

LOG-CANONICAL THRESHOLDS ON DEL PEZZO SURFACES OF DEGREES ≥ 2

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Abstract. We compute the global log-canonical thresholds (lct) of del Pezzo surfaces of degrees ≥ 2 with du Val singularities.

§1. Introduction

Unless otherwise mentioned, all varieties are assumed to be projective, normal, and defined over \mathbb{C} .

Let X be a variety with at worst log-canonical singularities, and let D be an effective divisor on X . The log-canonical threshold $c_p(X, D)$ of D at a point p in X is defined as

$$c_p(X, D) = \sup\{c \mid \text{the pair } (X, cD) \text{ is log-canonical at the point } p\}.$$

The log-canonical threshold $c(X, D)$ of the divisor D is defined as

$$c(X, D) = \sup\{c \mid \text{the pair } (X, cD) \text{ is log-canonical}\} = \inf_{p \in X} \{c_p(X, D)\}.$$

The log-canonical threshold, like multiplicity, measures how singular a divisor is. It has many amazing properties and has important applications to various areas such as birational geometry and Kähler geometry.

The following theorem is one of the motivations of this article.

THEOREM 1.1. *Suppose that X is an n -dimensional Fano orbifold. If there is a positive real number ϵ such that, for every effective \mathbb{Q} -divisor D numerically equivalent to $-K_X$, the pair $(X, (n + \epsilon)/(n + 1)D)$ is Kawamata log-terminal, then X has a Kähler-Einstein metric.*

Proof. See [4, Theorem 1.17] and [6, page 549]. □

Received May 11, 2009. Revised November 16, 2009. Accepted February 10, 2010.

2000 Mathematics Subject Classification. Primary 14J26, 14J17.

The authors' work supported by Korea Research Foundation grant KRF-2008-313-C00024.

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This means that it is worthwhile for us to define the following numerical invariants.

DEFINITION 1.2. Let X be a Fano variety with at worst log-terminal singularities. The m th global log-canonical threshold of X is defined by the number

$$\text{lct}_m(X) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the pair } (X, \frac{\lambda}{m}D) \text{ is log-canonical} \\ \text{for any effective divisor } D \in |-mK_X| \end{array} \right. \right\}.$$

The global log-canonical threshold is defined by $\text{lct}(X) = \inf \{ \text{lct}_m(X) \mid m \in \mathbb{N} \}$. Here, we do not define the m th global log-canonical threshold of X if the linear system $|-mK_X|$ is empty.

We can see that $\text{lct}(X)$ is the supremum of the values c such that the pair (X, cD) is log-canonical for every effective \mathbb{Q} -divisor D numerically equivalent to $-K_X$. Using the global log-canonical threshold, Theorem 1.1 can be read as meaning that the Fano manifold X admits a Kähler-Einstein metric if

$$\text{lct}(X) > \frac{\dim(X)}{\dim(X) + 1}.$$

Pukhlikov [14] also shows that the global log-canonical threshold plays an important role in rationality problems.

In this article, we study the global log-canonical thresholds of del Pezzo surfaces. The global log-canonical thresholds of smooth del Pezzo surfaces have been computed already. It turns out that they coincide with the first global log-canonical thresholds.

THEOREM 1.3. *Let X be a smooth del Pezzo surface. Then*

$$\text{lct}(X) = \text{lct}_1(X)$$

$$= \begin{cases} 1/3 & \text{when } X \cong \mathbb{F}_1 \text{ or } K_X^2 \in \{7, 9\}, \\ 1/2 & \text{when } X \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_X^2 \in \{5, 6\}, \\ 2/3 & \text{when } K_X^2 = 4, \\ 2/3 & \text{when } X \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point,} \\ 3/4 & \text{when } X \text{ is a cubic in } \mathbb{P}^3 \text{ without Eckardt points,} \\ 3/4 & \text{when } K_X^2 = 2 \text{ and } |-K_X| \text{ has a tacnodal curve,} \\ 5/6 & \text{when } K_X^2 = 2 \text{ and } |-K_X| \text{ has no tacnodal curves,} \\ 5/6 & \text{when } K_X^2 = 1 \text{ and } |-K_X| \text{ has a cuspidal curve,} \\ 1 & \text{when } K_X^2 = 1 \text{ and } |-K_X| \text{ has no cuspidal curves.} \end{cases}$$

Proof. See [2, Theorem 1.7] and [10, Corollary 3.3]. \square

The global log-canonical thresholds of del Pezzo surfaces with du Val singularities have been studied in [2], [3], and [9]. In [3], the global log-canonical thresholds of cubic surfaces with du Val singularities have been computed. Kosta [9] computes the global log-canonical thresholds of del Pezzo surfaces of degree 1 with du Val singularities and del Pezzo surfaces of Picard rank 1 with du Val singularities.

In this paper, we compute the global log-canonical thresholds of all the del Pezzo surfaces of degree ≥ 2 with du Val singularities. Even though the global log-canonical thresholds of all cubic surfaces with du Val singularities and del Pezzo surfaces of Picard rank 1 with du Val singularities have already been computed, we also compute them again here, since this article provides a simpler method.

Throughout, we call an algebraic surface S with ample anticanonical divisor a *del Pezzo surface of degree d* if it has at worst du Val singularities and if the self-intersection number of the anticanonical divisor is d . Also, we call a smooth algebraic surface \tilde{S} with nef and big anticanonical divisor a *weak del Pezzo surface*.

For the global log-canonical thresholds, we need to distinguish some singularity types of del Pezzo surfaces of degree 2 with the same dual graphs. We distinguish A_5 singularities into two types: one has a -1 curve intersecting the -2 curve corresponding to the vertex v in the dual graph of A_5 such that $A_5 - v = 2A_2$ on the minimal resolution of the del Pezzo surface, and the other does not. In the former case, the type of singularities are denoted by A'_5 , and in the latter case, by A''_5 . For singularity types A_5 and $A_5 + A_1$ on del Pezzo surfaces of degree 2, there are two types for each (see [15, page 590]): one is for A'_5 , and the other is for A''_5 . For singularity type $A_5 + A_2$ on del Pezzo surfaces of degree 2, there is only one type (see [15, page 590]). The singularity A_5 in this type is A'_5 .

Also, there are two types of singularities on del Pezzo surfaces of degree 2 with the dual graph $3A_1$ (resp., $4A_1$; see [15, page 590]): one has a -1 curve on the del Pezzo surface which passes through three A_1 singular points (denoted by $(3A_1)'$; resp., $(4A_1)'$), and the other does not (denoted by $(3A_1)''$; resp., $(4A_1)''$). For singularity type $A_2 + 3A_1$ on del Pezzo surfaces of degree 2, there is only one type (see [15, page 590]). The singularities $3A_1$ in this type are $(3A_1)'$.

The first global log-canonical threshold may be a cornerstone to get the global log-canonical threshold. For a del Pezzo surface S , the first global log-canonical threshold $\text{lct}_1(S)$ is meaningful by itself. It has a nice application to birational maps between del Pezzo fibrations (see [10] or [11]). In [11] and [12], the first global log-canonical thresholds of all del Pezzo surfaces have been computed. For convenience, we state all the first global log-canonical thresholds of del Pezzo surfaces of degrees ≥ 2 .

