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ABSTRACT

The commutative differential graded algebra $A_{\text{PL}}(X)$ of polynomial forms on a simplicial set X is a crucial tool in rational homotopy theory. In this note, we construct an integral version $A^{\mathcal{I}}(X)$ of $A_{\text{PL}}(X)$. Our approach uses diagrams of chain complexes indexed by the category of finite sets and injections \mathcal{I} to model E_{∞} differential graded algebras (dga) by strictly commutative objects, called commutative \mathcal{I} -dgas. We define a functor $A^{\mathcal{I}}$ from simplicial sets to commutative \mathcal{I} -dgas and show that it is a commutative lift of the usual cochain algebra functor. In particular, it gives rise to a new construction of the E_{∞} dga of cochains. The functor $A^{\mathcal{I}}$ shares many properties of A_{PL} , and can be viewed as a generalization of A_{PL} that works over arbitrary commutative ground rings. Working over the integers, a theorem by Mandell implies that $A^{\mathcal{I}}(X)$ determines the homotopy type of X when X is a nilpotent space of finite type.

1. Introduction

Determining the homotopy type of a topological space is a difficult task in general. One possibility of simplifying the problem is to aim for algebraic models of spaces, so that the study of homotopy types reduces to an algebraic question. If one is interested in the homotopy type of a rational nilpotent space of finite type, then this is possible, and the Sullivan cochain algebra is such an algebraic model: the algebra of rational singular cochains of a space, $C(X; \mathbb{Q})$, is quasi-isomorphic to the commutative differential graded algebra (cdga) $A_{\text{PL}}(X)$ of polynomial forms on X , which is a very powerful tool in rational homotopy theory [BG76, Sul77]. The functor A_{PL} has a contravariant adjoint, called the Sullivan realization in [FHT01, § 17]. With the help of this adjoint pair of functors one can determine the homotopy type of rational nilpotent spaces of finite type (see [BG76, Chapter 9], or [Hes07, Theorem 1.25] for the simply connected case).

For a general commutative ring k the cochains $C(X; k)$ on a space X with values in k form a differential graded algebra whose cohomology is the singular cohomology $H^*(X; k)$ of X . The multiplication of $C(X; k)$ induces the cup product on $H^*(X; k)$. However, for general k , there is no cdga which is quasi-isomorphic to $C(X; k)$, for example because the Steenrod operations witness the non-commutativity of $C(X; \mathbb{F}_p)$. So it seems that we cannot hope for a strictly

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commutative model for the cochains of a space that determines the homotopy type. However, $C(X; k)$ is always commutative up to coherent homotopy. This can be encoded using the language of operads [May72]: the multiplication of $C(X; k)$ extends to the action of an E_∞ operad in chain complexes turning $C(X; k)$ into an E_∞ dga.

This gives rise to an algebraic model for the homotopy type of a space by a result of Mandell. He shows that the cochain functor $C(-; \mathbb{Z})$ to E_∞ dgas classifies nilpotent spaces of finite type up to weak equivalence [Man06, Main Theorem]. But here, the algebraic model consists of the cochain algebra together with its E_∞ -algebra structure, so this algebraic model is rather involved.

One can describe homotopy coherent commutative multiplications on chain complexes using diagram categories instead of operads. Let \mathcal{I} be the category with objects the finite sets $\mathbf{m} = \{1 \dots, m\}$, $m \geq 0$, with the convention that $\mathbf{0}$ is the empty set. Morphisms in \mathcal{I} are the injections. Concatenation in \mathcal{I} and the tensor product of chain complexes of k -modules give rise to a symmetric monoidal product \boxtimes on the category $\text{Ch}_k^{\mathcal{I}}$ of \mathcal{I} -diagrams in Ch_k . A commutative \mathcal{I} -dga is a commutative monoid in $(\text{Ch}_k^{\mathcal{I}}, \boxtimes)$ or, equivalently, a lax symmetric monoidal functor $\mathcal{I} \rightarrow \text{Ch}_k$. Equipped with suitable model structures, the category of commutative \mathcal{I} -dgas, $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$, is Quillen equivalent to the category of E_∞ dgas [RS17, § 9]. This is analogous to the situation in spaces, where commutative monoids in \mathcal{I} -diagrams of spaces are equivalent to E_∞ spaces [SS12, § 3].

Chasing the E_∞ dga of cochains $C(X; k)$ on a space X through the chain of Quillen equivalences relating E_∞ dgas and commutative \mathcal{I} -dgas shows that $C(X; k)$ can be represented by a commutative \mathcal{I} -dga. The purpose of this paper is to construct a direct point-set level model $A^{\mathcal{I}}(X)$ for the quasi-isomorphism type of commutative \mathcal{I} -dgas determined by $C(X; k)$ that should be viewed as an integral generalization of $A_{\text{PL}}(X)$. Despite the fact that $A_{\text{PL}}(X)$ was introduced more than 40 years ago and has been widely studied, it appears that a direct integral counterpart was neither known nor expected to exist.

If E is a commutative \mathcal{I} -dga, then its Bousfield–Kan homotopy colimit $E_{h\mathcal{I}}$ has a canonical action of the Barratt–Eccles operad, which is an E_∞ operad built from the symmetric groups. The commutative \mathcal{I} -dga $A^{\mathcal{I}}(X)$ thus gives rise to an E_∞ dga $A^{\mathcal{I}}(X)_{h\mathcal{I}}$ which can be compared to the usual cochains without referring to model structures.

THEOREM 1.1. *The contravariant functors $X \mapsto A^{\mathcal{I}}(X)_{h\mathcal{I}}$ and $X \mapsto C(X; k)$ from simplicial sets to E_∞ dgas are naturally quasi-isomorphic.*

We prove the theorem using Mandell’s uniqueness result for cochain theories [Man02, Main Theorem].

Since our definition of $A^{\mathcal{I}}$ does not rely on the existing constructions of E_∞ structures on cochains, the theorem implies that our approach provides an alternative model of the E_∞ dga $C(X; k)$, namely $A^{\mathcal{I}}(X)_{h\mathcal{I}}$ with its canonical action of the Barratt–Eccles operad. If k is a field of characteristic 0, then there is a natural quasi-isomorphism $A^{\mathcal{I}}(X)_{h\mathcal{I}} \rightarrow A_{\text{PL}}(X)$ relating our approach to the classical polynomial forms (see Theorem 5.9).

The passage through commutative \mathcal{I} -dgas has the advantage that it provides a rather simple E_∞ model $A^{\mathcal{I}}(X)_{h\mathcal{I}}$ for the cochain algebra of a space X . In contrast, the existing constructions of E_∞ structures on the standard model for the cochain algebra are involved: based on work of Hinich and Schechtman [HS87], Mandell [Man02, § 5] lifts the action of the acyclic Eilenberg–Zilber operad to the action of an actual E_∞ operad. McClure and Smith generalize Steenrod’s cup- i -products to multivariable operations that give the cochains of a space the structure of an E_∞ -algebra via the action of the surjection operad [MS03]. Berger and Fresse [BF04]

use elaborate combinatorial arguments to define an action of the Barratt–Eccles operad that extends the action of the surjection operad. Another approach to capture the commutativity of $C(X; k)$ has been pursued by Karoubi [Kar09] who introduces a notion of *quasi-commutative* dgas that is based on a certain reduced tensor product, constructs a quasi-commutative model for the cochains, and uses Mandell’s results to relate it to ordinary cochains.

Since it is often easier to work with strictly commutative objects rather than E_∞ objects, we also expect that the commutative \mathcal{I} -dga $A^{\mathcal{I}}(X)$ will be a useful replacement of the E_∞ dga $C(X; k)$ in applications. For instance, iterated bar constructions for E_∞ algebras as developed in [Fre11] are rather involved whereas iterated bar construction for commutative monoids are straightforward. Commutative \mathcal{I} -dgas are tensored over simplicial sets whereas enrichments for E_∞ monoids are more complicated because the coproduct is not just the underlying monoidal product. This allows for constructions such as higher-order Hochschild homology [Pir00] for commutative \mathcal{I} -dgas.

Writing $A^{\mathcal{I}}(X; \mathbb{Z})$ for $A^{\mathcal{I}}(X)$ when working over $k = \mathbb{Z}$, Theorem 1.1 leads to the following reformulation of the main theorem of Mandell [Man06] that highlights the usefulness of $A^{\mathcal{I}}$.

THEOREM 1.2. *Two finite type nilpotent spaces X and Y are weakly equivalent if and only if $A^{\mathcal{I}}(X; \mathbb{Z})$ and $A^{\mathcal{I}}(Y; \mathbb{Z})$ are weakly equivalent in $\text{Ch}_{\mathbb{Z}}^{\mathcal{I}}[\mathcal{C}]$.*

1.3 Outline of the construction

Our chain complexes are homologically graded so that cochains are concentrated in non-positive degrees. We model spaces by simplicial sets and consider the singular complex of a topological space if necessary.

The functor $A_{\text{PL}}: \text{sSet}^{\text{op}} \rightarrow \text{cdga}_{\mathbb{Q}}$ of polynomial forms used in rational homotopy theory (see e.g. [BG76, § 1]) motivates our definition of $A^{\mathcal{I}}$. We recall that A_{PL} arises by Kan extending the functor $A_{\text{PL}, \bullet}: \Delta^{\text{op}} \rightarrow \text{cdga}_{\mathbb{Q}}$ sending $[p]$ in Δ to the algebra of polynomial differential forms

$$A_{\text{PL}, p} = \Lambda(t_0, \dots, t_p; dt_0, \dots, dt_p) / (t_0 + \dots + t_p = 1, dt_0 + \dots + dt_p = 0). \tag{1.1}$$

Here Λ is the free graded commutative algebra over \mathbb{Q} , the generators t_i have degree 0, and the dt_i have degree -1 (in our homological grading). Setting $d(t_i) = dt_i$ extends to a differential that turns $A_{\text{PL}, q}$ into a commutative dga, and addition of the t_i and insertion of 0 define the simplicial structure of $A_{\text{PL}, \bullet}$.

The topological standard p -simplex can be written as

$$\Delta^p = \{(t_0, \dots, t_p), t_0 + \dots + t_p = 1, t_i \geq 0\}$$

and as

$$\Delta^p = \{(x_0, x_1, \dots, x_p, x_{p+1}), 0 = x_0 \leq x_1 \leq \dots \leq x_p \leq x_{p+1} = 1\}.$$

Setting $x_i = t_0 + \dots + t_{i-1}$ for $1 \leq i \leq p$ yields an isomorphism

$$A_{\text{PL}, p} \cong \Lambda(x_1, \dots, x_p, dx_1, \dots, dx_p) \cong \Lambda(x_1, dx_1) \otimes \dots \otimes \Lambda(x_p, dx_p) \tag{1.2}$$

and this simple but crucial trick gives rise to the following reformulation: let $\mathbb{C}D^0$ be the free commutative \mathbb{Q} -dga on the chain complex D^0 with $(D^0)_i = 0$ if $i \neq 0, -1$ and $d_0: (D^0)_0 \rightarrow (D^0)_{-1}$ being $\text{id}_{\mathbb{Q}}$. Moreover, let S^0 in $\text{Ch}_{\mathbb{Q}}$ be the monoidal unit, i.e., the chain complex with a copy of \mathbb{Q} concentrated in degree 0. Sending $1 \in (\mathbb{C}D^0)_0$ to either 0 or 1 in \mathbb{Q} defines two commutative

CD^0 -algebra structures on S^0 that we denote by S_0^0 and S_1^0 . We argue in § 5.8 that the simplicial \mathbb{Q} -cdga $A_{\text{PL},\bullet}$ is isomorphic to the two-sided bar construction

$$B_{\bullet}(S_0^0, \mathbb{C}D^0, S_1^0) = ([p] \mapsto S_0^0 \otimes (\mathbb{C}D^0)^{\otimes p} \otimes S_1^0) \tag{1.3}$$

whose face maps are provided by the algebra structures on S_1^0 and S_0^0 and the multiplication of CD^0 , and whose degeneracy maps are induced by the unit of CD^0 .

