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ON THE HERMITIAN-EINSTEIN TENSOR OF A COMPLEX HOMOGENOUS VECTOR BUNDLE

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ABSTRACT We prove that any holomorphic, homogenous vector bundle admits a homogenous minimal metric—a metric for which the Hermitian-Einstein tensor is diagonal in a suitable sense The concept of minimality depends on the choice of the Jordan-Holder filtration of the corresponding parabolic module We show that the set of all admissible Hermitian-Einstein tensors of certain class of minimal metrics is a convex subset of the euclidean space As an application, we obtain an algebraic criterion for semistability of homogenous holomorphic vector bundles

1. **Introduction.** Let G denote a complex connected Lie group and P a parabolic subgroup of G.

We shall fix once and forever a maximal compact subgroup K of G and K-invariant Kähler form ω on G/P. This data determines an algebraic trace operator Λ of type $(-1,-1)$ acting on the Dolbeault complex of G/P as the adjoint of multiplication by ω. Let V denote a rational representation of P and let $\mathbf{V}$ be the vector bundle $G \times_P V$. The module V can always be filtered:

$$(*) \qquad 0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

by P submodules with $Q_i = V_i/V_{i-1}$ irreducible. We shall fix such a filtration once and forever. Recall that a P-module Q is irreducible if and only if the unipotent radical of P acts trivially on Q, and that irreducible module remains irreducible when restricted to $H = P \cap K$.

DEFINITION 1.1. A metric $\mathbf{V}$ is *minimal* if for $i = 1, 2, \ldots, n-1$ its second fundamental form β_i corresponding to the extension:

$$(1.2) \qquad V_i \rightarrowtail V \twoheadrightarrow V/V_i$$

is harmonic in $\Omega^{01}\big(\mathrm{Hom}(V/V_i, V_i)\big)$.

The first result of this paper is the existence theorem for minimal metrics.

THEOREM A. *For any choice of K-invariant metrics on $\mathbf{Q}_i$, $i = 1, 2, \ldots, n$, there exists a* minimal *(K-invariant) metric on $\mathbf{V}$ inducing this choice.*

Notice that if all Q_i's are pairwise non-isomorphic then any K-invariant metric on $\mathbf{V}$ is automatically minimal. In a forthcoming paper we shall demonstrate existence and

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uniqueness theorem for an arbitrary *filtered* holomorphic vector bundle on a Kähler manifold. Let h denote a K-invariant metric on $\mathbf{V}$ and $F \in \Omega^{11}\big(\mathrm{Hom}(\mathbf{V},\mathbf{V})\big)$ be its hermitian curvature tensor. The endomorphism $c\Lambda F$, where

$$c = \frac{i}{2n\pi}\int_{G/P}\omega^n, \quad n = \dim G/P,$$

will be referred to as *normalized Hermitian-Einstein tensor*. The constant c is chosen so that:

$$\mathrm{Tr}(c\Lambda F) = r\mu(\mathbf{V}) = \int_{G/P} c_1\omega^{n-1} \tag{1.4}$$

where c_1 denotes the first Chern class of V and $r = \mathrm{rank}(V)$. The quantity $\mu(V) = \frac{1}{r}\deg(\mathbf{V}) = \frac{1}{r}\int_{G/P} c_1\omega^{n-1}$ is the "slope" in Mumford-Takemoto concept of stability.

The metric h together with filtration $(*)$ determine an orthogonal, K-invariant decomposition:

$$\mathbf{V} = \mathbf{Q}_1 \oplus \mathbf{Q}_2 \oplus \cdots \oplus \mathbf{Q}_n \tag{1.5}$$

It turns out that h is minimal if and only if $c\Lambda F$ is diagonal relative to (1.5). Let $t = (t_1, t_2, \ldots, t_n)$ be a vector of positive real numbers. We shall write h^t for the metric obtained from h by rescaling it by $t_\imath$ on each factor $Q_\imath$ in (1.5). Let T denote the multiplicative group of such scalings. The main result of this paper describes the effect of scaling on the Hermitian-Einstein tensor of minimal metrics. To give the precise statement we shall need some more notation.