THEOREM 1.4. *Let S_d be a del Pezzo surface of degree d , and let Σ_d be the set of singular points in S_d . Suppose that $\Sigma_d \neq \emptyset$. Then*

$$\text{lct}_1(S_2) = \begin{cases} 1/6 & \text{if } \Sigma_2 = \{E_7\}; \\ 1/4 & \text{if } \Sigma_2 = \{E_6\}, \Sigma_2 \supseteq \{D_6\}; \\ 1/3 & \text{if } \Sigma_2 \supseteq \{D_5\}, \{(A_5)'\}; \\ 1/2 & \text{if } \Sigma_2 \supseteq \{(3A_1)'\}, \{(4A_1)'\}, \{5A_1\}, \{A_3\}, \{A_4\}, \{(A_5)''\}; \\ & \{A_6\}, \{A_7\}, \{D_4\}; \\ 2/3 & \text{otherwise.} \end{cases}$$

$$\text{lct}_1(S_3) = \begin{cases} 1/6 & \text{if } \Sigma_3 = \{E_6\}; \\ 1/4 & \text{if } \Sigma_3 \supseteq \{A_5\}, \Sigma_3 = \{D_5\}; \\ 1/3 & \text{if } \Sigma_3 \supseteq \{A_4\}, \{2A_2\}, \Sigma_3 = \{D_4\}; \\ 2/3 & \text{if } \Sigma_3 = \{A_1\}; \\ 1/2 & \text{otherwise.} \end{cases}$$

$$\text{lct}_1(S_4) = \begin{cases} 1/6 & \text{if } \Sigma_4 = \{D_5\}; \\ 1/4 & \text{if } \Sigma_4 \supseteq \{A_1 + A_3\}, \Sigma_4 = \{A_4\}, \Sigma_4 = \{D_4\}; \\ 1/3 & \text{if } \Sigma_4 = \{A_3\}, \Sigma_4 \supseteq \{A_1 + A_2\}; \\ 1/2 & \text{otherwise.} \end{cases}$$

$$\text{lct}_1(S_5) = \begin{cases} 1/6 & \text{if } \Sigma_5 = \{A_4\}; \\ 1/4 & \text{if } \Sigma_5 = \{A_3\}, \Sigma_5 = \{A_1 + A_2\}; \\ 1/3 & \text{if } \Sigma_5 = \{A_2\}, \{2A_1\}; \\ 1/2 & \text{if } \Sigma_5 = \{A_1\}. \end{cases}$$

$$\text{lct}_1(S_6) = \begin{cases} 1/6 & \text{if } \Sigma_6 = \{A_1 + A_2\}; \\ 1/4 & \text{if } \Sigma_6 = \{A_2\}, \Sigma_6 = \{2A_1\}; \\ 1/3 & \text{if } \Sigma_6 = \{A_1\}. \end{cases}$$

$$\text{lct}_1(S_7) = 1/4 \quad \text{if } \Sigma_7 = \{A_1\}.$$

In this article, we prove the following two theorems that complete the results of [2] and [9].

THEOREM 1.5. *Let S be a del Pezzo surface of degree ≥ 3 . Then $\text{lct}_1(S) = \text{lct}(S)$.*

THEOREM 1.6. *Let S be a del Pezzo surface of degree 2. Then*

$$\text{lct}(S_2) = \begin{cases} 1/6 & \text{if } \Sigma_2 = \{E_7\}; \\ 1/4 & \text{if } \Sigma_2 = \{E_6\}, \Sigma_2 \supseteq \{D_6\}; \\ 1/3 & \text{if } \Sigma_2 \supseteq \{D_5\}, \{(A_5)'\}, \{A_7\}; \\ 2/5 & \text{if } \Sigma_2 = \{A_6\}; \\ 1/2 & \text{if } \Sigma_2 \supseteq \{(3A_1)'\}, \{(4A_1)'\}, \{5A_1\}, \{A_3\}, \{A_4\}, \{(A_5)''\}, \\ & \{A_6\}, \{D_4\}; \\ 2/3 & \text{otherwise.} \end{cases}$$

If the singularity type of S is neither A_7 nor A_6 , then $\text{lct}(S) = \text{lct}_1(S)$.

From the proof of Theorem 1.6, we can notice that $\text{lct}(S) = \text{lct}_2(S) \neq \text{lct}_1(S)$ if the del Pezzo surface S of degree 2 has either an A_7 or A_6 singular point.

§2. Preliminaries

For the rest of this article, a del Pezzo surface will always be denoted by S , and its minimal resolution will be denoted by $\pi : \tilde{S} \rightarrow S$. The surface \tilde{S} is a weak del Pezzo surface. For a constant λ and an effective divisor C on S , we have

$$\pi^*(K_S + \lambda C) = K_{\tilde{S}} + \lambda \pi^*(C).$$

The pair $(S, \lambda C)$ is log-canonical if and only if the pair $(\tilde{S}, \lambda \pi^*(C))$ is log-canonical. Since every effective \mathbb{Q} -divisor numerically equivalent to $-K_{\tilde{S}}$ (resp., $-K_S$) is the pullback (resp., pushforward) of an effective \mathbb{Q} -divisor numerically equivalent to $-K_S$ (resp., $-K_{\tilde{S}}$) by the birational morphism π , we have $\text{lct}(S) = \text{lct}(\tilde{S})$. Thus, it is sufficient to consider effective \mathbb{Q} -divisors numerically equivalent to $-K_{\tilde{S}}$ on \tilde{S} to compute $\text{lct}(S)$.

LEMMA 2.1. *Let D_1 and D_2 be effective \mathbb{Q} -divisors on \tilde{S} with $D_1 \equiv D_2$. Suppose that the pair (\tilde{S}, D_1) is not log-canonical at a point $p \in \tilde{S}$, while*

the pair (\tilde{S}, D_2) is log-canonical at the point p . Then there is an effective \mathbb{Q} -divisor D on \tilde{S} such that

- $D \equiv D_1$,
- at least one irreducible component of D_2 is not contained in the support of D ,
- the pair (\tilde{S}, D) is not log-canonical at the point p .

Moreover, if $D_1 \equiv -\lambda K_{\tilde{S}}$ for some positive number λ and if the point p lies on a -2 curve, then the support of D must contain the support of every -2 curve over the singular point $\pi(p)$.

Proof. Write $D_2 = \sum_{i=1}^r b_i C_i$, where b_i are positive rational numbers and C_i are distinct irreducible and reduced divisors. Also, we write $D_1 = \Delta + \sum_{i=1}^r e_i C_i$, where e_i are nonnegative rational numbers and Δ is an effective \mathbb{Q} -divisor whose support contains no C_i .