While polynomial differential forms appear to have no obvious counterpart in commutative \mathcal{I} -dgas, their description in terms of a two-sided bar construction easily generalizes to commutative \mathcal{I} -dgas over an arbitrary commutative ground ring k . For this we consider the left adjoint

$$\mathbb{C}F_1^{\mathcal{I}}: \text{Ch}_k \rightarrow \text{Ch}_k^{\mathcal{I}}[\mathcal{C}], \quad A \mapsto \left(\mathbf{m} \mapsto \bigoplus_{s \geq 0} \left(\left(\bigoplus_{\mathcal{I}(\mathbf{1} \sqcup_s, \mathbf{m})} A^{\otimes s} \right) / \Sigma_s \right) \right)$$

to the evaluation of a commutative \mathcal{I} -dga at the object $\mathbf{1}$ in \mathcal{I} and recall that the unit $U^{\mathcal{I}}$ in $\text{Ch}_k^{\mathcal{I}}$ is the constant \mathcal{I} -diagram on the unit S^0 in Ch_k . As above, we form $\mathbb{C}F_1^{\mathcal{I}}D^0$, observe that $U^{\mathcal{I}}$ gives rise to two commutative $\mathbb{C}F_1^{\mathcal{I}}D^0$ algebras $U_0^{\mathcal{I}}$ and $U_1^{\mathcal{I}}$, and define $A_{\bullet}^{\mathcal{I}}: \Delta^{\text{op}} \rightarrow \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ to be the two-sided bar construction

$$B_{\bullet}(U_0^{\mathcal{I}}, \mathbb{C}F_1^{\mathcal{I}}(D^0), U_1^{\mathcal{I}}) = ([p] \mapsto U_0^{\mathcal{I}} \boxtimes (\mathbb{C}F_1^{\mathcal{I}}(D^0))^{\boxtimes p} \boxtimes U_1^{\mathcal{I}}).$$

At this point it is central to work with strictly commutative objects since the multiplication map of an E_{∞} object is typically not an E_{∞} map. It is also important to use $\mathbf{1}$ rather than $\mathbf{0}$ in the above left adjoint since this ensures that $A_p^{\mathcal{I}}(\mathbf{m})$ is contractible. This is related to J. Smith’s insight that one has to use *positive* model structures for commutative symmetric ring spectra. Using that \boxtimes is the coproduct in commutative \mathcal{I} -dgas, we get an isomorphism

$$A_p^{\mathcal{I}} \cong \mathbb{C}F_1^{\mathcal{I}}(D^0 \oplus \dots \oplus D^0)$$

identifying the simplicial degree p part of $A_{\bullet}^{\mathcal{I}}$ with a free commutative \mathcal{I} -dga on p generators. We also describe the simplicial structure maps of $A_{\bullet}^{\mathcal{I}}$ in terms of these generators (see § 3.4).

Via Kan extension and restriction along the canonical functor $\Delta^{\text{op}} \rightarrow \text{sSet}^{\text{op}}$, this $A_{\bullet}^{\mathcal{I}}$ gives rise to functors $A^{\mathcal{I}}: \text{sSet}^{\text{op}} \rightarrow \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ and $\langle - \rangle_{\mathcal{I}}: \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]^{\text{op}} \rightarrow \text{sSet}$ (see Definition 3.6). More explicitly, the evaluation of $A^{\mathcal{I}}(X)$ at \mathcal{I} -degree \mathbf{m} and chain complex level q is the k -module of simplicial set morphisms $\text{sSet}(X, A_{\bullet}^{\mathcal{I}}(\mathbf{m})_q)$. For every E in $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$, we set $\langle E \rangle_{\mathcal{I}} = \text{Ch}_k^{\mathcal{I}}[\mathcal{C}](E, A_{\bullet}^{\mathcal{I}})$. The functors $A^{\mathcal{I}}$ and $\langle - \rangle_{\mathcal{I}}$ are contravariant right adjoint in the sense that there are natural isomorphisms $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}](E, A^{\mathcal{I}}(X)) \cong \text{sSet}(X, \langle E \rangle_{\mathcal{I}})$. They are integral analogues of the functor of polynomial forms and of the Sullivan realization functor.

1.4 Homotopical analysis of $A^{\mathcal{I}}$

We equip simplicial sets with the standard model structure and the category of commutative \mathcal{I} -dgas $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ with the *descending \mathcal{I} -model structure* making it Quillen equivalent to E_{∞} dgas (see § 4 for details).

THEOREM 1.5. *Both $A^{\mathcal{I}}$ and $\langle - \rangle_{\mathcal{I}}$ send cofibrations to fibrations and acyclic cofibrations to acyclic fibrations. They induce functors on the corresponding homotopy categories $\mathbb{R}\langle - \rangle_{\mathcal{I}}: \text{Ho}(\text{Ch}_k^{\mathcal{I}}[\mathcal{C}])^{\text{op}} \rightarrow \text{Ho}(\text{sSet})$ and $\mathbb{R}A^{\mathcal{I}}: \text{Ho}(\text{sSet})^{\text{op}} \rightarrow \text{Ho}(\text{Ch}_k^{\mathcal{I}}[\mathcal{C}])$ that are related by a*

natural isomorphism

$$\mathrm{Ho}(\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}])(E, \mathbb{R}A^{\mathcal{I}}(X)) \cong \mathrm{Ho}(\mathrm{sSet})(X, \mathbb{R}\langle E \rangle_{\mathcal{I}}).$$

Here, $\mathbb{R}(-)$ indicates that we right-derive the functors on $(\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}])^{\mathrm{op}}$ and $\mathrm{sSet}^{\mathrm{op}}$. So no fibrant replacement is necessary before applying $A^{\mathcal{I}}$ since all simplicial sets are cofibrant, while a cofibrant replacement in $\mathrm{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ is necessary to derive $\langle - \rangle_{\mathcal{I}}$.

A similar result for $A_{\mathrm{PL}}: \mathrm{sSet}^{\mathrm{op}} \rightarrow \mathrm{cdga}_{\mathbb{Q}}$ has been established by Bousfield and Gugenheim [BG76, § 8]. Mandell [Man02, § 4] constructed an analogous adjunction between simplicial sets and E_{∞} dgas using the E_{∞} structure on cochains as input. The functor of homotopy categories $\mathbb{R}A^{\mathcal{I}}$ fails to be full for the same reason as its counterpart studied by Mandell (see the discussion after [Man06, Theorem 0.2]). One of the referees of this paper raised the interesting question whether there exists a modification of our diagrammatic approach that remedies this shortcoming.

Since all simplicial sets are cofibrant, the statement of Theorem 1.5 implies that each $A^{\mathcal{I}}(X)$ is descending \mathcal{I} -fibrant. Writing \mathcal{I}_+ for the full subcategory of \mathcal{I} on objects \mathbf{m} with $|\mathbf{m}| \geq 1$, this means that for each morphism $\mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{I}_+ and each $q \geq -|\mathbf{m}|$, the induced map $H_q(A^{\mathcal{I}}(X)(\mathbf{m})) \rightarrow H_q(A^{\mathcal{I}}(X)(\mathbf{n}))$ is an isomorphism. Hence each chain complex $A^{\mathcal{I}}(X)(\mathbf{m})$ with \mathbf{m} in \mathcal{I}_+ captures the cohomology groups of X up to degree $|\mathbf{m}|$. This is the maximal information to be expected from $A^{\mathcal{I}}(X)(\mathbf{m})$ as it is a chain complex concentrated in degrees between 0 and $-|\mathbf{m}|$. Since the descending \mathcal{I} -model structure is the left Bousfield localization of a descending level model structure, it also follows that weak homotopy equivalences $X \rightarrow Y$ induce isomorphisms $H_q(A^{\mathcal{I}}(Y)(\mathbf{m})) \rightarrow H_q(A^{\mathcal{I}}(X)(\mathbf{m}))$ if \mathbf{m} is in \mathcal{I}_+ and $q \geq -|\mathbf{m}|$.

Analogous to the corresponding statement about A_{PL} , the proof of the theorem is based on the observation that the simplicial sets $A_{\bullet}^{\mathcal{I}}(\mathbf{m})_q$ are contractible for a fixed \mathbf{m} in \mathcal{I}_+ and a fixed chain level q with $0 \geq q \geq -|\mathbf{m}|$.

Remark 1.6. After a first version of the present manuscript was made available, the authors learned from Dan Petersen that he recently found another construction of a commutative \mathcal{I} -dga that models the cochain algebra of a space [Pet20]. His approach applies to locally contractible topological spaces, uses sheaf cohomology, and has applications in the study of configuration spaces.

1.7 Notation and conventions

Throughout the paper, k denotes a commutative ring with unit, and Ch_k denotes the category of unbounded homologically graded chain complexes of k -modules. For $q \in \mathbb{Z}$, we as usual write S^q for the chain complex with k concentrated in degree q , and D^q for the chain complex with $(D^q)_i = k$ if $i \in \{q, q - 1\}$, with $(D^q)_i = 0$ for all other i , and with $d_q = \mathrm{id}_k$.

1.8 Organization

In § 2 we study homotopy colimits of commutative \mathcal{I} -dgas. Section 3 provides the construction of the functor $A^{\mathcal{I}}$. We review model structures on \mathcal{I} -chain complexes and commutative \mathcal{I} -dgas in § 4. In § 5 we establish the homotopical properties of $A^{\mathcal{I}}$, prove a comparison to the usual cochains disregarding multiplicative structures, and prove Theorem 1.5. In the final § 6, we prove the E_{∞} comparison from Theorem 1.1 as Theorem 6.2 and explain how to derive Theorem 1.2.

2. Homotopy colimits of \mathcal{I} -chain complexes

Let \mathcal{I} be the category with objects the finite sets $\mathbf{m} = \{1, \dots, m\}$ for $m \geq 0$ and with morphisms the injective maps. In this section we study multiplicative properties of the homotopy colimit functor for \mathcal{I} -diagrams of chain complexes.

DEFINITION 2.1. An \mathcal{I} -chain complex is a functor $\mathcal{I} \rightarrow \text{Ch}_k$, and $\text{Ch}_k^{\mathcal{I}}$ denotes the resulting functor category.