We shall say that a subset $S \subset \{1, 2, \ldots, n\}$ is a *submodule* if and only if the H-submodule $\oplus_{\imath\in S}\mathbf{Q}_\imath$ is a P-submodule of V. Let $r_\imath = \mathrm{rank}(\mathbf{Q}_\imath)$ and let Σ be a subset of vectors $x = (x_1, x_2, \ldots, x_n)$ in R^n characterized by the following two properties:

$$\sum_{\imath=1}^{n} r_\imath x_\imath = 0 \tag{1.6}$$

$$\sum_{\imath\in S} r_\imath x_\imath > 0, \text{ for all proper submodules } S. \tag{1.7}$$

THEOREM B. *A minimal metric remains minimal under scaling. If V does not decompose as a direct sum of P-modules, then the map:*

$$t \dashrightarrow c\Lambda F(h^t)$$

induces a diffeomorphism of T/T^+ onto the convex subset of R^n:

$$(c\Lambda F_1, c\Lambda F_2, \ldots, c\Lambda F_n) + \Sigma.$$

Here $F_\imath$ denotes the curvature of the induced metric on $\mathbf{Q}_\imath$ (hence it does not depend on t); we write the diagonal matrix $c\Lambda F(h^t)$ as an n-vector, and let T^+ denote the diagonal subgroup of T.

As a corollary of Theorem B one obtains a uniformization theorem which can be deduced from the proof of the general existence theorem for Hermitian-Einstein metric proved by Uhlenbeck and Yau in [2].

THEOREM 1.9. *Suppose that $\mu(\mathbf{U}) < \mu(\mathbf{V})$ for every P-submodule U of V. Then* **V** *admits K-invariant Hermitian-Einstein metric.*

PROOF. We have to show that the system of equations $\mu = f_i + x_i$, $i = 1, 2, \ldots, n$, has a solution $x \in \Sigma$, where

$$f_i = c\Lambda F_i$$
$$\mu = \mu(\mathbf{V}) = \frac{1}{r_1 + r_2 + \cdots + r_n} \sum_{i=1}^{n} r_i f_i.$$

By the definition of Σ, this is equivalent to

$$\sum_{i \in S} r_i(\mu - f_i) > 0,$$

or to

$$\mu > \frac{1}{\sum_{i \in S} r_i} \sum_{i \in S} r_i f_i = \mu\Big(\bigoplus_{i \in S} \mathbf{Q}_i\Big), \text{ for all proper submodules } S.$$

REMARK. Notice that it suffices to check the inequalities for just a finite number of submodules of V; those submodules are determined by the minimal metric (and the choice of the filtration $(*)$). It would be interesting to find somewhat more algebraic characterization of this particular family of submodules.

This paper is organized in the following way. In §2 we prove Theorem A by studying the Donaldson functional corresponding to the secondary characteristic class of c_2. In §3 we derive the transformation law (3.8) of $c\Lambda F$, describing the effect of scaling a minimal metric. In §4 we study the equation (3.8) and prove Theorem B, it turns out to be convenient to work within the framework of graph theory. The last section contains an example indicating the range of applicability of Theorem 1.9 for deciding stability of a homogenous vector bundle.

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2. **Existence of minimal metrics.** Let $h_1, h_2, \ldots, h_n$ denote a collection of K-invariant hermitian products on $Q_1, Q_2, \ldots, Q_n$ and let $\mathfrak{H}$ be the set of all K-invariant hermitian products on V inducing $(h_1, h_2, \ldots, h_n)$ on respective quotients. The unipotent radical of flag-preserving automorphisms is the group:

$$N = \{a \in \mathrm{Aut}^H(V) \mid (a - I)(V_i) \subset V_{i-1} \text{ for all } i\};$$

it acts freely and transitively on $\mathfrak{H}$ via $h^a = h(a_-, a_-)$. Let $N_i = \{a \in N \mid (a - I)(V) \subset V_i \cap \mathrm{Ker}(a - I)\}$ be a collection of abelian subgroups of N. The map:

$$(2.1) \qquad (a_{n-1}, a_{n-2}, \ldots, a_1) \dashrightarrow a_{n-1}a_{n-2} \cdots a_1$$

is a diffeomorphism of $N_{n-1} \times N_{n-2} \times \cdots \times N_1$ onto N. The subgroups N_i are isomorphic to their Lie algebras $n_i = \mathrm{Hom}^H(V/V_i, Q_i)$ via $a \dashrightarrow a - 1$. In particular, the choice of $h \in \mathfrak{H}$ induces a diffeomorphism:

$$(2.2) \qquad n = \oplus n_i \approx \mathfrak{H}$$
$$\mathbf{n} = (\mathbf{n}_{n-1}, \mathbf{n}_{n-2}, \ldots, \mathbf{n}_1) \dashrightarrow h^{(1+\mathbf{n}_{n-1})(1+\mathbf{n}_{n-2}), \ ,(1+\mathbf{n}_1)},$$

inducing on $\mathfrak{H}$ the structure of trivial H-module with hermitian product induced by h, the corresponding metric will be denoted by $| \ |$ The decomposition (2 2) of $\mathfrak{H}$ is orthogonal relative to this metric The operator $\bar{\partial}\ G \times_P n \to \Omega^{01}(G \times_P n)$ induces orthogonal decomposition

$$n = \operatorname{Ker} \bar{\partial} \oplus n^0$$

respecting (2 2)

THEOREM 2 3 *The vector space n^0 contains a* minimal *metric (for arbitrary choice of h)*

The proof of this theorem occupies the rest of §2 It relies on the variational characterization of minimality, the functional being the Donaldson's functional $R_2(\_, h)$ corresponding to the secondary characteristic class for c_2 We refer to [1] for its definition and properties, but for convenience of the reader we state the property which relevant in our application

THEOREM 2 4 [1, PROPOSITION 7] *Let $S \rightarrowtail V \twoheadrightarrow Q$ be an extension of holomorphic vector bundles on a Kahler manifold M A choice of metric h on V determines a functional $R_2(\ , h)$ on the space of metrics on V with the property that for two metrics k, l inducing the same metrics on Q and S*

$$R_2(k, h) - R_2(l, h) = |\bar{\partial} s_1|^2 - |\bar{\partial} s_2|^2,$$

where s_1 (resp s_2) denote the splitting $Q \to V$ of this extension induced by k (resp l)

PROOF OF THEOREM 2 3 We shall write $R(\mathbf{n}) = R_2(h^{(1+\mathbf{n}_{n\ 1})(1+\mathbf{n}_{n\ 2}) \ (1+\mathbf{n}_1)}, h)$ It suffices to show the following two facts

PROPOSITION 2 5 *The critical points of $R(\ )$ on n^0 correspond to* minimal *metrics*

PROPOSITION 2 6 *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|n| > \delta$ for $n \in n^0$ then $R(n) > \varepsilon$ In particular, $R(\_)$ attains its minimum on n^0*

PROOF OF PROPOSITION 2 5 Let $n \in n^0$ be a critical point, let $s\ V/V_i \to V$ denote the corresponding H-invariant splitting of $V_i \rightarrowtail V \twoheadrightarrow V/V_i$, and $\beta = \bar{\partial} s$ Since for all $u \in n^0 \cap \operatorname{Hom}^H(V/V_i, V)$,

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t-0} R(n + tu) = \frac{d}{dt}\Big|_{t=0} \big(R_2(n + tu, h) - R_2(n, h)\big) \\ &= \frac{d}{dt}\Big|_{t\ 0} (|\beta + t\bar{\partial} u|_n^2 - |\beta|_n^2) = (\bar{\partial}^* \beta, u) + (u, \bar{\partial}^* \beta) \end{aligned}$$

Hence β is harmonic

PROOF OF PROPOSITION 2 6 Applying Theorem 2 4 inductively to the extensions

$$V_i/V_{i\ 1} \rightarrowtail V/V_{i-1} \twoheadrightarrow V/V_i, \quad i = 1, 2, \quad n,$$

one obtains an identity:

$$\begin{aligned} &R\big((\mathbf{n}_{n-1},\mathbf{n}_{n-2},\ldots,\mathbf{n}_1)\big) - R(0) \\ &\quad = |\bar\partial\mathbf{n}_{n-1} + \bar\partial s_{n-1}|_0^2 + |\bar\partial\mathbf{n}_{n-2} + \bar\partial s_{n-2}|_{\mathbf{n}_{n-1}}^2 + \cdots + |\bar\partial\mathbf{n}_1 + \bar\partial s_1|_{\mathbf{n}_{n-1}\cdots\mathbf{n}_2}^2 \\ &\qquad - (|\bar\partial s_{n-1}|_0^2 + \cdots + |\bar\partial s_1|_0^2). \end{aligned}$$