Let

$$\alpha = \min \left\{ \frac{e_i}{b_i} \mid i = 1, 2, \dots, r \right\}.$$

Then the nonnegative rational number α is less than 1 since $D_1 \equiv D_2$. Put

$$\begin{aligned} D &= \frac{1}{1-\alpha} D_1 - \frac{\alpha}{1-\alpha} D_2 \\ &= \frac{1}{1-\alpha} \Delta + \sum_{i=1}^r \left(\frac{e_i - \alpha b_i}{1-\alpha} \right) C_i. \end{aligned}$$

It is easy to see that the divisor D satisfies the first two conditions. If the pair (\tilde{S}, D) is log-canonical at the point p , then the pair $(\tilde{S}, D_1) = (\tilde{S}, (1-\alpha)D + \alpha D_2)$ must be log-canonical at the point p . Therefore, the divisor D also satisfies the last condition. For the last statement, do the same with the divisors $\pi(D_1)$ and $\pi(D_2)$ on S and then take the pullback of the obtained divisor by the birational morphism π . □

LEMMA 2.2. *Let X be a smooth surface, and let B be an effective \mathbb{Q} -divisor on X . If the pair (X, B) is not log-canonical at a point $p \in X$, then $\text{mult}_p(B) > 1$. For a smooth curve C on X and a nonnegative number $m \leq 1$, if the pair $(X, mC + B)$ is not log-canonical at a point $p \in C$, then $C \cdot B > 1$.*

Proof. This immediately follows from [8, Theorem 17.7]. □

LEMMA 2.3. *Let \tilde{S} be a weak del Pezzo surface, and let D be an effective \mathbb{Q} -divisor numerically equivalent to $-K_{\tilde{S}}$. For a positive number $\lambda < 1$, the locus where the pair $(\tilde{S}, \lambda D)$ is not Kawamata log-terminal is connected.*

Proof. See [8, Theorem 17.4]. □

COROLLARY 2.4. *Let C be an effective \mathbb{Q} -divisor on \mathbb{P}^2 with $\deg C < 2$. If the pair (\mathbb{P}^2, C) is not Kawamata log-terminal, then it is not Kawamata log-terminal along a curve.*

Proof. Suppose that the locus of non-Kawamata log-terminal singularities of the pair (\mathbb{P}^2, C) is 0-dimensional. It follows from Lemma 2.3 that the locus consists of a single point p . Let L be a general line on \mathbb{P}^2 . Put

$$D = \frac{3}{1 + \deg(C)}(C + L), \quad \lambda = \frac{1 + \deg(C)}{3}.$$

The divisor D is an effective \mathbb{Q} -divisor numerically equivalent to $-K_{\mathbb{P}^2}$, and $\lambda < 1$. However, the locus of non-Kawamata log-terminal singularities of the pair $(\mathbb{P}^2, \lambda D)$ consists of the point p and the line L . Since these two components are disconnected, it is a contradiction. □

The following variant of [4, Lemma 4.9] will be useful here. In fact, the proof of [4, Lemma 4.9] is also based on Lemma 2.3.

COROLLARY 2.5. *Let S be a smooth del Pezzo surface of degree 7. Let $L_1, L_2,$ and L_3 be the three -1 curves on S with $L_1 \cdot L_3 = 0$. For an effective \mathbb{Q} -divisor D on S numerically equivalent to $-K_S$ and with $\lambda \leq 1/2$, if the pair $(S, \lambda D)$ is not log-canonical at some point p , then it is not log-canonical along the curve L_2 .*

Proof. For a sufficiently small positive real number ϵ , the pair $(S, (\lambda - \epsilon)D)$ is not log-canonical at the point p . Then [4, Lemma 4.9] implies that the pair $(S, (\lambda - \epsilon)D)$ is not Kawamata log-terminal along the curve L_2 . Therefore, the pair $(S, \lambda D)$ is not log-canonical along the curve L_2 . □

LEMMA 2.6. *Let S be a del Pezzo surface of degree ≥ 2 . For an effective \mathbb{Q} -divisor D numerically equivalent to $-K_S$ and for a positive number λ with $\lambda K_S^2 \leq 1$, the pair $(S, \lambda D)$ is log-canonical at every smooth point.*

Proof. Suppose that the pair $(S, \lambda D)$ is not log-canonical at some smooth point p . Then $\lambda \text{mult}_p D > 1$. We can choose an irreducible curve C in the anticanonical linear system $|-K_S|$ such that it passes through the point p but its support is not contained in the support of D . However,

$$\lambda K_S^2 = \lambda D \cdot C \geq \lambda \text{mult}_p(D) > 1.$$

This is a contradiction. □

The following lemma is the main tool for this article.

LEMMA 2.7. *Let \tilde{S} be a weak del Pezzo surface. Suppose that the surface \tilde{S} has mutually disjoint -1 curves L_1, \dots, L_t and M_1, \dots, M_s . Let \tilde{S}_L be the smooth surface obtained by contracting all the curves L_i , and let \tilde{S}_M be the smooth surface obtained by contracting all the curves M_j . Then*

$$\text{lct}(\tilde{S}) \geq \min\{\text{lct}(\tilde{S}_L), \text{lct}(\tilde{S}_M)\}.$$

In particular, if $\text{lct}_1(\tilde{S}) \leq \min\{\text{lct}(\tilde{S}_L), \text{lct}(\tilde{S}_M)\}$, then $\text{lct}_1(\tilde{S}) = \text{lct}(\tilde{S})$.

Proof. Let $\pi_L : \tilde{S} \rightarrow \tilde{S}_L$ be the contraction of the -1 curves L_1, \dots, L_t , and let $\pi_M : \tilde{S} \rightarrow \tilde{S}_M$ be the contraction of the -1 curves M_1, \dots, M_s . For every effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$, the effective divisors $\pi_L(D)$ and $\pi_M(D)$ are effective \mathbb{Q} -divisors numerically equivalent to $-K_{\tilde{S}_L}$ and $-K_{\tilde{S}_M}$, respectively.

For an arbitrary positive number $\lambda \leq \min\{\text{lct}(\tilde{S}_L), \text{lct}(\tilde{S}_M)\}$ and for an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$, the pairs $(\tilde{S}_L, \lambda\pi_L(D))$ and $(\tilde{S}_M, \lambda\pi_M(D))$ are log-canonical. The birational morphism π_L is an isomorphism in the outside of $\bigcup_{i=1}^t L_i$. The birational morphism π_M is an isomorphism in the outside of $\bigcup_{i=1}^s M_i$. Since $(\bigcup_{i=1}^t L_i) \cap (\bigcup_{i=1}^s M_i) = \emptyset$, the pair $(\tilde{S}, \lambda D)$ is log-canonical. This implies the first inequality.

The second statement is obvious since $\text{lct}_1(\tilde{S}) \geq \text{lct}(\tilde{S})$. □

The proofs of Theorems 1.5 and 1.6 are inductive. If we compute all the global log-canonical thresholds of del Pezzo surfaces of degrees $> d$, we can easily compute the global log-canonical thresholds of almost all del Pezzo surfaces of degree d by using Lemma 2.7.

To use Lemma 2.7, we need to find some -1 curves on weak del Pezzo surfaces. The configurations of -2 curves and -1 curves on weak del Pezzo surfaces of degrees ≥ 4 can be found in [5].