For each \mathbf{m} in \mathcal{I} there is an adjunction $F_{\mathbf{m}}^{\mathcal{I}}: \text{Ch}_k \rightleftarrows \text{Ch}_k^{\mathcal{I}}: \text{Ev}_{\mathbf{m}}$ with right adjoint the evaluation functor $\text{Ev}_{\mathbf{m}}(P) = P(\mathbf{m})$ and left adjoint

$$F_{\mathbf{m}}^{\mathcal{I}}: \text{Ch}_k \rightarrow \text{Ch}_k^{\mathcal{I}}, \quad A \mapsto \left(\mathbf{n} \mapsto \bigoplus_{\mathcal{I}(\mathbf{m}, \mathbf{n})} A \right). \tag{2.1}$$

The functor $F_{\mathbf{0}}^{\mathcal{I}}$ is isomorphic to the constant functor since $\mathbf{0}$ is initial in \mathcal{I} .

2.2 Homotopy colimits

Our next aim is to define Bousfield–Kan style homotopy colimits for \mathcal{I} -diagrams of chain complexes. For the subsequent multiplicative analysis, we fix notation and conventions about bicomplexes.

DEFINITION 2.3. Let $\text{Ch}_k(\text{Ch}_k)$ be the category of chain complexes in Ch_k . Its objects are $\mathbb{Z} \times \mathbb{Z}$ -graded k -modules $(Y_{p,q})_{p,q \in \mathbb{Z}}$ with k -linear horizontal differentials, $d_h: Y_{p,q} \rightarrow Y_{p-1,q}$, and k -linear vertical differentials, $d_v: Y_{p,q} \rightarrow Y_{p,q-1}$, such that

$$d_h \circ d_h = 0 = d_v \circ d_v \quad \text{and} \quad d_v \circ d_h = d_h \circ d_v.$$

A morphism $g: Y \rightarrow Z$ in $\text{Ch}_k(\text{Ch}_k)$ is a family $(g_{p,q}: Y_{p,q} \rightarrow Z_{p,q})_{p,q \in \mathbb{Z} \times \mathbb{Z}}$ of k -linear maps that commute with the horizontal and vertical differentials, i.e.,

$$d_h \circ g_{p,q} = g_{p-1,q} \circ d_h \quad \text{and} \quad d_v \circ g_{p,q} = g_{p,q-1} \circ d_v$$

for all $p, q \in \mathbb{Z}$.

Since we require horizontal and vertical differentials to commute, an additional sign is needed to form the total complex.

DEFINITION 2.4. Let Y be an object in $\text{Ch}_k(\text{Ch}_k)$. Its associated total complex $\text{Tot}(Y)$ is the chain complex with $\text{Tot}(Y)_n = \bigoplus_{p+q=n} Y_{p,q}$ in chain degree $n \in \mathbb{Z}$ and with differential $d_{\text{Tot}}(y) = d_h(y) + (-1)^p d_v(y)$ for every homogeneous $y \in Y_{p,q}$.

Let sCh_k be the category of simplicial objects in Ch_k .

DEFINITION 2.5. For $A \in \text{sCh}_k$ we denote by $C_*(A)$ the chain complex in chain complexes with $(C_*(A))_{p,q} = A_{p,q}$. We define the horizontal differential on $C_*(A)$, $d_h: A_{p,q} \rightarrow A_{p-1,q}$, as

$$d_h = \sum_{i=0}^p (-1)^i d_i$$

where the d_i are the simplicial face maps of A . The vertical differential on $C_*(A)$ is given by the differential d^A on A .

As the d_i commute with d^A , this gives indeed a chain complex in chain complexes whose horizontal part is concentrated in non-negative degrees.

CONSTRUCTION 2.6. Let $P: \mathcal{I} \rightarrow \text{Ch}_k$ be an \mathcal{I} -chain complex. The *simplicial replacement* of P is the simplicial chain complex $\text{srep}(P): \Delta^{\text{op}} \rightarrow \text{Ch}_k$ given in simplicial degree $[p]$ by

$$\text{srep}(P)[p] = \bigoplus_{(\mathbf{n}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{n}_p) \in N(\mathcal{I})_p} P(\mathbf{n}_p).$$

The last face map sends the copy of $P(\mathbf{n}_p)$ indexed by $(\alpha_1, \dots, \alpha_p)$ via $P(\alpha_p)$ to the copy of $P(\mathbf{n}_{p-1})$ indexed by $(\alpha_1, \dots, \alpha_{p-1})$. The other face and degeneracy maps are induced by the identity on $P(\mathbf{n}_p)$ and corresponding simplicial structure maps of the nerve $\mathcal{N}(\mathcal{I})$ of \mathcal{I} .

The homotopy colimit functor $(-)_h\mathcal{I}: \text{Ch}_k^{\mathcal{I}} \rightarrow \text{Ch}_k$ is defined by

$$P_{h\mathcal{I}} = \text{Tot } C_*(\text{srep}(P)).$$

A bicomplex spectral sequence argument shows that $P_{h\mathcal{I}} \rightarrow Q_{h\mathcal{I}}$ is a quasi-isomorphism if each $P(\mathbf{m}) \rightarrow Q(\mathbf{m})$ is a quasi-isomorphism. There is a canonical map $P_{h\mathcal{I}} \rightarrow \text{colim}_{\mathcal{I}} P$, and one can show by cell induction that it is a quasi-isomorphism if P is cofibrant in the projective level model structure on $\text{Ch}_k^{\mathcal{I}}$. Together this shows that $P_{h\mathcal{I}}$ is a model for the homotopy colimit of P . A more elaborate argument that shows that $P_{h\mathcal{I}}$ is a corrected homotopy colimit can be found in [RG14]. A version of the above homotopy colimit for functors with values in modules can be found in [DL98, Definition 3.13].

2.7 Commutative \mathcal{I} -dgas

The ordered concatenation of ordered sets $\mathbf{m} \sqcup \mathbf{n} = \mathbf{m} + \mathbf{n}$ equips \mathcal{I} with a symmetric strict monoidal structure that has $\mathbf{0}$ as a strict unit and the block permutations as symmetry isomorphisms. If $P, Q: \mathcal{I} \rightarrow \text{Ch}_k$ are \mathcal{I} -chain complexes, then the left Kan extension of

$$\mathcal{I} \times \mathcal{I} \xrightarrow{P \times Q} \text{Ch}_k \times \text{Ch}_k \xrightarrow{\otimes} \text{Ch}_k$$

along $\sqcup: \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$ provides an \mathcal{I} -chain complex $P \boxtimes Q$. This defines a symmetric monoidal product \boxtimes on $\text{Ch}_k^{\mathcal{I}}$, the Day convolution product, with unit the constant \mathcal{I} -diagram $U^{\mathcal{I}} = F_{\mathbf{0}}^{\mathcal{I}}(S^0)$.

DEFINITION 2.8. A *commutative \mathcal{I} -dga* is a commutative monoid in $(\text{Ch}_k^{\mathcal{I}}, \boxtimes, U^{\mathcal{I}})$, i.e., a lax symmetric monoidal functor $(\mathcal{I}, \sqcup, \mathbf{0}) \rightarrow (\text{Ch}_k, \otimes, S^0)$. The resulting category of commutative \mathcal{I} -dgas is denoted by $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$.

We write $\mathbb{C}: \text{Ch}_k^{\mathcal{I}} \rightleftarrows \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]: U$ for the adjunction with right adjoint the forgetful functor and left adjoint the free functor \mathbb{C} given by

$$\mathbb{C}(P) = \bigoplus_{s \geq 0} P^{\boxtimes s} / \Sigma_s. \tag{2.2}$$

The definition of \boxtimes as a left Kan extension implies the existence of a natural isomorphism $F_{\mathbf{n}_1}^{\mathcal{I}}(A^1) \boxtimes F_{\mathbf{n}_2}^{\mathcal{I}}(A^2) \cong F_{\mathbf{n}_1 \sqcup \mathbf{n}_2}^{\mathcal{I}}(A^1 \otimes A^2)$. This shows that in the case $P = F_{\mathbf{1}}^{\mathcal{I}}(A)$, we have an isomorphism $F_{\mathbf{1}}^{\mathcal{I}}(A)^{\boxtimes s} \cong F_{\mathbf{1} \sqcup s}^{\mathcal{I}}(A^{\otimes s})$ of Σ_s -equivariant objects where Σ_s acts on the target by permuting both the \otimes -powers of A and the index set of the sum. The commutative \mathcal{I} -dga

$\mathbb{C}(F_1^{\mathcal{I}}(A))$ will be of particular importance for us, and we note that the above implies

$$\mathbb{C}(F_1^{\mathcal{I}}(A))(\mathbf{m}) \cong \bigoplus_{s \geq 0} \left(\left(\bigoplus_{\mathcal{I}(\mathbf{1} \sqcup^s, \mathbf{m})} A^{\otimes s} \right) / \Sigma_s \right). \tag{2.3}$$

2.9 Homotopy colimits of commutative \mathcal{I} -dgas

We will now construct an operad action on the homotopy colimit of a commutative \mathcal{I} -dga. Our construction involves a symmetric monoidal structure on simplicial chain complexes.

DEFINITION 2.10. Let A and B be two simplicial chain complexes. Their tensor product $A \hat{\otimes} B$ is the simplicial chain complex with

$$\bigoplus_{\ell+m=n} A_{p,\ell} \otimes B_{p,m}$$

in simplicial degree p and chain degree n . The simplicial structure maps act coordinatewise and the differential $d^{\hat{\otimes}}$ is

$$d^{\hat{\otimes}}(a \otimes b) = d(a) \otimes b + (-1)^\ell a \otimes d(b)$$

for $a \otimes b \in A_{p,\ell} \otimes B_{p,m}$. The symmetry isomorphism $c: A \hat{\otimes} B \rightarrow B \hat{\otimes} A$ sends a homogeneous element $a \otimes b$ as above to $(-1)^{\ell \cdot m} b \otimes a$.

We denote by $\tilde{\Sigma}_s$ the translation category of the symmetric group Σ_s . Its objects are elements $\sigma \in \Sigma_s$ and $\tau \in \Sigma_s$ is the unique morphism from σ to $\tau \circ \sigma$ in $\tilde{\Sigma}_s$. Since there is exactly one morphism between each pair of objects, we get a functor

$$\tilde{\Sigma}_s \times \tilde{\Sigma}_{j_1} \times \cdots \times \tilde{\Sigma}_{j_s} \rightarrow \tilde{\Sigma}_{j_1 + \cdots + j_s} \tag{2.4}$$

by specifying that $(\sigma; \tau_1, \dots, \tau_s)$ is sent to the composite

$$(\tau_{\sigma^{-1}(1)} \sqcup \cdots \sqcup \tau_{\sigma^{-1}(s)}) \circ \sigma(j_1, \dots, j_s)$$

of the block permutation $\sigma(j_1, \dots, j_s): \mathbf{j}_1 \sqcup \cdots \sqcup \mathbf{j}_s \rightarrow \mathbf{j}_{\sigma^{-1}(1)} \sqcup \cdots \sqcup \mathbf{j}_{\sigma^{-1}(s)}$ induced by σ and the concatenation of the $\tau_{\sigma^{-1}(j)}$ (see [May74, §4] and [CLM76, Correction 34 on p. 490]).