The subscripts in this expression indicate the dependence of metrics on $\mathbf{n} = (\mathbf{n}_{n-1}, \mathbf{n}_{n-2}, \ldots, \mathbf{n}_1)$, and $\{s_i\}$ denotes the collection of respective splittings corresponding to h. Since $\operatorname{Ker}\bar\partial \cap n^0 = 0$ and n^0 is finite-dimensional, for any $C > 0$, there exist $A, B > 0$ such that for all k, $0 < k < n$, and $n \in n^0$:
if

$$|(n_{n-1}, n_{n-2}, \ldots, n_{n-k+1})| < C$$

then

$$A|n_{n-k}| - B < |\bar\partial n_{n-k} + \bar\partial s_{n-k}|_{n_{n-1}n_{n-2}\cdots n_{n-k+1}}.$$

Using this estimate and the induction on n, it is easy to see that the right hand side of our formula has the desired growth.

3. **Hermitian-Einstein tensor of minimal metric.** It is well known that the set of K-invariant hermitian metrics on $\mathbf{V}$ is diffeomorphic to the set of H-invariant hermitian products on V. The decomposition $(*)$ together with a choice of such products induces the orthogonal decomposition (1.5) of V and the corresponding decomposition of $\operatorname{End}^H(V)$.

PROPOSITION 3.1. *A metric h is* minimal *if and only if ΛF is diagonal relative to (1.5).*

PROOF. Recall that the hermitian curvature tensor of h relative to the extension $V_i \rightarrowtail V \twoheadrightarrow V/V_i$ has the following form:

$$\begin{bmatrix} F' - \beta\beta^* & D\beta \\ -\bar\partial\beta^* & F'' - \beta^*\beta \end{bmatrix} \tag{3.2}$$

Here F' (resp. F'') denotes the curvature tensor of $\mathbf{V}_i$ (resp. $\mathbf{V}/\mathbf{V}_i$), D is the $(1,0)$-component of the induced hermitian connection on $(\mathbf{V}/\mathbf{V}_i)^* \otimes \mathbf{V}_i$ and $b = \bar\partial s$ where $s\colon V/V_i \longrightarrow V$ is the splitting induced by h. By the Kähler identity $\bar\partial^* = -i[\Lambda, D]$, one has $\Lambda D\beta = \bar\partial^*\beta$; hence β is harmonic if and only if ΛF is diagonal with respect to the decomposition $V = V_i \oplus V/V_i$ induced by h. Repeating this reasoning for $i = 1, 2, \ldots, n$, completes the proof.

Let $T = \{t = (t_1, t_2, \ldots, t_n) \mid t_i > 0, \text{ all } i\}$ denote the real n-dimensional torus and let T^+ be its diagonal subgroup. For given metric h let h^t denote the metric obtained by scaling factors in (1.5) with $(t_1, t_2, \ldots, t_n)$.

PROPOSITION 3.3. *If a metric h is* minimal *then the metric h^t is* minimal.

PROOF. Let $t^\iota = (1, 1, \ldots \overset{\iota}{1}, a, a, \ldots, a)$ for a positive scalar a. Since an element t of T can be written as a product $t = t^1 t^2 \cdots t^n$ with suitable a's, it suffices to prove the claim for $t = t^\iota$. However, since the forms β, F', F'' are not affected by such scaling and $(\ )^{*t} = a(\ )^*$, the claim follows directly from (3.2).

Suppose now that h is a minimal K-invariant metric. We shall write an explicit expression for the function $t \mapsto c\Lambda F(h^t)$. Let $\beta_j = \bar{\partial} s_j$ where s_j is the induced splitting of $V_j \rightarrowtail V_{j+1} \twoheadrightarrow Q_j$. Then the tensor $\beta_j \in \Omega^{01}(Q_j^* \otimes V_{j-1})$ decomposes relative to (1.5):

$$\beta_j = \bigoplus_{\iota\, \iota<j} (\beta_j)_\iota, \text{ with } (\beta_j)_\iota \in \Omega^{01}(Q_j^* \otimes Q_\iota).$$

Similarly, the tensor $b_j = -c\Lambda(\beta_j \beta_j^*) \in \mathrm{End}^H(V_j)$, decomposes as:

$$b_j = \bigoplus_{(\iota,k)\ \iota,k<j} (b_j)^\iota_k \text{ with } (b_j)^\iota_k \in \mathrm{Hom}^H(Q_k, Q_\iota).$$