For weak del Pezzo surfaces of degrees ≤ 3 , we refer the reader to [15, Table], which completely classifies subsystems of the root systems E_6 and E_7 up to actions of their Weyl groups. Furthermore, [15] shows that singularity types of del Pezzo surfaces of degree 2 and classes of subsystems of the root system E_7 (except for the subsystem of type $7A_1$) are in one-to-one correspondence. It is also well known that singularity types of del Pezzo surfaces of degree 3 and classes of subsystems of the root system E_6 are in one-to-one correspondence (see [1] or [13]). Since these correspondences preserve the intersection forms for the Picard groups of weak del Pezzo

surfaces of del Pezzo surfaces and for the root subsystems, we can conclude from [7, théorème III.2 and corollaire] that a given singularity type has a unique configuration of -1 curves and -2 curves. (The configurations of -1 curves and -2 curves in [5] have been obtained by the same method.) Consequently, for a given singularity type of del Pezzo surfaces of degree d in [15, Table] except $7A_1$ of E_7 , we find one weak del Pezzo surface of degree d whose corresponding singular del Pezzo surface has the given singularity type. This weak del Pezzo surface gives us the configuration of -1 curves and -2 curves for the given singularity type since every del Pezzo surface with the same singularity type has the same configuration of -1 curves and -2 curves on its weak del Pezzo surface, as explained above. These configurations are usually complicated since they may have too many -1 curves. Fortunately, to prove Theorems 1.5 and 1.6, we do not have to know the complete configuration of -1 curves on a given weak del Pezzo surface; instead, we need information only on appropriate -1 curves on a given weak del Pezzo surface that make Lemma 2.7 work. Such -1 curves can be found basically by using [12]. For the reader's convenience, in the appendix we list configurations of -2 curves and appropriate -1 curves on weak del Pezzo surfaces of degrees 2 and 3 that make Lemma 2.7 applicable.

§3. Proof of Theorem 1.5

Throughout this article, a -1 curve is denoted by \circ , and a -2 curve is denoted by \bullet in every dual graph.

PROPOSITION 3.1. *The global log-canonical threshold of Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$, is $1/(n+2)$.*

Proof. Let C be the irreducible curve on \mathbb{F}_n with $C^2 = -n$. Let L be an irreducible curve with $L^2 = 0$. Then $-K_{\mathbb{F}_n} \equiv 2C + (n+2)L$. The pair $(\mathbb{F}_n, (2/(n+2))C + L)$ is log-canonical.

Suppose that $\text{let}(\mathbb{F}_n) < 1/(n+2)$. Then there is an effective \mathbb{Q} -divisor D on \mathbb{F}_n numerically equivalent to $-K_{\mathbb{F}_n}$ such that the pair $(\mathbb{F}_n, (1/(n+2))D)$ is not log-canonical at some point $p \in \mathbb{F}_n$. We may assume that L passes through the point p . If L is not contained in the support of D , then

$$2 = L \cdot D \geq \text{mult}_p(D) > (n+2)$$

by Lemma 2.2. This is a contradiction. Therefore, the curve L must be contained in the support of D . On the other hand, we may assume that

the curve C is not contained in the support of D by Lemma 2.1. Write $D = aL + \Omega$, where Ω is an effective \mathbb{Q} -divisor whose support does not contain the curve L . Then $2 - n = D \cdot C = a + C \cdot \Omega \geq a$. If $n \geq 2$, this is already a contradiction. If $n = 1$ or 0 , then we obtain an absurd inequality $n + 2 < (D - aL) \cdot L = 2$ from Lemma 2.2, since $(\mathbb{F}_n, L + (1/(n + 2))\Omega)$ is not log-canonical at the point p . \square

COROLLARY 3.2. *The global log-canonical threshold of a singular del Pezzo surface of degree 8 is $1/4$.*

Proof. Since the minimal resolution of the surface is the surface \mathbb{F}_2 , the statement immediately follows from Proposition 3.1. \square

PROPOSITION 3.3. *Let S be a singular del Pezzo surface of degree 7. Then $\text{lct}(S) = 1/4$.*

Proof. The surface has one singular point that is of type A_1 . Since $\text{lct}_1(S) = 1/4$, we have $\text{lct}(S) \leq 1/4$. The minimal resolution \tilde{S} of S contains two -1 curves L_1 and L_2 and one -2 curve E , with $L_1 \cdot L_2 = 1$, $L_1 \cdot E = 1$, and $L_2 \cdot E = 0$ (see [5, Proposition 8.1]).

Suppose that $\text{lct}(S) < 1/4$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (1/4)D)$ is not log-canonical at some point $p \in \tilde{S}$.

By contracting the curve L_1 , we obtain a birational morphism of \tilde{S} to \mathbb{F}_1 . On the other hand, by contracting the curve L_2 , we obtain a birational morphism of \tilde{S} to \mathbb{F}_2 . Since $\text{lct}(\mathbb{F}_1) = 1/3$ and $\text{lct}(\mathbb{F}_2) = 1/4$, the point p must be the intersection point of L_1 and L_2 . Furthermore, the multiplicity of D along the curve L_2 must be at most 3.

Write $D = aL_2 + \Omega$, where Ω is an effective \mathbb{Q} -divisor whose support does not contain the curve L_2 . Since $a \leq 3$, the pair $(\tilde{S}, L_2 + \Omega/4)$ is not log-canonical at the point p . Therefore, $(\Omega/4) \cdot L_2 = ((D - aL_2)/4) \cdot L_2 = ((1 + a)/4) > 1$ by Lemma 2.2, and hence $a > 3$. This is a contradiction. \square

PROPOSITION 3.4. *Let S be a singular del Pezzo surface of degree 6. Then $\text{lct}(S) = \text{lct}_1(S)$.*

Proof. Let \tilde{S} be the minimal resolution of S . Unless the singularity type of S is $A_1 + A_2$, there are two disjoint -1 curves L_1 and L_2 on \tilde{S} that intersect a -2 curve (see [5, Proposition 8.3]). By contracting L_1 , we get a weak del Pezzo surface \tilde{S}' of degree 7 with $\text{lct}(\tilde{S}') \geq \text{lct}_1(S)$. By contracting

L_2 , we get a weak del Pezzo surface \tilde{S}'' of degree 7 with $\text{lct}(\tilde{S}'') \geq \text{lct}_1(S)$. Therefore, $\text{lct}(S) = \text{lct}_1(S)$ by Lemma 2.7.

Now suppose that the singularity type of S is $A_1 + A_2$. Then we have one -1 curve L on \tilde{S} . It intersects the -2 curve F over the singular point of type A_1 and the chain of two -2 curves $E_1 + E_2$ over the singular point of type A_2 . We may assume that L intersects E_1 but not E_2 .

Suppose that $\text{lct}(S) < 1/6$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (1/6)D)$ is not log-canonical at some point $p \in \tilde{S}$. By contracting the -1 curve L , we can see that the point p must belong to the curve L since the global log-canonical threshold of a del Pezzo surface of degree 7 is at least $1/4$. Since $6L + 2E_2 + 4E_1 + 3F \sim -K_{\tilde{S}}$ (see [12, Proposition 2.1]), we may assume that the curve L is not contained in the support of D by Lemma 2.1. Then

$$1 = D \cdot L \geq \text{mult}_p(D).$$

This is a contradiction. □

LEMMA 3.5. *Let S be a singular del Pezzo surface of degree 5. Then $\text{lct}(S) = \text{lct}_1(S)$.*

Proof. Let \tilde{S} be the minimal resolution of S . Unless the singularity type of S is A_4 , there are two disjoint -1 curves L_1 and L_2 on \tilde{S} for which we can apply Lemma 2.7 (see [5, Proposition 8.5]) to show that $\text{lct}(S) = \text{lct}_1(S)$.