The action (2.4) is associative, unital, and symmetric. It turns the collection of categories $(\tilde{\Sigma}_n)_{n \geq 0}$ into an operad $\tilde{\Sigma}$ in the category cat of small categories. For the next definition, we use that the nerve functor $\mathcal{N}: \text{cat} \rightarrow \text{sSet}$ and the k -linearization $k\{-\}: \text{sSet} \rightarrow \text{sMod}_k$ are strong symmetric monoidal and that the associated chain complex functor $C_*: \text{sMod}_k \rightarrow \text{Ch}_k$ is lax symmetric monoidal (compare Proposition 2.16 below).

DEFINITION 2.11. The Barratt–Eccles operad is the E_∞ operad \mathcal{E} in Ch_k with $\mathcal{E}_n = C_*(k\{\mathcal{N}(\tilde{\Sigma}_n)\})$ and operad structure induced by the functor (2.4).

The commutativity operad \mathcal{C} in Ch_k is the operad with $\mathcal{C}_n = S^0$ concentrated in chain complex level 0. The operad \mathcal{E} admits a canonical operad map $\mathcal{E} \rightarrow \mathcal{C}$ which is a quasi-isomorphism in each level. Moreover, \mathcal{E}_n is a free $k[\Sigma_n]$ -module for each n . Thus \mathcal{E} is an E_∞ operad in Ch_k in the terminology of [Man02, Definition 4.1].

Applying the nerve to $\tilde{\Sigma}$ defines an operad in sSet that is more commonly referred to as the Barratt–Eccles operad. It is well known that the latter operad acts on the nerve of a permutative

category [May74, Theorem 4.9]. The next lemma recalls the underlying action of $\widetilde{\Sigma}$ for the permutative category \mathcal{I} .

LEMMA 2.12. *The operad $\widetilde{\Sigma}$ in cat acts on \mathcal{I} . On objects σ in $\widetilde{\Sigma}_n$ and \mathbf{m}_i in \mathcal{I} , the action is given by $(\sigma; \mathbf{m}_1, \dots, \mathbf{m}_n) \mapsto \mathbf{m}_{\sigma^{-1}(1)} \sqcup \dots \sqcup \mathbf{m}_{\sigma^{-1}(n)}$.*

Proof. This is a special case of [May74, Lemmas 4.3 and 4.4]. Functoriality in morphisms of $\widetilde{\Sigma}_n$ uses the symmetry isomorphism of \mathcal{I} while the functoriality in \mathcal{I} is the evident one. \square

The next result is our main motivation for considering the Barratt–Eccles operad. It is analogous to the result about \mathcal{I} -diagrams in spaces established in [Sch09, Proposition 6.5].

THEOREM 2.13. *For every commutative \mathcal{I} -dga E , the chain complex $E_{h\mathcal{I}}$ has a natural action of the Barratt–Eccles operad \mathcal{E} .*

Proof. We can view the simplicial k -module $k\{\mathcal{N}(\widetilde{\Sigma}_s)\}$ as a simplicial chain complex concentrated in chain degree 0. The operad structure of $\widetilde{\Sigma}$ turns these simplicial k -modules into an operad in sMod_k and in sCh_k . We construct an action

$$k\{\mathcal{N}(\widetilde{\Sigma}_s)\} \hat{\otimes} \text{srep}(E) \rightarrow \text{srep}(E).$$

It is enough to specify the action of a q -simplex $\sigma_0 \xleftarrow{\tau_1} \sigma_1 \leftarrow \dots \leftarrow \xleftarrow{\tau_q} \sigma_q$ in $\mathcal{N}(\widetilde{\Sigma}_s)$ on a collection of elements $(\alpha_1^i, \dots, \alpha_q^i; x^i)$ in $\text{srep}(E)[q]_{p_i}$ where $\alpha_j^i: \mathbf{n}_j^i \rightarrow \mathbf{n}_{j-1}^i$ is a map in \mathcal{I} and x^i is an element in $E(\mathbf{n}_q^i)_{p_i}$. On the indices $(\alpha_1^i, \dots, \alpha_q^i)$ for the sums in the simplicial replacement, we use the action of (τ_1, \dots, τ_q) provided by the previous lemma. As element in $E(\mathbf{n}_q^{\sigma_q^{-1}(1)} \sqcup \dots \sqcup \mathbf{n}_q^{\sigma_q^{-1}(s)})_{p_1+\dots+p_s}$ we take the product $x^{\sigma_q^{-1}(1)} \dots x^{\sigma_q^{-1}(s)}$. Since E is commutative, this does indeed define an operad action in sCh_k . By Propositions 2.16 and 2.17 below, the composite $\text{Tot } C_*$ is lax symmetric monoidal. Hence it follows that \mathcal{E} acts on $E_{h\mathcal{I}}$. \square

2.14 Monoidality of C_* and Tot

It remains to verify the monoidal properties of C_* and Tot that were used in the proof of Theorem 2.13.

DEFINITION 2.15. Let Y and Z be two objects in $\text{Ch}_k(\text{Ch}_k)$. Their tensor product is $Y \otimes Z$ is the object in $\text{Ch}_k(\text{Ch}_k)$ with

$$(Y \otimes Z)_{p,q} = \bigoplus_{a_1+a_2=p} \bigoplus_{b_1+b_2=q} Y_{a_1,b_1} \otimes Z_{a_2,b_2}$$

and differentials $d_h^\otimes(y \otimes z) = d_h(y) \otimes z + (-1)^{a_1} y \otimes d_h(z)$ and $d_v^\otimes(y \otimes z) = d_v(y) \otimes z + (-1)^{b_1} y \otimes d_v(z)$. The symmetry isomorphism $\tau: Y \otimes Z \rightarrow Z \otimes Y$ sends a homogeneous element $y \otimes z \in Y_{a_1,b_1} \otimes Z_{a_2,b_2}$ to $(-1)^{a_1 a_2 + b_1 b_2} z \otimes y$.

PROPOSITION 2.16. *The functor $C_*: \text{sCh}_k \rightarrow \text{Ch}_k(\text{Ch}_k)$ is lax symmetric monoidal.*

Proof. As in [Mac63, Theorem VIII.8.8] we denote (p, q) -shuffles as two disjoint subsets $\mu_1 < \dots < \mu_p$ and $\nu_1 < \dots < \nu_q$ of $\{0, \dots, p + q - 1\}$. For simplicial chain complexes A and B we

define maps

$$\text{sh}_{A,B}: C_*(A) \otimes C_*(B) \rightarrow C_*(A \hat{\otimes} B)$$

that turn C_* into a lax symmetric monoidal functor: if $a \otimes b$ is a homogeneous element in $A_{r_1,r_2} \otimes B_{s_1,s_2}$ we set

$$\text{sh}_{A,B}(a \otimes b) = \sum_{(\mu,\nu)} \text{sgn}(\mu,\nu) s_{\nu_{s_1}} \circ \cdots \circ s_{\nu_1}(a) \otimes s_{\mu_{r_1}} \circ \cdots \circ s_{\mu_1}(b).$$

Here, the sum runs over all (r_1, s_1) -shuffles (μ, ν) and $\text{sgn}(\mu, \nu)$ denotes the signum of the associated permutation.

As the simplicial structure maps of A and B commute with d^A and d^B , it follows that sh commutes with the vertical differential. The proof that the horizontal differential is compatible with sh is the same as for sh in the context of simplicial modules.

It remains to show that sh turns C_* into a lax symmetric monoidal functor, i.e., we have to show that

$$C_*(c) \circ \text{sh}(a \otimes b) = \text{sh} \circ \tau(a \otimes b) \tag{2.5}$$

for any homogeneous element $a \otimes b \in A_{r_1,r_2} \otimes B_{s_1,s_2}$. As $\tau(a \otimes b) = (-1)^{r_1 s_1 + r_2 s_2} b \otimes a$, the right-hand side of equation (2.5) is

$$\sum_{(\xi,\zeta)} (-1)^{r_1 s_1 + r_2 s_2} \text{sgn}(\xi, \zeta) s_{\zeta_{s_1}} \circ \cdots \circ s_{\zeta_1}(b) \otimes s_{\xi_{r_1}} \circ \cdots \circ s_{\xi_1}(a)$$

with (ξ, ζ) being (s_1, r_1) -shuffles, whereas the left-hand side of the equation gives

$$(-1)^{r_2 s_2} \sum_{(\mu,\nu)} \text{sgn}(\mu, \nu) s_{\mu_{r_1}} \circ \cdots \circ s_{\mu_1}(b) \otimes s_{\nu_{s_1}} \circ \cdots \circ s_{\nu_1}(a)$$

because τ introduces the sign $(-1)^{r_2 s_2}$. Precomposing with the permutation that exchanges the blocks $0 < \cdots < r_1 - 1$ and $r_1 < \cdots < r_1 + s_1 - 1$ gives a bijection between the summation indices and introduces the sign $(-1)^{r_1 s_1}$. Hence the two sides agree. \square

PROPOSITION 2.17. *The functor Tot is strong symmetric monoidal.*

Proof. Spelling out what $\text{Tot}(Y) \otimes \text{Tot}(Z)$ is in degree n we obtain

$$(\text{Tot}(Y) \otimes \text{Tot}(Z))_n \cong \bigoplus_{r_1+r_2+s_1+s_2=n} Y_{r_1,r_2} \otimes Z_{s_1,s_2},$$

and we send a homogeneous element $y \otimes z \in Y_{r_1,r_2} \otimes Z_{s_1,s_2}$ to the element

$$(-1)^{r_2 s_1} y \otimes z \in \text{Tot}(Y \otimes Z)_n \cong \bigoplus_{r_1+s_1+r_2+s_2=n} Y_{r_1,r_2} \otimes Z_{s_1,s_2}.$$

This gives isomorphisms

$$\varphi_{Y,Z}: \text{Tot}(Y) \otimes \text{Tot}(Z) \rightarrow \text{Tot}(Y \otimes Z)$$

that are associative. It is clear that Tot respects the unit up to isomorphism.

The maps $\varphi_{Y,Z}$ are compatible with the differential. Let $y \otimes z$ be a homogeneous element in $Y_{r_1,r_2} \otimes Z_{s_1,s_2}$. The composition $d_{\text{Tot}} \circ \varphi$ applied to $y \otimes z$ gives

$$\begin{aligned} d_{\text{Tot}} \circ \varphi(y \otimes z) &= (-1)^{r_2 s_1} d_h^\otimes(y \otimes z) + (-1)^{r_2 s_1} (-1)^{r_1 + s_1} d_v^\otimes(y \otimes z) \\ &= (-1)^{r_2 s_1} d_h(y) \otimes z + (-1)^{r_2 s_1 + r_1} y \otimes d_h(z) \\ &\quad + (-1)^{r_2 s_1 + r_1 + s_1} d_v(y) \otimes z + (-1)^{r_2 s_1 + r_1 + s_1 + r_2} y \otimes d_v(z). \end{aligned}$$

First applying the differential to $y \otimes z$ and then φ yields

$$\begin{aligned} \varphi(d_{\text{Tot}}(y) \otimes z + (-1)^{r_1 + r_2} y \otimes d_{\text{Tot}}(z)) &= \varphi(d_h(y) \otimes z + (-1)^{r_1} d_v(y) \otimes z + (-1)^{r_1 + r_2} y \otimes d_h(z) + (-1)^{r_1 + r_2 + s_1} y \otimes d_v(z)) \\ &= (-1)^{r_2 s_1} d_h(y) \otimes z + (-1)^{r_1 + (r_2 - 1) s_1} d_v(y) \otimes z + (-1)^{r_1 + r_2 + r_2 (s_1 - 1)} y \otimes d_h(z) \\ &\quad + (-1)^{r_1 + r_2 + s_1 + r_2 s_1} y \otimes d_v(z), \end{aligned}$$

and thus both terms agree.

