CORRESPONDENCE

To the Editor of the *Mathematical Gazette.*

SIR,—We have read with interest, but also with considerable uneasiness, the translation of the German publication DIN 1303 on Vector Notation in the article by G. Windred on that subject in the July number of the *Mathematical Gazette.* While agreeing with Mr. Windred on the importance of achieving a general agreement on the subject of vector notation, we wish to express the hope that the notation for products of vectors proposed in DIN 1303 (sections 5 and 6) will not be generally followed, since it is unsatisfactory in several respects as we shall show, and we would suggest that the standardisation of an unsatisfactory notation might be a worse misfortune than the present lack of uniformity.

In our view, the notation for products of vectors suggested in DIN 1303 is often cramping and inconvenient in practical working, and may be ambiguous, or at least not easy to interpret, in any but the simplest algebraical combinations of vectors, and is not well suited to the natural extension of vector analysis. We wish to express our opinion, based on trial of several different notations, and practical experience, of the superiority of Gibbs' notation * in all these respects. Of the notations mentioned by Mr. Windred, the only two which, as far as our experience goes, are at all widely used now are those ascribed to Gans (in which the scalar and vector products are denoted by enclosing the symbols for the vectors in round and square brackets respectively) and to Gibbs (in which the two products are denoted by a dot and a cross, respectively, between the vector symbols). We feel that the standardisation of a notation differing from either of these, as the notation of DIN 1303 does (though it is related to that of Gans), could only be justified on the ground of definite superiority, and we do not consider that the notation of DIN 1303 satisfies this condition at all.

The use of a particular kind of bracket to indicate a particular kind of product is in practice very inconvenient except for the simplest algebraical expressions. Brackets are already overworked symbols in mathematical notation, and this may already lead to ambiguity apart from any special use for products of vectors. For example, $g(x+y)$ may mean a function of the variable $(x+y)$, or g here may be a coefficient multiplying $(x+y)$, and only the context can decide between the two meanings. The ambiguity is still more pronounced in more elaborate expressions, such as

$$g[f(x+y)+\phi(x-y)],$$

for example.

Also, apart from the use of brackets to enclose the variable of a function, their use, in particular contexts, to indicate particular

* See Gibbs, *Collected Works*, Vol. II, section on Vector Analysis ; or Weatherburn, *Elementary Vector Analysis.*

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kinds of products (or other combinations of the quantities concerned) cramps very severely their use in their purely algebraical sense, of indicating the order in which operations of addition, multiplication, etc., are to be performed. It is a good general principle that in a connected piece of work each symbol should, as far as possible, be used in a single sense throughout , any departure from this principle introduces possibilities of confusion and ambiguity, or at least makes expressions more difficult to read, since if a symbol is used in two senses, the reader has to consider in what sense to read it each time it occurs. If a bracket notation is used for products of vectors, it is often impossible to avoid either violating this principle, perhaps even within the bounds of a single formula, or adopting some other symbol to serve the purpose for which the bracket is no longer available, which is a hardly less unsatisfactory procedure.

Thus to give particular meanings to particular kinds of brackets in particular contexts is cramping in the algebraical manipulation of any but fairly simple expressions, adds to the possibility of confusion and ambiguity, and makes expressions needlessly difficult to read.

For this reason we are glad to see that for a scalar product the use of round brackets has been discarded in the proposals of DIN 1303, but consider the retention of the square brackets for the vector product to be unsatisfactory The representation of a scalar product by the juxtaposition of the two vector symbols without a multiplication sign, as proposed in DIN 1303, seems harmless as far as vectors alone are concerned, but is not convenient for further developments as we shall explain later.

On the other hand, the dot and cross used to indicate the scalar and vector products in Gibbs' notation are not overworked symbols. They are only required as signs of multiplication in numerical, as distinct from algebraical, expressions, and their use in products of vectors is clear, unambiguous, and easy to read, and is convenient for the extension of vector analysis ; and it seems to us that the standardisation of a less convenient notation would be a retrograde step, likely to hinder rather than help the further use and development of vector analysis. We would recommend one small modification of Gibbs' notation, namely the use of $\mathbf{A}\wedge\mathbf{B}$ rather than $\mathbf{A}\times\mathbf{B}$ for the vector product, to avoid the possibility of confusion between the multiplication sign and the letter $\mathbf{x}$ or $\mathbf{X}$ in manuscript.

One feature of a good notation is that it can easily be adapted to extensions and developments of the subject for which it was originally designed, and, as Mr. Windred points out, it may even suggest such developments , and here also Gibbs' notation seems to have considerable advantages over a bracket notation.

The first natural extension of vector algebra is provided by the concept of a tensor (also called a " linear vector operator " or "dyadic ") and the development of an algebra involving such entities,* which is both interesting as a mathematical development, and useful in physical applications. It is perhaps significant as regards the relative values of the two notations from this point of view that, using the dot and cross notation for products of vectors, Gibbs did develop an algebra and calculus of tensors, whereas, so far as we are aware, no comparable development was made by anyone using a bracket notation.