Now suppose that the singularity type of S is A_4 . Then we have one -1 curve L on \tilde{S} . Let $E_i, i = 1, 2, 3, 4$, be the -2 curves over the singular point such that $E_1 \cdot E_2 = E_2 \cdot E_3 = E_3 \cdot E_4 = 1$ and $E_2 \cdot L = 1$.

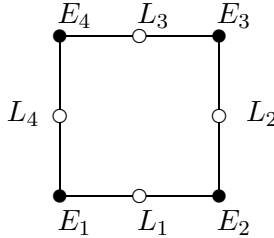
Suppose that $\text{lct}(S) < 1/6$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (1/6)D)$ is not log-canonical at some point $p \in \tilde{S}$. The contraction of L shows that the point p must lie on the curve L . Since $5L + 3E_1 + 6E_2 + 4E_3 + 2E_4 \sim -K_{\tilde{S}}$ (see [12, Proposition 2.1]), we may assume that the curve L is not contained in the support of D . Then

$$1 = D \cdot L \geq \text{mult}_p(D).$$

This is a contradiction. □

LEMMA 3.6. *Let S be a singular del Pezzo surface of degree 4 with singularity type $4A_1$. Then $\text{lct}(S) = \text{lct}_1(S) = 1/2$.*

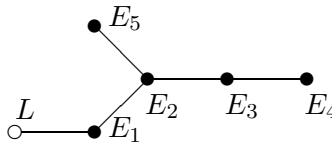
Proof. Let \tilde{S} be the minimal resolution of S . The configuration of -1 curves and -2 curves on \tilde{S} is as follows (see [5, Proposition 6.1]):



Suppose that $\text{lct}(S) < 1/2$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (1/2)D)$ is not log-canonical at some point $p \in \tilde{S}$. By contracting all the -1 curves L_i to $\mathbb{P}^1 \times \mathbb{P}^1$, we see that the point p must be contained in a -1 curve. We may assume that the point p belongs to the curve L_1 . Contracting the -1 curves L_2 and L_4 and then E_4 to a smooth del Pezzo surface of degree 7, we see that the pair $(\tilde{S}, (1/2)D)$ is not log-canonical along the curve E_2 by Corollary 2.5. This is a contradiction since the contraction of all the -1 curves L_i to $\mathbb{P}^1 \times \mathbb{P}^1$ shows that the pair $(\tilde{S}, (1/2)D)$ is log-canonical at a generic point of E_2 . \square

LEMMA 3.7. *Let S be a singular del Pezzo surface of degree 4 with singularity type D_5 . Then $\text{lct}(S) = \text{lct}_1(S) = 1/6$.*

Proof. Let \tilde{S} be the minimal resolution of S . The configuration of -1 curves and -2 curves on \tilde{S} is as follows (see [5, Proposition 6.1]):



Suppose that $\text{lct}(S) < 1/6$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (1/6)D)$ is not log-canonical at some point $p \in \tilde{S}$. The contraction of L shows that the point p must lie on the curve L . Since $4L + 5E_1 + 6E_2 + 4E_3 + 2E_4 + 3E_5 \sim -K_{\tilde{S}}$ (see [12, Proposition 2.1]), we may assume that the curve L is not contained in the support of D by Lemma 2.1. Then

$$1 = D \cdot L \geq \text{mult}_p(D).$$

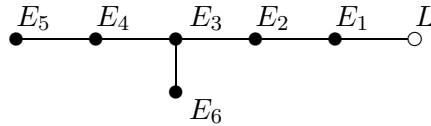
This is a contradiction. □

PROPOSITION 3.8. *Let S be a singular del Pezzo surface of degree 4. Then $\text{lct}(S) = \text{lct}_1(S)$.*

Proof. Unless the singular type of S is $4A_1$ or D_5 , there are two disjoint -1 curves L_1 and L_2 on the minimal resolution of S for which we can apply Lemma 2.7 (see [5, Proposition 6.1]). Therefore, $\text{lct}(S) = \text{lct}_1(S)$. □

LEMMA 3.9. *Let S be a singular del Pezzo surface of degree 3 with singularity type E_6 . Then $\text{lct}(S) = \text{lct}_1(S) = 1/6$.*

Proof. Let \tilde{S} be the minimal resolution of S . From [12, Proposition 2.1] we obtain the configuration of all the -2 curves and some -1 curves on \tilde{S} as follows:

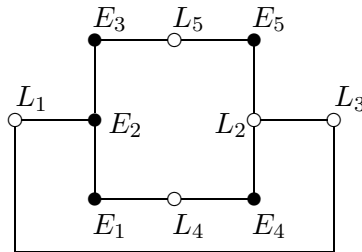


In fact, the curve L is the only -1 curve on \tilde{S} . However, we do not need this fact for our proof.

Suppose that $\text{lct}(S) < 1/6$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (1/6)D)$ is not log-canonical at some point $p \in \tilde{S}$. Since $2L + 4E_1 + 5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E_6 \sim -K_{\tilde{S}}$ (see [12, Proposition 2.1]), we may assume that the curve L is not contained in the support of D by Lemma 2.1. The same argument as in the proof of Lemma 3.7 gives us a contradiction. □

LEMMA 3.10. *Let S be a singular del Pezzo surface of degree 3 with singularity type $A_3 + 2A_1$. Then $\text{lct}(S) = \text{lct}_1(S) = 1/2$.*

Proof. Let \tilde{S} be the minimal resolution of S . The configuration of -1 curves and -2 curves on \tilde{S} is as follows (see the appendix, Table 2):



Suppose that $\text{lct}(S) < 1/2$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (1/2)D)$ is not log-canonical at some point $p \in \tilde{S}$. By contracting the -1 curve L_1 , we see that the point p must be contained in L_1 . Contracting the -1 curves L_4 and L_5 and then E_1 and E_3 to a smooth del Pezzo surface of degree 7, we see that the pair $(\tilde{S}, (1/2)D)$ is not log-canonical along the curve L_3 by Corollary 2.5. This is a contradiction. \square

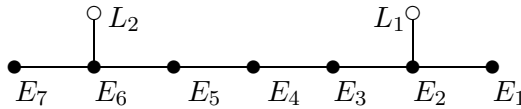
PROPOSITION 3.11. *Let S be a singular del Pezzo surface of degree 3. Then $\text{lct}(S) = \text{lct}_1(S)$.*

Proof. Unless the singularity type of S is $A_3 + 2A_1$ or E_6 , there are two disjoint -1 curves L_1 and L_2 on the minimal resolution of S for which we can apply Lemma 2.7 (see the appendix, Table 2). Therefore, $\text{lct}(S) = \text{lct}_1(S)$. \square

§4. Proof of Theorem 1.6

PROPOSITION 4.1. *Let S be a singular del Pezzo surface of degree 2 with singularity type A_7 . Then $\text{lct}(S) = 1/3$.*

Proof. Let $\pi : \tilde{S} \rightarrow S$ be the minimal resolution. From [12, Proposition 2.12] we obtain the configuration of all the -2 curves and some -1 curves on \tilde{S} as follows:



In fact, the curves L_1 and L_2 are the only -1 curves on \tilde{S} .