We denote the symmetry isomorphism in the category of chain complexes by χ . Then

$$\varphi \circ \chi(e \otimes f) = \varphi((-1)^{(r_1 + r_2)(s_1 + s_2)} f \otimes e) = (-1)^{r_1 s_1 + r_2 s_2 + s_1 r_2 + 2s_2 r_1} f \otimes e$$

and this is equal to

$$\text{Tot}(\tau) \circ \varphi(e \otimes f) = \text{Tot}(\tau)((-1)^{r_2 s_1} e \otimes f) = (-1)^{r_2 s_1 + r_1 s_1 + r_2 s_2} f \otimes e. \quad \square$$

Remark 2.18. One can also consider a symmetric monoidal structure on $\text{Ch}_k(\text{Ch}_k)$ with the same underlying tensor product but with symmetry isomorphism

$$y \otimes z \mapsto (-1)^{(r_1 + r_2)(s_1 + s_2)} z \otimes y$$

for homogeneous elements $y \otimes z \in Y_{r_1,r_2} \otimes Z_{s_1,s_2}$. Then one can take φ in Proposition 2.17 to be the identity. However, this symmetry isomorphism is *not* compatible with the shuffle transformation from the proof of Proposition 2.16.

Remark 2.19. For a simplicial chain complex A one can also consider a normalized object $N(A) \in \text{Ch}_k(\text{Ch}_k)$ where one divides out by the subobject generated by degenerate elements. As the simplicial structure maps commute with the differential of A , this is well defined, and the proof of Proposition 2.16 can be adapted as in [Mac63, Corollary VIII.8.9] to show that the functor $N: \text{sCh}_k \rightarrow \text{Ch}_k(\text{Ch}_k)$ is also lax symmetric monoidal. Consequently, one can also use N instead of C_* in the definition of the Barratt–Eccles operad \mathcal{E} and the homotopy colimit $P_{h\mathcal{I}}$ so that Theorem 2.13 remains valid.

3. Cochain functors with values in \mathcal{I} -chain complexes

In this section we construct the functor $A^{\mathcal{I}}$ discussed in the introduction and a version of the ordinary cochains with values in \mathcal{I} -chain complexes.

3.1 Adjunctions induced by simplicial objects

We briefly recall an ubiquitous construction principle for adjunctions that we will later apply to simplicial objects in the categories of commutative \mathcal{I} -dgas and \mathcal{I} -chain complexes in order to

define the commutative \mathcal{I} -dga of polynomial forms on a simplicial set and an integral version of the Sullivan realization functor (see Definition 3.6).

CONSTRUCTION 3.2. Let $D_\bullet: \Delta^{\text{op}} \rightarrow \mathcal{D}$ be a simplicial object in a complete category \mathcal{D} . Passing to opposite categories, D_\bullet gives rise to a functor $\tilde{D}^\bullet: \Delta \rightarrow \mathcal{D}^{\text{op}}$. Since \mathcal{D} is complete, \mathcal{D}^{op} is cocomplete. Hence restriction and left Kan extension along $\Delta \rightarrow \text{sSet}, [p] \mapsto \Delta^p$ define an adjunction

$$\tilde{D}: \text{sSet} \rightleftarrows \mathcal{D}^{\text{op}}: K_D.$$

Writing $D: \text{sSet}^{\text{op}} \rightarrow \mathcal{D}$ for the opposite of \tilde{D} , this implies that for a simplicial set X and an object E of \mathcal{D} , we have a natural isomorphism

$$\mathcal{D}(E, D(X)) = \mathcal{D}^{\text{op}}(\tilde{D}(X), E) \cong \text{sSet}(X, K_D(E)) \tag{3.1}$$

exhibiting D and K_D as *contravariant right adjoint* functors. Unraveling definitions, the contravariant functors K_D and D are given by $K_D(E)_\bullet = \mathcal{D}(E, D_\bullet)$ and $D(X) = \lim_{\Delta^p \rightarrow X} D_p$ where the limit is taken over the category of elements of X . In the special case $\mathcal{D} = \text{Set}$, writing X as a colimit of representable functors indexed over its category of elements provides a natural bijection $D(X) \cong \text{sSet}(X, D)$.

The functor D extends the original functor D_\bullet in that there is a natural isomorphism $D_\bullet \cong D(\Delta^\bullet)$. The construction is also functorial in D_\bullet , i.e., a natural transformation $D_\bullet \rightarrow D'_\bullet$ of functors $\Delta^{\text{op}} \rightarrow \mathcal{D}$ induces a natural transformation $D \rightarrow D'$ of functors $\text{sSet}^{\text{op}} \rightarrow \mathcal{D}$.

We note an immediate consequence of having the adjunction (\tilde{D}, K_D) .

LEMMA 3.3. *The functor D takes colimits in sSet to limits in \mathcal{D} , and K_D takes colimits in \mathcal{D} to limits in sSet .*

When $D_\bullet: \Delta^{\text{op}} \rightarrow \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ is a simplicial object in commutative \mathcal{I} -dgas, we may apply Construction 3.2 both to D_\bullet and to its composite $D'_\bullet = UD_\bullet$ with the forgetful functor $U: \text{Ch}_k^{\mathcal{I}}[\mathcal{C}] \rightarrow \text{Ch}_k^{\mathcal{I}}$. Since the extensions of D_\bullet and D'_\bullet to functors on sSet are defined by limit constructions and U commutes with limits, we have a natural isomorphism $U(D(X)) \cong D'(X)$ for a simplicial set X . The adjoints K_D and $K_{D'}$ are related by a natural isomorphism $K_{D'} \cong K_D \circ \mathbb{C}: (\text{Ch}_k^{\mathcal{I}})^{\text{op}} \rightarrow \text{sSet}$. An analogous remark applies to simplicial objects of algebras in $\text{Ch}_k^{\mathcal{I}}$ over a more general operad than the commutativity operad.

For $D_\bullet: \Delta^{\text{op}} \rightarrow \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$, the fact that $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}] \rightarrow \text{Set}, E \mapsto E(\mathbf{m})_q$ commutes with limits implies that the underlying set of $D(X)(\mathbf{m})_q$ is $\text{sSet}(X, D_\bullet(\mathbf{m})_q)$. The pointwise k -module structure, differentials and multiplications on these sets give rise to the commutative \mathcal{I} -dga structure on $D(X)$.

3.4 The commutative \mathcal{I} -dga version of polynomial forms

Composing the left adjoints in the adjunctions $(F_1^{\mathcal{I}}, \text{Ev}_1)$ and (\mathbb{C}, U) introduced in (2.1) and (2.2) provides a left adjoint $\mathbb{C}F_1^{\mathcal{I}}: \text{Ch}_k \rightarrow \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ made explicit in (2.3). We are particularly interested in the commutative \mathcal{I} -dga $\mathbb{C}F_1^{\mathcal{I}}(D^0)$. For an element $i \in k$, the k -module map $(D^0)_0 = k \rightarrow k = \text{Ev}_1(U^{\mathcal{I}})_0$ determined by $1 \mapsto i$ gives rise to a map $\varepsilon_i: \mathbb{C}F_1^{\mathcal{I}}(D^0) \rightarrow U^{\mathcal{I}}$. We write $U_0^{\mathcal{I}}$ and $U_1^{\mathcal{I}}$ for the two commutative $\mathbb{C}F_1^{\mathcal{I}}(D^0)$ -algebras resulting from the elements $0, 1 \in k$.

DEFINITION 3.5. We let $A_{\bullet}^{\mathcal{I}}: \Delta^{\text{op}} \rightarrow \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ be the simplicial commutative \mathcal{I} -dga given by the two-sided bar construction

$$[p] \mapsto A_p^{\mathcal{I}} = B_p(U_0^{\mathcal{I}}, \mathbb{C}F_1^{\mathcal{I}}(D^0), U_1^{\mathcal{I}}) = U_0^{\mathcal{I}} \boxtimes \mathbb{C}F_1^{\mathcal{I}}(D^0)^{\boxtimes p} \boxtimes U_1^{\mathcal{I}}. \tag{3.2}$$

As with the space-level version (see e.g. [May72]), the outer face maps are provided by the module structures of $U_0^{\mathcal{I}}$ and $U_1^{\mathcal{I}}$ resulting from the above algebra structures, the inner face maps come from the multiplication of $\mathbb{C}F_1^{\mathcal{I}}(D^0)$, and the degeneracy maps are induced by its unit.

To make this simplicial object more explicit, we write D_r^0 for the chain complex with copies of k on generators r in degree 0 and on dr in degree -1 and 0 elsewhere. Its non-zero differential is $d(a \cdot r) = a \cdot dr$. Since $\mathbb{C}F_1^{\mathcal{I}}$ is left adjoint and $U^{\mathcal{I}}$ is the unit for \boxtimes , commuting $\mathbb{C}F_1^{\mathcal{I}}$ with coproducts provides an isomorphism of commutative \mathcal{I} -dgas

$$A_p^{\mathcal{I}} \cong \mathbb{C}F_1^{\mathcal{I}}(D_{r_1(p)}^0 \oplus \cdots \oplus D_{r_p(p)}^0)$$

where the generators $r_1(p), \dots, r_p(p)$ correspond to the p copies of $\mathbb{C}F_1^{\mathcal{I}}(D^0)$. By adjunction, maps $f: \mathbb{C}F_1^{\mathcal{I}}(D_{r_1(p)}^0 \oplus \cdots \oplus D_{r_p(p)}^0) \rightarrow E$ in $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ correspond to families of elements $f(r_1(p)), \dots, f(r_p(p)) \in E(\mathbf{1})_0$.

We now set $r_0(p) = 0$ and define $r_{p+1}(p)$ to be the image of 1 under the map

$$k = U^{\mathcal{I}}(\mathbf{1})_0 \rightarrow \mathbb{C}F_1^{\mathcal{I}}(D_{r_1(p)}^0 \oplus \cdots \oplus D_{r_p(p)}^0)(\mathbf{1})_0$$

induced by the unit. With this notation, the simplicial structure maps of the two-sided bar construction (3.2) are determined by requiring

$$d_i(r_j(p)) = \begin{cases} r_j(p-1) & \text{if } j \leq i, \\ r_{j-1}(p-1) & \text{if } j > i, \end{cases} \quad s_i(r_j(p)) = \begin{cases} r_j(p+1) & \text{if } j \leq i, \\ r_{j+1}(p+1) & \text{if } j > i. \end{cases} \tag{3.3}$$

Applying Construction 3.2, we obtain the following pair of adjoint functors.

