) \in H$. If $z \notin \mathbb{R}$ then, by lemma 2 (b), there exists $k \geq 0$ such that $f^k(z) \notin S$. Therefore there exists a subsequence of the sequence $(f^n|_V)_{n=0}^{\infty}$ convergent to a map f_1 and a point $w \in V$ such that $f_1(w) \in H$ or $f_1(w) \notin S$. Then there exists a connected open neighbourhood U of w such that $U \subset V$ and $f^n(U) \subset H$ or $f^n(U) \cap S = \varnothing$ for infinitely many ns. Thus, by lemmas 4 and 5, some image of U intersects the real axis – a contradiction. □

THEOREM. *The set $F(f)$ is the whole plane.*

Proof. Suppose that $F(f)$ is not the whole plane. Then there exists a non-empty connected open set U such that $\{f^n|_U\}_{n=0}^{\infty}$ is a normal family of functions. By lemma 6, the set of points which are mapped eventually into $\mathbb{R}$ is dense in U. By lemma 2 (a), images of all these points converge to infinity. Hence, the sequence $(f^n|_U)_{n=0}^{\infty}$ converges uniformly to infinity. Since the set $f(\mathbb{C}\backslash H)$ is bounded, almost all sets $f^n(U)$ are contained in H. Also, almost all of them intersect the real axis. By lemma 2 (b), infinitely many of them are not contained in S. Since the boundary of S is disjoint from $W \cap H$ (see the proof of lemma 5), infinitely many of the images of U are not contained in W. Hence, infinitely many of the images of U are not contained in the strip $\{z : |\text{Im}\,z| \leq 2\pi\}$. Consequently, infinitely many of them intersect one of the lines $\mathbb{R} \pm \pi i$. But the second image of these lines is contained in the unit disk – a contradiction. □

In view of Montel's theorem we obtain immediately:

COROLLARY. The map f is topologically transitive. □

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