Since the Picard group of S is \mathbb{Z} , we can easily check that

$$4L_1 + 3E_1 + 6E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7 \equiv -2K_{\tilde{S}},$$

$$4L_2 + E_1 + 2E_2 + 3E_3 + 4E_4 + 5E_5 + 6E_6 + 3E_7 \equiv -2K_{\tilde{S}}.$$

Therefore, $\text{lct} S \leq 1/3$.

Suppose that $\text{lct}(S) < 1/3$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (1/3)D)$ is not log-canonical at some point $p \in \tilde{S}$. By Lemma 2.1, we may assume that neither the curve L_1 nor the curve L_2 is contained in the support of D . Write $D = aE_2 + bE_6 + \Omega$, where Ω is an effective \mathbb{Q} -divisor whose support does not contain the curves E_2, E_6 .

Lemma 2.6 shows that the pair $(\tilde{S}, (1/3)D)$ is log-canonical in the outside of the -2 curves. By contracting L_1, L_2 and then E_2, E_6 to a weak del Pezzo surface of degree 6 with only one -2 curve, we can see that the pair $(\tilde{S}, (1/3)D)$ is log-canonical in the outside of E_2 and E_6 since the global log-canonical threshold of a weak del Pezzo surface of degree 6 with only one -2 curve is $1/3$. We may assume that the point p is contained in the curve E_2 . Since L_1 is not contained in the support of D , we have

$$1 = D \cdot L_1 \geq aE_2 \cdot L_1 = a.$$

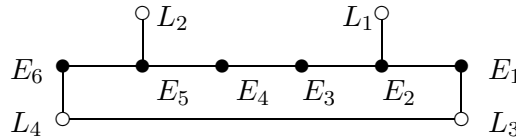
However, the pair $(\tilde{S}, E_2 + (b/3)E_6 + (1/3)\Omega)$ is not log-canonical at the point p , and hence

$$3 < (D - aE_2) \cdot E_2 = 2a$$

by Lemma 2.2. This is a contradiction. □

PROPOSITION 4.2. *Let S be a singular del Pezzo surface of degree 2 with singularity type A_6 . Then $\text{lct}(S) = 2/5$.*

Proof. Let $\pi : \tilde{S} \rightarrow S$ be the minimal resolution. The configuration of -1 curves and -2 curves on \tilde{S} is as follows (see the appendix, Table 1):



The Picard group of S is $\mathbb{Z} \oplus \mathbb{Z}$, and the lines $\pi(L_1)$ and $\pi(L_3)$ are linearly independent in the Picard group of S . Therefore, there must be two rational numbers m and n such that $m\pi(L_1) + n\pi(L_3) \equiv -K_S$. We can check that $m = 3/2$ and $n = 1/2$. Therefore,

$$3L_1 + L_3 + 3E_1 + 5E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 \equiv -2K_{\tilde{S}}.$$

This implies that $\text{lct}(S) \leq 2/5$.

Suppose that $\text{lct}(S) < 2/5$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (2/5)D)$ is not log-canonical at some point $p \in \tilde{S}$. Write $D = aL_1 + bL_3 + c_1E_1 + c_2E_2 + c_3E_3 + \Omega$, where Ω is an effective \mathbb{Q} -divisor whose support does not contain the curves E_1, E_2, E_3, L_1, L_3 . Lemma 2.6 shows that the pair $(\tilde{S}, (2/5)D)$ is log-canonical in the outside of the -2 curves. By contracting L_1, L_2 and

then E_2, E_5 to a smooth del Pezzo surface of degree 6, we can see that the pair $(\tilde{S}, (2/5)D)$ is log-canonical in the outside of E_2 and E_5 since the global log-canonical threshold of a smooth del Pezzo surface of degree 6 is $1/2$. We may assume that the point p belongs to E_2 .

If $a = 0$, then $1 = D \cdot L_1 \geq c_2$. However, we have $5/2 < (D - c_2E_2) \cdot E_2 = 2c_2$ by Lemma 2.2. This is a contradiction. Therefore, $a > 0$, and hence we may assume that $b = 0$ by Lemma 2.1. Then we have $1 = D \cdot L_3 \geq c_1$.

Suppose that the point p is the intersection point of E_1 and E_2 . Then we obtain

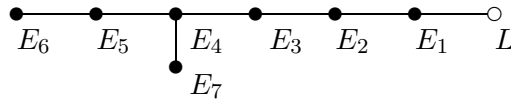
$$\frac{5}{2} < (D - c_1E_1) \cdot E_1 = 2c_1$$

from Lemma 2.2, and hence $5/4 < c_1$. This is a contradiction.

By contracting L_3, E_1, L_2, E_5, E_4 to a smooth del Pezzo surface of degree 7, we can see that Corollary 2.5 implies that the pair $(\tilde{S}, (2/5)D)$ is not log-canonical along the curve E_2 . This is a contradiction since the pair $(\tilde{S}, (2/5)D)$ is log-canonical at the intersection point of E_1 and E_2 . \square

LEMMA 4.3. *Let S be a singular del Pezzo surface of degree 2 with singularity type E_7 . Then $\text{lct}(S) = \text{lct}_1(S) = 1/6$.*

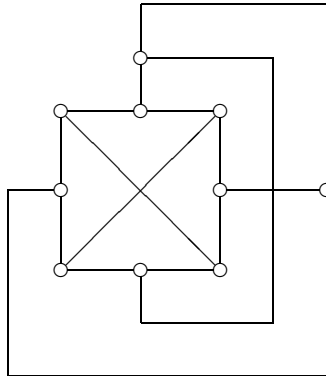
Proof. Let \tilde{S} be the minimal resolution of S . From [12, Proposition 2.1] we obtain the configuration of all the -2 curves and some -1 curves on \tilde{S} as follows:



Suppose that $\text{lct}(S) < 1/6$. Then there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, (1/6)D)$ is not log-canonical at some point $p \in \tilde{S}$. Since $2L + 3E_1 + 4E_2 + 5E_3 + 6E_4 + 4E_5 + 2E_6 + 3E_7 \sim -K_{\tilde{S}}$ (see [12, Proposition 2.1]), we may assume that the curve L is not contained in the support of D due to Lemma 2.1. Then, the same argument as in the proof of Lemma 3.7 gives us a contradiction. \square

LEMMA 4.4. *Let S be a smooth del Pezzo surface of degree 5. For a point p in S , there are four disjoint -1 curves that do not pass through the point p .*

Proof. The surface S has ten -1 curves. Their configuration is as follows:



This can be obtained by contracting the -1 curve l_{24} in the configuration of $2A_1$ with nine lines in [5, Proposition 6.1]. The configuration of the ten -1 curves immediately implies the statement. \square

LEMMA 4.5. *Let S be a del Pezzo surface of degree 2 with at least two singular points. Let $\pi : \tilde{S} \rightarrow S$ be the minimal resolution of S , and let D be an effective \mathbb{Q} -divisor numerically equivalent to $-K_{\tilde{S}}$. If $\text{lct}_1(S) \geq 2/3$, then for a positive number $\lambda < 2/3$, the pair $(\tilde{S}, \lambda D)$ is log-canonical in the outside of a single point $p \in \tilde{S}$.*

Proof. We have a double cover $\rho : S \rightarrow \mathbb{P}^2$ ramified along a quartic curve R with simple singularities in \mathbb{P}^2 . The pullback of a line in \mathbb{P}^2 by the morphism ρ is an effective anticanonical divisor on S .