DEFINITION 3.6.

- (i) The commutative \mathcal{I} -dga of polynomial forms on a simplicial set X , $A^{\mathcal{I}}(X)$, is defined as

$$A^{\mathcal{I}}(X) = \text{sSet}(X, A_{\bullet}^{\mathcal{I}}).$$

This defines a functor $A^{\mathcal{I}}: \text{sSet}^{\text{op}} \rightarrow \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$.

- (ii) Its adjoint functor $\langle - \rangle_{\mathcal{I}}: \text{Ch}_k^{\mathcal{I}}[\mathcal{C}]^{\text{op}} \rightarrow \text{sSet}$ sends a commutative \mathcal{I} -dga E to

$$\langle E \rangle_{\mathcal{I}} = \text{Ch}_k^{\mathcal{I}}[\mathcal{C}](E, A_{\bullet}^{\mathcal{I}}).$$

The simplicial set $\langle E \rangle_{\mathcal{I}}$ is the Sullivan realization of E .

For a simplicial k -module $Z: \Delta^{\text{op}} \rightarrow \text{Mod}_k$, extra degeneracies are a family of k -linear maps $s_{p+1}: Z_p \rightarrow Z_{p+1}$ satisfying: $d_{p+1}s_{p+1} = \text{id}_{Z_p}$ if $p \geq 0$; $d_i s_{p+1} = s_p d_i: Z_p \rightarrow Z_p$ if $p \geq 1$ and $0 \leq i \leq p$; and $0 = d_0 s_1: Z_0 \rightarrow Z_0$. The presence of extra degeneracies implies that Z is contractible to 0 (in the sense that $Z \rightarrow 0$ is a weak equivalence in sMod_k) since the maps $(-1)^{p+1} s_{p+1}$ define a contracting homotopy for the chain complex $C_*(Z)$.

The following lemma is the technical backbone for our homotopical analysis of the prolongation $A^{\mathcal{I}}$ of $A_{\bullet}^{\mathcal{I}}$ in §5. It is analogous to [BG76, Proposition 1.1].

LEMMA 3.7. Let \mathbf{m} be an object of \mathcal{I} with $m = |\mathbf{m}| \geq 1$. Then for all integers q satisfying $0 \geq q > -m$, the simplicial k -module $A_{\bullet}^{\mathcal{I}}(\mathbf{m})_q$ is contractible to 0.

Remark 3.8. The statement of the lemma does not hold for general \mathbf{m} and q . The easiest case is $\mathbf{m} = \mathbf{0}$ and $q = 0$ where $A_{\bullet}^{\mathcal{I}}(\mathbf{0})_0$ is the constant simplicial object on k , which is not contractible. One can also show that $\pi_1(A_{\bullet}^{\mathcal{I}}(\mathbf{1})_{-1})$ is non-trivial.

Proof of Lemma 3.7. Let A be a chain complex. The canonical bijection

$$\mathcal{I}(\mathbf{1}^{\sqcup s}, \mathbf{m})/\Sigma_s \rightarrow \{T \subseteq \mathbf{m} \mid |T| = s\}, \quad [\alpha] \mapsto \text{im}(\alpha)$$

induces natural isomorphisms

$$\mathbb{C}(F_1^{\mathcal{I}}(A))(\mathbf{m}) = \bigoplus_{s \geq 0} \left(\bigoplus_{\mathcal{I}(\mathbf{1}^{\sqcup s}, \mathbf{m})} A^{\otimes s}/\Sigma_s \right) \cong \bigoplus_{T \subseteq \mathbf{m}} A^{\otimes T} \cong (A \oplus S^0)^{\otimes \mathbf{m}}. \tag{3.4}$$

Here the last isomorphism sends the tensor power indexed by T to an iterated tensor product of copies of A and S^0 with copies of A placed at the entries indexed by T .

The isomorphism (3.4) specializes to an isomorphism

$$A_p^{\mathcal{I}}(\mathbf{m})_q \cong \left(\left(D_{r_1(p)}^0 \oplus \cdots \oplus D_{r_p(p)}^0 \oplus S_{r_{p+1}(p)}^0 \right)^{\otimes \mathbf{m}} \right)_q$$

where we now write $r_{p+1}(p)$ for the generator of $(S^0)_0$. Under this identification, the simplicial structure maps of $[p] \mapsto A_p^{\mathcal{I}}(\mathbf{m})_q$ are again determined by (3.3).

As a first step, we now notice that $A_{\bullet}^{\mathcal{I}}(\mathbf{1})_0$ is contractible since $s_{p+1}(r_j(p)) = r_j(p+1)$ defines extra degeneracies for this simplicial object. For the case of a general \mathbf{m} and $0 \geq q > -m$, we notice that the above isomorphisms induce an isomorphism of simplicial objects

$$A_p^{\mathcal{I}}(\mathbf{m})_q \cong \bigoplus_{\substack{q_1 + \cdots + q_m = q \\ q_i = 0, -1}} A_p^{\mathcal{I}}(\mathbf{1})_{q_1} \otimes \cdots \otimes A_p^{\mathcal{I}}(\mathbf{1})_{q_m}.$$

Since $q > -m$, each summand has at least one tensor factor that is of chain complex degree 0 and thus contractible by the previous step. Since the shuffle map is a chain homotopy equivalence [Mac63, Theorem VIII.8.1], it follows that each summand and thus the whole sum is contractible. □

3.9 Ordinary cochains

Let $C(X; k)$ be the cochains with values in k on the simplicial set X , viewed as a homologically graded chain complex concentrated in non-positive degrees. (At this point, we disregard its cup product structure.) So for $q \geq 0$, we have $C(X; k)_{-q} = \text{Set}(X_q, k)$ with the pointwise k -module structure and differential induced by the face maps of X . The cochains on the standard n -simplices assemble to a functor $C_{\bullet}: \Delta^{\text{op}} \rightarrow \text{Ch}_k, [p] \mapsto C(\Delta^p; k)$. The following lemma is well known (see e.g. [FHT01, Lemmas 10.11 and 10.12(ii)]).

LEMMA 3.10.

- (i) The extension of C_\bullet to a functor $\text{sSet}^{\text{op}} \rightarrow \text{Ch}_k$ resulting from Construction 3.2 is naturally isomorphic to $C(-; k)$.
- (ii) For all $q \in \mathbb{Z}$, the simplicial k -module $C_{\bullet, q} = C(\Delta^\bullet; k)_q$ is contractible to 0.

Proof. For (i), we note that the description of the extension as $\lim_{\Delta^p \rightarrow X} C(\Delta^p; k)$ implies that there is a natural map from $C(X; k)$. Writing X as a colimit of representable functors over its category of elements, the evaluation of this map at q is a bijection since taking maps into k turns colimits into limits.

For (ii), we only need to consider the case $q \leq 0$, set $n = -q$ and define

$$s_{p+1}: C(\Delta^p; k)_q \rightarrow C(\Delta^{p+1}; k)_q$$

on $f: (\Delta^p)_n \rightarrow k$ as follows: We set $s_{p+1}(f): (\Delta^{p+1})_n \rightarrow k$ to be 0 on all n -simplices not in the image of $d^{p+1}: \Delta^p \rightarrow \Delta^{p+1}$ and require that $s_{p+1}(f)$ restricts to f on the last face. Identifying Δ_n^{p+1} with $\Delta([n], [p+1])$, this means that $s_{p+1}(f)(d^{p+1}\alpha') = f(\alpha')$ and $s_{p+1}(f)(\alpha) = 0$ if $p+1 \in \alpha([n])$. Then for $\beta: [n] \rightarrow [p]$, the equation $d_{p+1}(s_{p+1}(f))(\beta) = \beta$ holds by definition, and $d_0s_1 = 0$ in simplicial degree 0 is also immediate. Now assume $p \geq 1$. If β has p in its image, then $d_i s_{p+1}(f)(\beta) = 0 = s_p d_i(f)(\beta)$. Otherwise, we must have $\beta = d^p \beta'$ and thus

$$\begin{aligned} d_i s_{p+1}(f)(d^p \beta') &= s_{p+1}(f)(d^i d^p \beta') = s_{p+1}(f)(d^{p+1} d^i \beta') \\ &= f(d^i \beta') = (d_i f)(\beta') = s_p d_i(f)(d^p \beta'). \end{aligned} \quad \square$$

For later use, we lift C_\bullet to \mathcal{I} -chain complexes by defining

$$C_\bullet^\mathcal{I}: \Delta^{\text{op}} \rightarrow \text{Ch}_k^\mathcal{I}, \quad [p] \mapsto F_0^\mathcal{I}(C(\Delta^p; k)).$$

COROLLARY 3.11.

- (i) The extension $C^\mathcal{I}$ of $C_\bullet^\mathcal{I}$ to a functor $\text{sSet}^{\text{op}} \rightarrow \text{Ch}_k^\mathcal{I}$ resulting from Construction 3.2 is naturally isomorphic to $X \mapsto F_0^\mathcal{I}C(X; k)$.
- (ii) For all $q \in \mathbb{Z}$ and \mathbf{m} in \mathcal{I} , the simplicial k -module $C_{\bullet, q}^\mathcal{I}(\mathbf{m}) = F_0^\mathcal{I}(C(\Delta^\bullet; k)_q)$ is contractible to 0.

4. Homotopy theory of \mathcal{I} -chain complexes and commutative \mathcal{I} -dgas

In this section we review and set up results about model category structures on \mathcal{I} -chain complexes and commutative \mathcal{I} -dgas. Much of this is motivated by (and analogous to) the corresponding results for space valued functors developed in [SS12, § 3].

We continue to consider the category of unbounded chain complexes Ch_k and equip it with the projective model structure whose weak equivalences are the quasi-isomorphisms and whose fibrations are the level-wise surjections [Hov99, Theorem 2.3.11]. It has the inclusions $S^{q-1} \hookrightarrow D^q$ as generating cofibrations and the maps $0 \rightarrow D^q$ as generating acyclic cofibrations. We will also need the following variant of this model structure.

PROPOSITION 4.1. *Let s be an integer. Then Ch_k admits an s -truncated model structure where a map $f: A \rightarrow B$ is a weak equivalence if $H_q(f): H_q(A) \rightarrow H_q(B)$ is an isomorphism for all $q \geq s$*

and a fibration if $f_q: A_q \rightarrow B_q$ is an epimorphism for all $q > s$. The s -truncated model structure is combinatorial and right proper.

Proof. By shifting it is enough to consider the case $s = 0$. The smart truncation

$$\tau_{\geq 0}: \text{Ch}_k \rightarrow \text{Ch}_k^{\geq 0}, \quad A \mapsto (\cdots \rightarrow A_2 \rightarrow A_1 \rightarrow \ker d_0^A)$$

to non-negatively graded chain complexes is right adjoint to the functor that adds copies of 0 in negative degrees. The desired model structure arises by applying [Hir03, Theorem 11.3.2] to this adjunction and the standard projective model structure on $\text{Ch}_k^{\geq 0}$ [DS95, §7]. The assumptions of the theorem are trivially satisfied. The resulting model structure is combinatorial since Ch_k is, and right proper because all objects are fibrant. \square

Remark 4.2. Since the usual long exact sequence argument is not applicable, we do not know if the s -truncated model structure is left proper. We do not investigate this further since it is not relevant for our applications.