Since Q_ι's are irreducible K-modules, by Schur's lemma $(b_\iota)^l_k$ are scalars. Moreover, since

$$r_\iota (b_j)^\iota_\iota = -\mathrm{Tr}\big((b_j)^\iota_\iota\big) = -\mathrm{Tr}(\beta_j)_\iota \big((\beta_j)_\iota\big)^* = \frac{1}{2\pi n} \mathrm{vol}(G/P) |(\beta_j)_\iota|^2 > 0,$$

we can write the relevant, diagonal part of this decomposition as a strictly upper triangular matrix with non-negative entries:

$$\big(F(h)\big)_{\iota j} = \begin{cases} (b_j)^\iota_\iota & \text{if } i < j \leq n \\ 0 & \text{if } n \geq i \geq j. \end{cases} \tag{3.4}$$

We have the following transformation law:

$$F(h^t) = tF(h)t^{-1}. \tag{3.5}$$

The element $c\Lambda(\beta_\iota^* \beta_\iota) \in \mathrm{End}^H(Q_\iota)$ is again a real positive scalar which can be computed as:

$$\begin{aligned} c\Lambda(\beta_\iota^*\beta_\iota) &= \frac{1}{r_\iota} \mathrm{Tr}\Big(c\Lambda(\beta_\iota^*\beta_\iota) = -\frac{1}{r_\iota}\mathrm{Tr}\Big(c\Lambda(\beta_\iota^*\beta_\iota)\Big) \\ &= \frac{1}{r_\iota} \sum_{k=1}^{k=\iota-1} \Big(F(h)\Big)_{k\iota} r_k = \sum_{k=1}^{k=\iota-1} \Big(([r]F(h)[r]^{-1}\Big)_{k\iota}\Big). \end{aligned}$$

We recall that $r_\iota = \mathrm{rank}(Q_\iota)$ and that $[r]$ denotes the diagonal matrix with entries $(r_1, r_2, \ldots, r_n)$.

Using (3.2) and the minimality of h one can write:

$$c\Lambda F(h) = \Big\{[c\Lambda F_1, c\Lambda F_2, \ldots, c\Lambda F_n] + \Big(F(h) - ([r]F(h)[r]^{-1}\Big)^{\mathrm{tr}}\Big\}\{1\} \tag{3.6}$$

where $\{1\} = e_1 + e_2 + \cdots + e_n$ ($\{e_\iota\}$ denotes the standard basis in R^n), F_ι denotes the curvature of the induced metric on Q_ι, and we write diagonal matrices as vectors. As usual $(\ )^{\mathrm{tr}}$ denotes the transposition.

To simplify the notation, we shall write (3.6) as:

$$c\Lambda F = f + \big(F - (F^r)^{\mathrm{tr}}\big)\{1\}. \tag{3.7}$$

Combining (3.5) and (3.6) one obtains the transformation law describing the effect of scaling on the reduced Hermitian-Einstein tensor of a minimal metric:

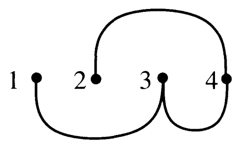
$$c\Lambda F^t = f + \big(tFt^{-1} - (tF^r t^{-1})^{\mathrm{tr}}\big)\{1\}. \tag{3.8}$$

Theorem B is a formal consequence of the above equation; we shall study it carefully in the next section.

4. **Formalism of graphs—proof of the main theorem.** Let F be a strictly upper-triangular $n \times n$ matrix with non-negative entries, and $[r] = (r_1, r_2, \ldots, r_n)$ a vector of positive integers. We shall associate to F a graph with vertices $\{1, 2, \ldots, n\}$, where (i,j) are connected by an edge if and only if $F_{ij} > 0$. For example, for

$$F = \begin{bmatrix} 0010 \\ 0001 \\ 0001 \\ 0000 \end{bmatrix},$$

the corresponding graph F has the form:



From now on we shall restrict the term "graph" to graphs obtained by this procedure.

DEFINITION 4.1. We say that a vertex $\{i\}$ *dominates* $\{j\}$ if $i < j$ and $\{i\}$ is connected to $\{j\}$ by an edge.