When this extension is taken into account, there are not only the two products of two vectors mentioned by Mr. Windred, but also a third, the complete or " dyadic " product, for which a notation is required. This product is not mentioned in DIN 1303, which seems to have been framed from the point of view of vector algebra and analysis only, without any reference to this further development. As far as we can see, the notation of DIN 1303 cannot be extended consistently and satisfactorily to deal with tensors, as will be shown below, whereas Gibbs' notation, in which the complete product is represented by the juxtaposition of the vector symbols without a multiplication sign, permits this extension consistently, clearly, and without the introduction of any new symbol.

The following are examples of the difficulties of extending the notation of DIN 1303 to tensor algebra and analysis.

(*a*) Following the notation of DIN 1303, in which the scalar product of two vectors is indicated by the juxtaposition of the vector symbols without a multiplication sign, the direct or " scalar " product of two tensors ($T.S$ in Gibbs' notation) would presumably be written TS without a multiplication sign. But then there is no further modification (other than the introduction of a special symbol) for the double scalar product, which in Gibbs' notation is naturally written $T{:}S$.

(*b*) In any notation other than Gibbs' for the dyadic product of two vectors, the elegant form of the identity $(\mathbf{AB}).\mathbf{C} = \mathbf{A}(\mathbf{B}.\mathbf{C})$, and the consequent possibility of writing this vector unambiguously as $\mathbf{AB}.\mathbf{C}$ without brackets, would be lost.

(*c*) The absence of a dot between ∇ and $\mathbf{A}$ in the tensor $\nabla\mathbf{A}$ (in Gibbs' notation) is the counterpart of the absence of a dot between ∇ and ϕ (a scalar) in the vector $\nabla\phi$. This correspondence would be lost if $\nabla\mathbf{A}$ were taken to mean the (formal) scalar product of ∇ and $\mathbf{A}$, *i.e.* div $\mathbf{A}$, as it would do according to the notation of DIN 1303.

These are the reasons for which the representation of the scalar product by the simple juxtaposition of the vector symbols, suggested in DIN 1303, seems unsatisfactory.

(*d*) Using the bracket notation for vector products of tensors and vectors, it is not possible to make the notation show that the tensors which in Gibbs' notation (modified as suggested above) are written $(\mathbf{A}\wedge T)\wedge\mathbf{B}$ and $\mathbf{A}\wedge(T\wedge\mathbf{B})$ are the same.

We would like to emphasize that, as we ourselves originally learnt

* For the benefit of those readers who may be unacquainted with this development of vector algebra, and who have only heard of tensors in the rather formidable context of the mathematical theory of relativity, we may explain that the tensor algebra here referred to is not concerned with that elaborate development, but is much simpler.

the subject of vector algebra and analysis in terms of the bracket notation for products of vectors, our views on the subject of notation for these products are not the result of a mere conservatism, but are the fruit of experience, first of the unsatisfactory nature of the bracket notation, and then of trial and use of others. The notation which we recommend we have tested and used for several years in the course of systematic lecturing on vector algebra and analysis, and on their application to mechanics and other branches of applied mathematics , in this work and in research, we have found it to be entirely satisfactory, not only in developing and expressing general theorems, but also in the investigation and working out of particular problems.

Some readers, familiar with the " suffix notation " for the tensor calculus, as used, for example, in the general theory of relativity, may feel that the devising of a convenient suffix-free notation for vectors and tensors is superfluous. We would only say that, in many problems and investigations for which a suffix notation is unwieldy, much economy of thought and power of manipulation is achieved by a suffix-free notation. The translation from Gibbs' notation to the suffix notation (only rarely necessary) is provided by a few simple rules , for example, a dot is always to be replaced by suffixing the two symbols it separates by two consecutive identical (dummy) suffixes.

D. R. HARTREE.
S. CHAPMAN
E. A. MILNE.

MODELS OF AN ABACUS AN OFFER.

FOUR or five years ago Messrs. Platt & Co., Educational Publishers, of Wigan, under my directions produced a model of the Roman abacus in the British Museum (with an accompanying leaflet), suitable for wall decoration or for performing calculations to illustrate early arithmetic. Dr Rouse has recently become interested in it and the publishers, at his request, have made a reduction in price for members of his association (for the reform of Latin teaching). They have also improved the model so that the counters or beads are very easily manipulated. I thought that this concession having been made should be open to the members of the Mathematical Association, and the publishers have agreed to make the same reduction, viz. 15% from the net price of 30s. The model is hand-made, about 2 ft. by $1\frac{1}{2}$ ft., finished bronze colour to match the original, and framed in dark oak.

R. S. WILLIAMSON

1075. Herself a chemist, she worked with her husband, M. Pierre Curie, a physicist, in the new field, which in its early days called for all the help it could get from both sciences. Now the subject is getting worked out as far as the chemist is concerned, and even the physicist is being subordinated to the modern transcendental mathematician and his strange ways of economising, not to say avoiding, thought.—F Soddy, *The Interpretation of the Atom*, p. 7. [Per Mr. E. V Smith.]