4.3 Level model structures

We call an object \mathbf{m} of \mathcal{I} *positive* if $|\mathbf{m}| \geq 1$ and write \mathcal{I}_+ for the full subcategory of positive objects in \mathcal{I} . To ease notation, we write m for the cardinality of $\mathbf{m} = \{1, \dots, m\}$.

A map $f: P \rightarrow Q$ in $\text{Ch}_k^{\mathcal{I}}$ is an *absolute level equivalence* (respectively *absolute level fibration*) if $f(\mathbf{m})$ is a quasi-isomorphism (respectively a fibration) in Ch_k for all \mathbf{m} in \mathcal{I} . A map $f: P \rightarrow Q$ in $\text{Ch}_k^{\mathcal{I}}$ is a *descending level equivalence* (respectively *descending level fibration*) if for all \mathbf{m} in \mathcal{I}_+ , the map $f(\mathbf{m})$ is a weak equivalence (respectively fibration) in the $-m$ -truncated model structure on Ch_k .

PROPOSITION 4.4. *These maps define an absolute level and a descending level model structure on $\text{Ch}_k^{\mathcal{I}}$. Both model structures are combinatorial and right proper, and the absolute level model structure is in addition left proper.*

Proof. For integers $s \leq t$, the identity functor is a left Quillen functor from the t -truncated model structure to the s -truncated model structure since s -truncated weak equivalences (respectively fibrations) are t -truncated weak equivalences (respectively fibrations). To obtain the descending level model structure, we can therefore apply [HSS20, Proposition 3.10] to the constant functor $\mathcal{C}: \mathcal{I} \rightarrow \text{Cat}$ with value Ch_k where $\mathcal{C}(\mathbf{m})$ is equipped with the $-m$ -truncated model structure. The absolute level model structure arises by considering the same functor where $\mathcal{C}(\mathbf{m})$ carries the standard projective model structure for all \mathbf{m} . \square

The cofibrations in the absolute level model structure are the retracts of relative cell complexes built out of cells of the form $F_{\mathbf{m}}^{\mathcal{I}}(S^{q-1} \hookrightarrow D^q)$ with \mathbf{m} in \mathcal{I} and $q \in \mathbb{Z}$. Here $F_{\mathbf{m}}^{\mathcal{I}}$ is the free functor defined in (2.1). For the descending level model structure, it follows from the proof of the previous proposition that one may use

$$\{F_{\mathbf{m}}^{\mathcal{I}}(S^{q-1} \hookrightarrow D^q) \mid \mathbf{m} \in \mathcal{I}_+, q > -m\} \cup \{F_{\mathbf{m}}^{\mathcal{I}}(0 \hookrightarrow S^{-m}) \mid \mathbf{m} \in \mathcal{I}_+\}$$

as the set of generating cofibrations and

$$\{F_{\mathbf{m}}^{\mathcal{I}}(0 \hookrightarrow D^q) \mid \mathbf{m} \in \mathcal{I}_+, q > -m\}$$

as the set of generating acyclic cofibrations.

4.5 \mathcal{I} -model structures

We now again use the homotopy colimit $P_{h\mathcal{I}}$ from Construction 2.6. A map $P \rightarrow Q$ in $\text{Ch}_k^{\mathcal{I}}$ is an \mathcal{I} -equivalence if it induces a quasi-isomorphism $P_{h\mathcal{I}} \rightarrow Q_{h\mathcal{I}}$. An \mathcal{I} -chain complex P is *absolute \mathcal{I} -fibrant* if $\alpha_*: P(\mathbf{m}) \rightarrow P(\mathbf{n})$ is a quasi-isomorphism for all $\alpha: \mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{I} . It is *descending \mathcal{I} -fibrant* if for all $\alpha: \mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{I}_+ , the map $\alpha_*: P(\mathbf{m}) \rightarrow P(\mathbf{n})$ is a weak equivalence in the $-m$ -truncated model structure, that is, if for all $q \geq -m$, the map $H_q(\alpha_*): H_q(P(\mathbf{m})) \rightarrow H_q(P(\mathbf{n}))$ is an isomorphism.

The next lemma will be needed to identify the \mathcal{I} -equivalences as part of a descending model structure.

LEMMA 4.6. *A map $P \rightarrow Q$ between descending \mathcal{I} -fibrant objects in $\text{Ch}_k^{\mathcal{I}}$ is an \mathcal{I} -equivalence if and only if it is a descending level equivalence.*

Proof. For each positive \mathbf{m} we consider the following commutative diagram.

$$\begin{CD} P(\mathbf{m}) @>>> \text{hocolim}_{\mathcal{I}_{\geq m}} P @>\sim>> P_{h\mathcal{I}} \\ @VVV @VVV @VVV \\ Q(\mathbf{m}) @>>> \text{hocolim}_{\mathcal{I}_{\geq m}} Q @>\sim>> Q_{h\mathcal{I}} \end{CD}$$

The right hand horizontal maps are induced by the inclusion $\mathcal{I}_{\geq m} \rightarrow \mathcal{I}$ of the full subcategory of objects of cardinality at least m . Since this inclusion is homotopy cofinal (compare [SS12, Corollary 5.9]), they are quasi-isomorphisms. The left hand horizontal maps are induced by the inclusion of the object \mathbf{m} in $\mathcal{I}_{\geq m}$. They are $-m$ -truncated equivalences by [Dug01, Proposition 5.4] because the restrictions of P and Q to $\mathcal{I}_{\geq m}$ are diagrams of $-m$ -truncated equivalences. The claim then follows by two-out-of-three and the fact that a map is a quasi-isomorphism if and only if it is an s -truncated equivalence for all $s < 0$. □

Using this lemma, we can build the desired model structures.

PROPOSITION 4.7. *The absolute (respectively descending) level model structure on $\text{Ch}_k^{\mathcal{I}}$ admits a left Bousfield localization with fibrant objects the absolute (respectively descending) \mathcal{I} -fibrant objects. The weak equivalences in these two model structures coincide, and they are given by the \mathcal{I} -equivalences. Both model structures are left proper and combinatorial.*

Proof. For the absolute case, we apply [Dug01, Theorem 5.2]. Since the weak equivalences in the latter case are the maps that induce weak equivalences on the corrected homotopy colimits, they coincide with the \mathcal{I} -equivalences. The resulting model structure is left proper and combinatorial since these properties are preserved by left Bousfield localization.

Let \mathcal{N} be the subcategory of \mathcal{I} given by the subset inclusions and let P be an \mathcal{I} -chain complex. Since the homotopy colimit over \mathcal{N} is equivalent to the colimit of a suitable cofibrant replacement of the underlying \mathcal{N} -diagram, there is a natural isomorphism $\text{colim}_{\mathbf{m} \in \mathcal{N}} H_q(P(\mathbf{m})) \cong H_q(\text{hocolim}_{\mathbf{m} \in \mathcal{N}} P(\mathbf{m}))$. Hence any descending level equivalence $P \rightarrow Q$ induces a quasi-isomorphism $\text{hocolim}_{\mathcal{N}} P \rightarrow \text{hocolim}_{\mathcal{N}} Q$ and thus an \mathcal{I} -equivalence by [Shi00, Proposition 2.2.9]. So the weak equivalences of the descending level model structure are contained in those of the absolute \mathcal{I} -model structure, and the same is by definition true for the cofibrations. We can thus apply [Col06, Theorem 2.1] to obtain a model structure that has the weak equivalences and cofibrations of the second model structure to be constructed. Using Lemma 4.6, an object P is

descending \mathcal{I} -fibrant if and only if every absolute \mathcal{I} -fibrant replacement $P \rightarrow P'$ is a descending level equivalence. Hence the dual of [Col06, Proposition 3.6] shows that the fibrant objects of the second model structure are the descending \mathcal{I} -fibrant objects. The model structure is combinatorial by [Bar10, Lemma 4.6 and Proposition 2.2] and left proper since the absolute \mathcal{I} -model structure is. \square

We call these two model structures the *absolute* and *descending \mathcal{I} -model structures* on $\text{Ch}_k^{\mathcal{I}}$ and their fibrations *absolute* and *descending \mathcal{I} -fibrations*.

Remark 4.8. Analogous to [SS12, Proposition 6.16], \mathcal{I} -model structures on chain complexes can be constructed without relying on an abstract existence theorem for left Bousfield localizations. This has been done by Joachimi [Joa11] in the absolute case, and we expect that similar arguments apply in the descending case. The advantage of the direct approach is that it provides an explicit characterization of the \mathcal{I} -fibrations by a homotopy pullback condition like in [SS12, Proposition 3.2].

COROLLARY 4.9. *A map between descending \mathcal{I} -fibrant objects is a descending level fibration if and only if it is a descending \mathcal{I} -fibration.*

Proof. This follows from [Hir03, Proposition 3.3.16]. \square

The following useful consequence of Lemma 4.6 was already used in the proof of Proposition 4.7.

COROLLARY 4.10. *If P is descending \mathcal{I} -fibrant in $\text{Ch}_k^{\mathcal{I}}$, then a fibrant replacement $P \rightarrow P'$ in the absolute \mathcal{I} -model structure is a descending level equivalence.*

We also note that the homology groups of $P_{h\mathcal{I}}$ can be read off from a fibrant object P in the following way.

LEMMA 4.11. *If P is absolute \mathcal{I} -fibrant in $\text{Ch}_k^{\mathcal{I}}$ and \mathbf{m} is any object in \mathcal{I} , then the canonical map $P(\mathbf{m}) \rightarrow P_{h\mathcal{I}}$ is a quasi-isomorphism. If P is only descending \mathcal{I} -fibrant, then the induced map $H_q(P(\mathbf{m})) \rightarrow H_q(P_{h\mathcal{I}})$ is an isomorphism when \mathbf{m} is positive and $q \geq -m$.*

Proof. The absolute case follows from [Dug01, Proposition 5.4] since \mathcal{I} has contractible classifying space. With Corollary 4.10, the claim for P descending \mathcal{I} -fibrant follows from the absolute case. \square

As another consequence of [Dug01, Theorem 5.2], we note that the adjunction $\text{colim}_{\mathcal{I}}: \text{Ch}_k^{\mathcal{I}} \rightleftarrows \text{Ch}_k: \text{const}_{\mathcal{I}}$ is a Quillen equivalence when $\text{Ch}_k^{\mathcal{I}}$ is equipped with the absolute or descending \mathcal{I} -model structure. In particular, the composite of

$$(\text{const}_{\mathcal{I}}A)_{h\mathcal{I}} \rightarrow \text{colim}_{\mathcal{I}} \text{const}_{\mathcal{I}}A \rightarrow A \tag{4.1}$$

is always a quasi-isomorphism, and each P in $\text{Ch}_k^{\mathcal{I}}$ is related by a zig-zag of \mathcal{I} -equivalences

$$\text{const}_{\mathcal{I}} \text{colim}_{\mathcal{I}}(P^{\text{cof}}) \leftarrow P^{\text{cof}} \rightarrow P \tag{4.2}$$

to a constant \mathcal{I} -diagram $\text{colim}_{\mathcal{I}}(P)^{\text{cof}}$ where $P^{\text{cof}} \rightarrow P$ is a cofibrant replacement.