DEFINITION 4.2. A subset S of the set of vertices is a *submodule* if all vertices dominating a vertex in S are contained in S.

DEFINITION 4.3. Set S_F be a convex subset of $x = (x_1, x_2, \ldots, x_n) \in R^n$ defined by the following set of conditions:

i) $\sum_{i=1}^n r_i x_i = 0$

ii) $\sum_{i \in S} r_i x_i > 0$, for all proper submodules S in graph F.

Write $F' = [r]F[r]^{-1}$. The main result of this section is the following theorem.

THEOREM 4.4. *Suppose that graph F is connected. The map $f \colon T \longrightarrow R^n$ defined by:*

$$f(t) = \big(tF't^{-1} - (tF't^{-1})^{\mathrm{tr}}\big)\{1\}$$

induces diffeomorphism of T/T^+ onto S_F.

PROOF OF THEOREM B. Before explaining the details of the proof of the theorem, we point out that Theorem 4.4 combined with the transformation law (3.8) and the proposition below imply Theorem B.

PROPOSITION 4.6. *Let $V = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$ be the orthogonal decomposition corresponding to H-invariant hermitian product on V and to filtration $(*)$. Then the H-submodule $\oplus_{s \in S} Q_s$ is a P-submodule if and only if S is a submodule of graph F, where F denotes the matrix (3.4).*

PROOF. We prove the "only if" part of the proposition, leaving the other, easier part to the reader. From the inductive hypothesis we can assume that the proposition holds for V/V_1. It follows from the definition of matrix F, that the graph F^1 corresponding to V/V_1 with the induced metric can be constructed from graph F by erasing all edges originating at $\{1\}$. Let S graph F be a submodule of graph F^1. Since $S^1 = S \bigcap \{2, 3, \ldots, n\}$ is a submodule of graph F^1, $Q = \oplus_{s \in S'} Q_s$ is a P-submodule of V/V_1. If $S = S^1 \cup \{1\}$ then $\oplus_{s \in S} Q_s = p^{-1}(Q)$ where p denotes the projection $V \twoheadrightarrow V/V_1$; hence it is a submodule. If $S = S^1$ then there is no edge in the graph F which originates at $\{1\}$ and ends at a vertex in S. Hence from the definition of F we have:

$$0 = \operatorname{Tr} \Lambda\Big((\bar\partial s)^* \bar\partial s\Big) = \frac{1}{2n\pi} \operatorname{vol}(G/P) |\bar\partial s|^2,$$

where s denotes the map of vector bundles induced by an H-invariant embedding of P-modules:

$$\begin{array}{ccc} Q & \rightarrowtail & Q_1 \oplus Q \\ \cap & & \cap \\ V/V_1 & & V \end{array}.$$

It remains to show that if $\bar\partial s = 0$ then this embedding is P-equivariant.

LEMMA. *Let $U \dashrightarrow V$ be an H-invariant homomorphism of P-modules, and let s denote (smooth) induced map of vector bundles:*

$$\begin{array}{ccc} s\colon K \times_H U & \longrightarrow & K \times_H V \\ \| & & \| \\ G \times_P U & & G \times_P V. \end{array}$$

Then $\bar\partial s = 0$ if and only if s is P-equivariant.

PROOF. The Killing form induces the isomorphism $u = p^\perp \approx g/p = k/h$, where u (resp. g, p, k, h) denote the Lie algebra of the unipotent radical U of P (resp. Lie algebras of: G, P, K, H). Relative to this isomorphism and to the standard identification of $K \times_H k/h$ with the tangent bundle of $K/H = G/P$ we have:

$$i_v(\bar\partial s) = ([v, s] + i[v, s]), \text{ for all } v \in u,$$

where $i_{(\_)}$ is the contraction operator. Thus the lemma follows.