Since $\text{lct}_1(S) \geq 2/3$, the surface S has only A_1 or A_2 singularities by Theorem 1.4. Suppose that there is a line L on S that passes through three singular points. Then the line $\rho(L)$ passes through three singular points of the curve R , and hence $\rho(L)$ is a component of the quartic curve R . This contradicts $\text{lct}_1(S) \geq 2/3$ since $2L \sim -K_S$. Therefore, there is no line passing through three singular points on S .

Suppose that the pair $(\tilde{S}, \lambda D)$ is not log-canonical at a generic point of an irreducible curve C on \tilde{S} . Write $D = aC + \Omega$, where Ω is an effective \mathbb{Q} -divisor whose support does not contain the curve C . Then

$$2 = -K_{\tilde{S}} \cdot D \geq -aK_{\tilde{S}} \cdot C > \frac{3}{2}(-K_{\tilde{S}} \cdot C),$$

and hence the curve C must be either a -1 curve or a -2 curve.

Suppose that the curve C is a -1 curve. Then there is another -1 curve C' on \tilde{S} such that $\rho^*(\rho(\pi(C))) = \pi(C) + \pi(C') \sim -K_S$. We may assume

that C' is not contained in the support of D . (The proof of Lemma 2.1 shows that we can do this keeping $a\lambda > 1$.) Then

$$1 = D \cdot C' = a\pi(C) \cdot \pi(C') + \pi(\Omega) \cdot \pi(C') \geq a\pi(C) \cdot \pi(C') \geq \frac{2}{3}a.$$

This is a contradiction.

Suppose that the curve C is a -2 curve. Let L be the line on \mathbb{P}^2 passing through the point $\rho(\pi(C))$ and another singular point of R . Then $\rho^*(L)$ consists of two lines M_1 and M_2 on S such that $\rho^*(L) = M_1 + M_2 \sim -K_S$. Let L_i be the -1 curve on \tilde{S} with $\pi(L_i) = M_i$. Since the pair $(\tilde{S}, \lambda\pi^*(\rho^*(L)))$ is log-canonical, we may assume that L_2 is not contained in the support of D . (The proof of Lemma 2.1 shows that we can do this keeping $a\lambda > 1$.)

If $\pi(C)$ is an A_1 singular point, then $1 = D \cdot L_2 \geq aC \cdot L_2 = a$. This is a contradiction.

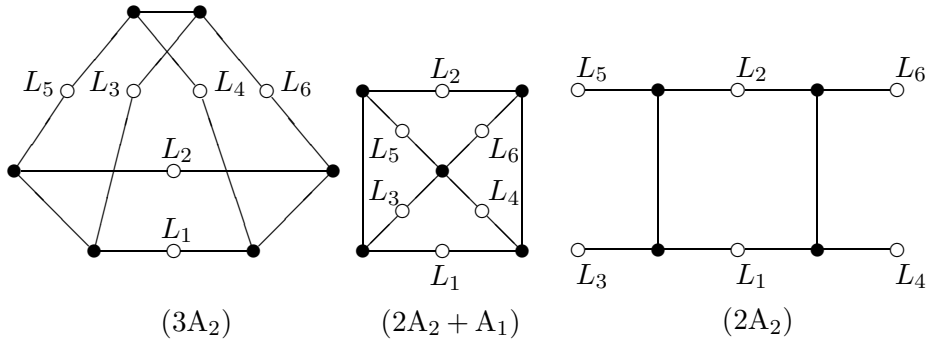
Suppose that $\pi(C)$ is an A_2 singular point. If L_2 intersects C , then we obtain a contradictory inequality $1 = D \cdot L_2 \geq aC \cdot L_2 = a$. If L_2 does not intersect C , then there is another -2 curve C' such that $C \cdot C' = 1$ and $C' \cdot L_2 = 1$. Write $D = aC + bC' + \Delta$, where Δ is an effective \mathbb{Q} -divisor whose support does not contain the curves C and C' . The pair $(\tilde{S}, \lambda D)$ is not log-canonical at the intersection point p of C and C' . Since $1 = D \cdot L_2 \geq bC' \cdot L_2 = b$, the pair $(\tilde{S}, C' + a\lambda C + \lambda\Delta)$ is not log-canonical at the point p . Therefore, Lemma 2.2 implies that

$$\frac{3}{2} < (D - bC') \cdot C' = 2b.$$

However, $0 \leq (-K_{\tilde{S}} - C - C') \cdot \Delta = 2 - (a + b)$, and hence $a < 5/4$. This is a contradiction. Therefore, the pair $(\tilde{S}, \lambda D)$ is log-canonical in the outside of finitely many points. Then Lemma 2.3 completes the proof. \square

LEMMA 4.6. *Let S be a singular del Pezzo surface of degree 2 with singularity type $3A_2$, $2A_2 + A_1$, or $2A_2$. Then $\text{lct}(S) = \text{lct}_1(S) = 2/3$.*

Proof. Let $\pi : \tilde{S} \rightarrow S$ be the minimal resolution of S . Then for each case the smooth surface \tilde{S} has six -1 curves L_i , $i = 1, 2, \dots, 6$, that have the following configuration with -2 curves (see the appendix, Table 1):



Suppose that $\text{lt}(S) < 2/3$. Then for $\text{lt}(S) < \lambda < \frac{2}{3}$ there is an effective \mathbb{Q} -divisor D on \tilde{S} numerically equivalent to $-K_{\tilde{S}}$ such that the pair $(\tilde{S}, \lambda D)$ is not log-canonical only at a single point $p \in \tilde{S}$ by Lemma 4.5.