PROOF OF THEOREM 4.4. Multiplying the equation of Theorem 4.4 on the left by $[r]$ one obtains more symmetric expression;

$$(4.8)' \qquad f(t) = \Big(tF''t^{-1} - (tF''t^{-1})^{\mathrm{tr}}\Big)\{1\} = (tF''t^{-1} - t^{-1}F''^{\mathrm{tr}}t)\{1\}$$

with $F'' = [r]F$, and therefore, without losing generality, we can assume that $[r]$ is the identity matrix (and $F = F''$). For a subset $S \subset \{1, 2, \ldots, n\}$ let $\{1_S\} \in R^n$ denote the vector $\sum_{\iota \in S} e_\iota$, let $\{1_\Delta\} = \{1\} - \{1_S\}$, and $(\_,\_)$ denote the standard euclidean inner product on R^n. Writing $F(t) = tF''t^{-1}$, we have

$$\begin{aligned}
\sum_{\iota \in S} f_\iota(t) &= \big(\{1_S\}, F(t) - F(t)^{\mathrm{tr}}\{1\}\big) \\
&= \big(\{1_S\}, F(t) - F(t)^{\mathrm{tr}}\{1_S\}\big) + \big(\{1_S\}, F(t) - F(t)^{\mathrm{tr}}\{1_\Delta\}\big) \\
&= \big(\{1_S\}, F(t)\{1_\Delta\}\big) - \big(\{1_\Delta\}, F(t)\{1_S\}\big).
\end{aligned} \tag{4.9}$$

Checking from definitions shows that the first term in this expression is always nonnegative, and it is positive providing that S is not a connected component of graph F. Moreover, the second term vanishes if S is a submodule. Summarizing, the image of f is contained in S_F.

To analyze this map in more detail it is convienient to view its domain T/T^+ as an open subset U of the real projective space P^{n-1}. Its boundary ∂U is a sum of hyperplanes $H_\iota = \{[x] \in P^{n-1} \mid x_\iota = 0\}$ and so, it is canonically stratified. The boundary of S_F is a subset of $x \in R^n$ characterized by the following three properties:

i) $\sum_{\iota=1}^n x_\iota = 0$

ii) $\sum_{\iota \in S} x_\iota \geq 0$, for all proper submodules of graph F

iii) at least one equality in ii) is the equality.

In particular, any convergent sequence $\{x(n)\} \subset S_F$ for which there exists a proper submodule S such that $\big(\sum_{\iota \in S} x_\iota(n)\big) \dashrightarrow 0$ has its limit in ∂S_F. To show that f is onto S_F it suffices to show that f is locally invertible, and that for any sequence $\{t(n)\} \subset U$ converging to ∂U, its image $\{f\big(t(n)\big)\}$ either diverges or it converges to ∂S_F. Once this is accomplished, f must be a covering map, and since both U and S_F are contractible, it has to be onto S_F and 1–1. Therefore, to complete the proof it suffices to show the following two facts.

PROPOSITION 4.10. *The function f is locally invertible at every $t \in T/T^+$.*

PROPOSITION 4.11. *Let $\{t(n)\} \subset U$ with $\big(t(n)\big) \dashrightarrow z \in \partial U$. Then either:*

i) there exists i such that $\big|f_\iota\big(t(n)\big)\big| \dashrightarrow \infty$ or

ii) there exists a proper submodule S such that $\Big(\sum_{\iota \in S} f_\iota\big(t(n)\big)\Big) \dashrightarrow 0$.

PROOF OF PROPOSITION 4.11. The point z determines a nontrivial partition of the set of vertices of graph F: $I = \{i \mid z_\iota = 0\}$, $I' = \{i \mid z_\iota \neq 0\}$. Hence every vertex is either a zero or non-zero vertex. The following lemma can be easily proved by induction on the number of zero vertices—we leave the details to the reader.

LEMMA. *For an arbitrary graph with non-trivial partition of the set of vertices either:*

i) there exists a non-zero vertex $\{i\}$ which is not dominated by any zero vertex and dominating a zero vertex $\{k\}$ or

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ii) there exists a submodule S consisting of zero vertices with the property that every vertex outside of S which is dominated by a vertex in S is non-zero.

We can now complete the proof of the proposition. Examining the equation $(4.8)'$ we have:

$$f_i(t) = \sum_{i<j} F_{ij} \frac{t_i}{t_j} - \sum_{i>j} F_{ji} \frac{t_j}{t_i}.$$

If a vertex $\{i\}$ has the property i) then the right sum is bounded uniformly for all n, and since $F_{ik} > 0$, the sum on the left diverges to $+\infty$; hence the alternative i) holds.