Contracting either three -1 curves L_1, L_5, L_6 or three -1 curves L_2, L_3, L_4 , we can obtain a birational morphism of \tilde{S} to a smooth del Pezzo surface of degree 5, which is an isomorphism around the point p . By Lemma 4.4, we can obtain an effective \mathbb{Q} -divisor C on \mathbb{P}^2 numerically equivalent to $-K_{\mathbb{P}^2}$ such that the pair $(\mathbb{P}^2, \lambda C)$ is not log-canonical only at a single point. However, this contradicts Corollary 2.4. \square

PROPOSITION 4.7. *Let S be a singular del Pezzo surface of degree 2. If the singularity type of S is neither A_7 nor A_6 , then $\text{lt}(S) = \text{lt}_1(S)$.*

Proof. If the singularity type of S is not $E_7, 3A_2, 2A_2 + A_1$, or $2A_2$, then there are disjoint -1 curves on the minimal resolution of S for which we can apply Lemma 2.7 (see the appendix, Table 1). Therefore, $\text{lt}(S) = \text{lt}_1(S)$. \square

§5. Appendix

The following tables show the configurations of the -2 curves and some -1 curves on weak del Pezzo surfaces of del Pezzo surfaces with given singularity types. The columns labeled “Example” show configurations of some effective divisors on certain blow-ups of \mathbb{P}^2 in order to show existence of the configurations in the second columns on weak del Pezzo surfaces corresponding to the given singularity types. In each example, solid lines, which denote the exceptional curves of blow-ups of \mathbb{P}^2 , show the manner of performing blow-ups from \mathbb{P}^2 . Among the solid lines, thin lines (always drawn horizontally) denote -1 curves and thick lines (always drawn diagonally) denote -2 curves. The dotted curves in each example are the strict transform of

a line, an irreducible conic, or an irreducible cubic via the blow-ups. The letters L , Q , and C beside the dotted curves mean that the corresponding curves are the strict transforms of a line, an irreducible conic, and an irreducible cubic, respectively. In Table 1, in the examples for $3A_2$, $2A_2 + A_1$, and $A_2 + 2A_1$, \circ means that the two curves with the circle do not intersect at the circled point.

Table 1: Degree 2

<i>Singularity type</i>	<i>Configuration</i>	<i>Example</i>
E_7		
E_6		
$D_6 + A_1$		
D_6		
$D_5 + A_1$		
D_5		
$D_4 + 3A_1$		
$D_4 + 2A_1$		
$D_4 + A_1$		
D_4		


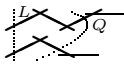
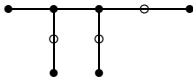
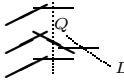


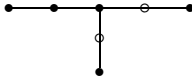
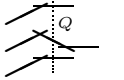

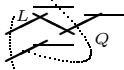
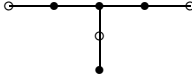
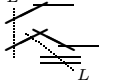

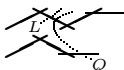
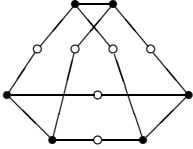
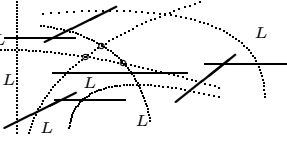
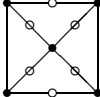
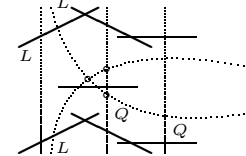
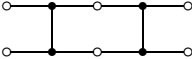
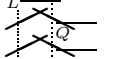

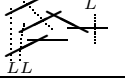
(continued)

Table 1: Degree 2 (*continued*)

A_7		
A_6		
$A_5 + A_2$		
$(A_5 + A_1)'$		
$(A_5 + A_1)''$		
$(A_5)'$		
$(A_5)''$		
$A_4 + A_2$		
$A_4 + A_1$		
A_4		
$2A_3 + A_1$		
$2A_3$		
$A_3 + A_2 + A_1$		

(*continued*)

Table 1: Degree 2 (continued)

$A_3 + A_2$		
$A_3 + 3A_1$		
$(A_3 + 2A_1)'$		
$(A_3 + 2A_1)''$		
$(A_3 + A_1)'$		
$(A_3 + A_1)''$		
A_3		
$3A_2$		
$2A_2 + A_1$		
$2A_2$		
$A_2 + 3A_1$		

(continued)

Table 1: Degree 2 (*continued*)

$A_2 + 2A_1$		
$A_2 + A_1$		
A_2		
$6A_1$		
$5A_1$		
$(4A_1)'$		
$(4A_1)''$		
$(3A_1)'$		
$(3A_1)''$		
$2A_1$		
A_1		

Table 2: Degree 3

<i>Singularity type</i>	<i>Configuration</i>	<i>Example</i>
E_6		
D_5		
D_4		
$A_5 + A_1$		
A_5		
$A_4 + A_1$		
A_4		
$A_3 + 2A_1$		
$A_3 + A_1$		
A_3		
$3A_2$		
$2A_2 + A_1$		
$2A_2$		

(continued)

Table 2: Degree 3 (*continued*)

$A_2 + 2A_1$		
$A_2 + A_1$		
A_2		
$4A_1$		
$3A_1$		
$2A_1$		
A_1		

REFERENCES

[1] V. Alexeev and V. Nikulin, *Del Pezzo and K3 surfaces*, MSJ Mem. **15**, Math. Soc. Japan, Tokyo, 2006.

[2] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces*, Geom. Funct. Anal. **18** (2008), 1118–1144.

[3] ———, *On singular cubic surfaces*, Asian J. Math. **13** (2009), 191–214.

[4] I. Cheltsov and K. Shramov, *Log-canonical thresholds of smooth Fano threefolds*, Russian Math. Surveys **63**, no. 5 (2008), 859–958.

[5] D. F. Coray and M. A. Tsfasman, *Arithmetic on singular del Pezzo surfaces*, Proc. Lond. Math. Soc. (3) **57** (1988), 25–87.

[6] J.-P. Demailly and J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) **34** (2001), 525–556.

[7] M. Demazure, “Surfaces de del Pezzo” in *Séminaire sur les singularités des surfaces (Palaiseau, France, 1977)*, Lecture Notes in Math. **777**, Springer, Berlin, 1980, 21–69.

[8] J. Kollár, “Adjunction and discrepancies” in *Flips and Abundance for Algebraic Threefolds (Salt Lake City, 1991)*, Astérisque **211**, Soc. Math. France, Paris, 1992, 183–192.

[9] D. Kosta, *Del Pezzo surfaces with Du Val singularities*, Ph.D. dissertation, University of Edinburgh, Edinburgh, 2009, arXiv:0904.0943v2 [math.AG]

- [10] J. Park, *Birational maps of del Pezzo fibrations*, J. Reine Angew. Math. **538** (2001), 213–221.
- [11] ———, *A note on del Pezzo fibrations of degree 1*, Comm. Algebra **31** (2003), 5755–5768.
- [12] J. Park and J. Won, *Log canonical thresholds on Gorenstein canonical del Pezzo surfaces*, to appear in Proc. Edinb. Math. Soc., preprint, arXiv:0904.4513 [math.AG]
- [13] H. C. Pinkham, “Simple elliptic singularities, del Pezzo surfaces and Cremona transformations” in *Several Complex Variables (Williamstown, Mass., 1975)*, Proc. Sympos. Pure Math. **30**, Amer. Math. Soc., Providence, 1977, 69–70.
- [14] A. V. Pukhlikov, *Birational geometry of Fano direct products*, Izv. Math. **69**, no. 6 (2005), 1225–1255.
- [15] T. Urabe, “On singularities on degenerate del Pezzo surfaces of degree 1, 2” in *Singularities, Part 2 (Arcata, Calif., 1981)*, Proc. Sympos. Pure Math. **40**, Amer. Math. Soc., Providence, 1983, 587–591.

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