Suppose now that S is a submodule of graph F for which ii) of the lemma holds. As we have explained in (4.9), to show that ii) of the proposition holds it suffices to prove that:

$$\Big(\{1_S\}, F\big(t(n)\big)\{1_\Delta\}\Big) \to 0.$$

Examining this quantity gives:

$$\big(\{1_S\}, F(t)\{1_\Delta\}\big) = \sum_{ij \in S\times\Delta} F_{ij} \frac{t_i}{t_j}, \text{ where } \Delta = \{1, 2, \ldots, n\} - S.$$

The condition ii) of the lemma implies that $F_{ij} \neq 0$ only when $\{j\}$ is a non-zero vertex, and the proposition follows.

PROOF OF PROPOSITION 4.10. Since for a diagonal matrix u one has: $\frac{d}{ds}|_{s=0} f\big(u \exp(st)\big) = [t, uFu^{-1} + (uFu^{-1})^{\mathrm{tr}}]$, the proof of the proposition follows from the following lemma.

LEMMA. *Let t denote the Lie algebra of trace-free diagonal matrices, and F be a strictly upper-triangular matrix with non-negative entries. Then:*

i) the symmetric bilinear form on t defined by: $s, t \to (s\{1\}, [t, F + F^{\mathrm{tr}}]\{1\})$ *is non-negative definite.*

ii) It is positive definite if graph F is connected.

PROOF. Since the form depends linearly on F, to prove i) it suffices to consider the case when F has one non-zero entry—we leave this as a simple exercise. The matrix F can be (non-uniquely) written as a sum $F = F' + F''$, where graph F' is minimal, *i.e.*, it has minimal number of edges relative to the property that it has the same connected components as graph F. By i) and the remark above, it suffices to consider F with a minimal graph. We proceed by induction on the number of vertices in graph F. Any minimal graph has a vertex, say $\{1\}$, which is connected to precisely one other vertex, say $\{k\}$. One writes $F = F_1 + ae_{1k}$, $a > 0$, where the graph F_1 has precisely two connected components $\{1\} \bigcup \{2, 3, \ldots, n\}$. Let $h = t_1 - \frac{1}{n-1}\sum_{i=2}^n t_i$. Since for all $i, j \in \{2, 3, \ldots, n\}$:

$$(h\{1\}, [t_i - t_j, F_1 + F_1^{\mathrm{tr}}]\{1\}) = 0$$

$$(h\{1\}, [h, ae_{1k} + ae_{k1}]\{1\}) > 0,$$

hence, the positivity of the form in question follows from positivity of its restriction to the hyperplane $t_1 = 0$, which follows in turn from the inductive hypothesis.

5. **An example.** It is clear that a K-invariant metric on $\mathbf{V}$ with V irreducible is automatically Hermitian-Einstein. Kobayashi proved in [3] that a homogenous vector bundle corresponding to irreducible representation is stable, *i.e.*, it does not decompose as a direct sum of nontrivial stable bundles. The following example shows that in general, a homogenous vector bundle for which the hypothesis of Theorem 1.9 holds may still decompose as a direct sum of subbundles.

EXAMPLE. Consider the extension of line bundles on P^1:

$$O(-1) \rightarrowtail O \oplus O \twoheadrightarrow O(1)$$

given by two sections in $O(1)$, say x_1, x_2. Tensoring this extension with $O(-1)$ and dualizing one obtains the extension:

$$0 \rightarrowtail O(1) \oplus O(1) \twoheadrightarrow O(2). \qquad (**)$$

There is the canonical identification of $H^0\big(O(1)\oplus O(1)\big)$ with $\mathrm{End}(C^2)$ and $\mathrm{SL}_2(C)$ acts on the latter by the adjoint representation. This action induces $\mathrm{SL}_2(C)$-homogenous structure on the extension $(**)$, for which the isotropy representation has a unique one-dimensional submodule which is trivial, so that Theorem 1.9 applies to $O(1)\oplus O(1)$ with such "exotic" homogenous structure.

REMARK. The concept of minimality on which our results heavily depend, clearly depends itself on the choice of filtration $(*)$. However, for a P-module V the set V^U consisting of vectors fixed by the unipotent radical of P is always non-empty (Borel's fixed-point theorem). Therefore, we have a particular filtration defined recursively as: $V_1 = V^U, \ldots, V_i = p^{-1}\big((V/V_{i-1})^U\big)$, where p denotes the projection $V \longrightarrow V/V_{i-1}$. It can be shown that the notion of minimality does not depend on the choice of direct sum decomposition of V_i/V_{i-1}, for each i.

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