We record the following lemma for later use.

LEMMA 4.12. *If $(P_j)_{j \in J}$ is a family of \mathcal{I} -chain complexes, then the canonical map*

$$\left(\prod_{j \in J} P_j \right)_{h\mathcal{I}} \rightarrow \prod_{j \in J} (P_j)_{h\mathcal{I}}$$

is a quasi-isomorphism provided that all the P_j are descending \mathcal{I} -fibrant.

Proof. Arbitrary products of weak equivalences between fibrant objects in a model category are weak equivalences. Therefore, using that (4.2) is a zig-zag of \mathcal{I} -equivalences between descending \mathcal{I} -fibrant objects under our assumptions allows us to assume that each P_j is of the form $\text{const}_{\mathcal{I}} A_j$. Forming the adjoint of the isomorphism $\prod_{j \in J} \text{const}_{\mathcal{I}} A_j \xrightarrow{\cong} \text{const}_{\mathcal{I}}(\prod_{j \in J} A_j)$ under the Quillen equivalence $(\text{colim}_{\mathcal{I}}, \text{const}_{\mathcal{I}})$ shows that the composite in

$$\left(\prod_{j \in J} \text{const}_{\mathcal{I}} A_j \right)_{h\mathcal{I}} \rightarrow \prod_{j \in J} (\text{const}_{\mathcal{I}} A_j)_{h\mathcal{I}} \xrightarrow{\sim} \prod_{j \in J} A_j$$

is a quasi-isomorphism. Since the second map is a product of quasi-isomorphisms, the claim follows by two-out-of-three. □

4.13 Commutative \mathcal{I} -dgas

Although essentially only our formulation of Theorem 1.5 depends on the existence of a lifted model structure on $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$, the following result is the main motivation for working with commutative \mathcal{I} -dgas.

THEOREM 4.14. *The category $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ admits a descending \mathcal{I} -model structure where a map is a weak equivalence (or fibration) if the underlying map in the descending \mathcal{I} -model structure on $\text{Ch}_k^{\mathcal{I}}$ is. With this model structure, $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ is Quillen equivalent to the category of E_{∞} dgas and to the category of commutative Hk -algebra spectra.*

Proof. The identification of \mathcal{I} -diagrams with generalized symmetric spectra (see [RS17, Proposition 9.1] or [PS19, § 3.2]) and [PS19, Propositions 3.2.2 and 3.3.1] imply that $\text{Ch}_k^{\mathcal{I}}$ admits a positive \mathcal{I} -model structure with weak equivalences the \mathcal{I} -equivalences. This model structure has more cofibrations than the descending \mathcal{I} -model structure and less cofibrations than the absolute \mathcal{I} -model structure. The positive \mathcal{I} -model structure lifts to $\text{Ch}_k^{\mathcal{I}}$ by [PS19, Theorem 4.1], and the resulting model category is Quillen equivalent to commutative Hk -algebra spectra and to E_{∞} dgas [RS17, Theorem 9.5, Corollary 8.3].

The descending \mathcal{I} -model structure on $\text{Ch}_k^{\mathcal{I}}$ is left proper, combinatorial, and has less cofibrations than the positive one. Hence the existence of the descending \mathcal{I} -model structure on $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ and the Quillen equivalence to the positive \mathcal{I} -model structure immediately follow from [Hir03, Theorem 11.3.2]. □

The equivalence of homotopy categories resulting from this theorem is actually induced by the homotopy colimit over \mathcal{I} with the \mathcal{E} -action from Theorem 2.13.

PROPOSITION 4.15. *The functor $(-)_h\mathcal{I}: \text{Ch}_k^{\mathcal{I}}[\mathcal{C}] \rightarrow \text{Ch}_k[\mathcal{E}]$ induces an equivalence of categories $\text{Ho}(\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]) \rightarrow \text{Ho}(\text{Ch}_k[\mathcal{E}])$.*

Proof. An \mathcal{I} -chain complex X admits a *bar resolution* $\overline{X} \rightarrow X$ defined by $\overline{X}(\mathbf{n}) = \text{hocolim}_{\mathcal{I} \downarrow \mathbf{n}}(X \circ \pi)$ where $\pi: \mathcal{I} \downarrow \mathbf{n} \rightarrow \mathcal{I}$ is the canonical projection from the overcategory forgetting the augmentation to \mathbf{n} . The inclusion of the terminal object in $\mathcal{I} \downarrow \mathbf{n}$ induces a map $\overline{X} \rightarrow X$ which is a level equivalence by a homotopy cofinality argument. The bar resolution has the property $\text{colim}_{\mathcal{I}} \overline{X} \cong X_{h\mathcal{I}}$. When M is an \mathcal{E} -algebra in $\text{Ch}_k^{\mathcal{I}}$, then \overline{M} inherits an \mathcal{E} -algebra structure with diagonal \mathcal{E} -action (compare Theorem 2.13 and the analogous space-level statement in [Sch09, Lemma 6.7]). When M is a commutative \mathcal{I} -dga, then the \mathcal{E} -algebra structure on $\text{colim}_{\mathcal{I}}(\overline{M})$ resulting from this observation and the strong monoidality of $\text{colim}_{\mathcal{I}}$ coincides with the one on $M_{h\mathcal{I}}$ provided by Theorem 2.13. We also note that if X is a cofibrant \mathcal{I} -chain complex, then the map $\text{colim}_{\mathcal{I}} \overline{X} \rightarrow \text{colim}_{\mathcal{I}} X$ is a quasi-isomorphism. This can be checked directly on free \mathcal{I} -chain complexes, and the general case follows because both sides preserve colimits and send generating cofibrations to levelwise injections.

To prove the proposition, we note that the chain of Quillen equivalences from Theorem 4.14 sends a commutative \mathcal{I} -dga M to $\text{colim}_{\mathcal{I}} M^{\text{cof}}$, the colimit over \mathcal{I} of a cofibrant replacement of M in $\text{Ch}_k^{\mathcal{I}}[\mathcal{E}]$. This colimit is related to $M_{h\mathcal{I}}$ by a natural zig-zag of \mathcal{E} -algebra maps

$$M_{h\mathcal{I}} \leftarrow (M^{\text{cof}})_{h\mathcal{I}} \xrightarrow{\cong} \text{colim}_{\mathcal{I}} \overline{M^{\text{cof}}} \rightarrow \text{colim}_{\mathcal{I}} M^{\text{cof}}$$

where the first map is a quasi-isomorphism since the cofibrant replacement is an \mathcal{I} -equivalence and the last map is a quasi-isomorphism by the above discussion since M^{cof} is a cofibrant \mathcal{I} -chain complex by [PS19, Theorem 4.4]. □

For later use we note that the commutative \mathcal{I} -dga $\mathbb{C}F_1^{\mathcal{I}}(A)$ from (2.3) has the following homotopical feature.

LEMMA 4.16. *Let A be a cofibrant acyclic chain complex. Then each $(\mathbb{C}F_1^{\mathcal{I}}(A))(\mathbf{m})$ is cofibrant in Ch_k , and the unit $U^{\mathcal{I}} \rightarrow \mathbb{C}F_1^{\mathcal{I}}(A)$ is an absolute level equivalence.*

Proof. This is an immediate consequence of the isomorphism (3.4). □

5. Comparison of cochain functors

We now define a simplicial \mathcal{I} -chain complex $B_{\bullet}^{\mathcal{I}}$ by setting $B_p^{\mathcal{I}} = A_p^{\mathcal{I}} \boxtimes C_p^{\mathcal{I}}$ in simplicial level p and using the \boxtimes -products of the simplicial structure maps of $A^{\mathcal{I}}$ and $C^{\mathcal{I}}$ as simplicial structure maps for $B_{\bullet}^{\mathcal{I}}$. There is a natural isomorphism

$$B_p^{\mathcal{I}}(\mathbf{m}) = (A_p^{\mathcal{I}} \boxtimes F_0^{\mathcal{I}}(C_p))(\mathbf{m}) \cong A_p^{\mathcal{I}}(\mathbf{m}) \otimes C_p \tag{5.1}$$

that results from the definition of \boxtimes as a left Kan extension.

The unit maps $U^{\mathcal{I}} \rightarrow C^{\mathcal{I}}$ and $U^{\mathcal{I}} \rightarrow A^{\mathcal{I}}$ induce a chain

$$A_{\bullet}^{\mathcal{I}} \rightarrow B_{\bullet}^{\mathcal{I}} \leftarrow C_{\bullet}^{\mathcal{I}} \tag{5.2}$$

of maps of simplicial objects in $\text{Ch}_k^{\mathcal{I}}$. By Construction 3.2, this chain gives rise to a chain of natural transformations $A^{\mathcal{I}} \rightarrow B^{\mathcal{I}} \leftarrow C^{\mathcal{I}}$ of functors $(\text{sSet})^{\text{op}} \rightarrow \text{Ch}_k^{\mathcal{I}}$.

THEOREM 5.1. *For every simplicial set X , the maps $A^{\mathcal{I}}(X) \rightarrow B^{\mathcal{I}}(X) \leftarrow C^{\mathcal{I}}(X)$ are descending level equivalences between descending \mathcal{I} -fibrant objects.*

We prove the theorem at the end of the section. The definition of $B^{\mathcal{I}}$ and our strategy of proof are motivated by the corresponding rational result in [FHT01, § 10].

COROLLARY 5.2. *If $X \rightarrow Y$ is a weak homotopy equivalence of simplicial sets, then $A^{\mathcal{I}}(Y) \rightarrow A^{\mathcal{I}}(X)$ is a descending level equivalence between descending \mathcal{I} -fibrant objects.*

Proof. The map $C^{\mathcal{I}}(Y) \rightarrow C^{\mathcal{I}}(X)$ is an \mathcal{I} -equivalence since $C(Y) \rightarrow C(X)$ is a quasi-isomorphism of chain complexes by the homotopy invariance of singular homology. By the theorem, the claim about $A^{\mathcal{I}}(Y) \rightarrow A^{\mathcal{I}}(X)$ follows. \square

Combining Theorem 5.1 with Lemma 4.11 does in particular imply that for positive objects \mathbf{m} , the chain complex $A^{\mathcal{I}}(X)(\mathbf{m})$ captures the cohomology groups of X in degrees between 0 and $|\mathbf{m}|$. It should not be surprising that there is a functor from spaces to chain complexes concentrated in degrees between 0 and $-|\mathbf{m}|$ which has this property: if one applies the smart truncation $\tau_{\geq -m}$ degreewise to the simplicial object $[p] \mapsto A_{\text{PL},p}$ and applies Construction 2.6 to the resulting simplicial object $\tau_{\geq -m}A_{\text{PL},\bullet}$, one gets back $\tau_{\geq -m}A_{\text{PL}}$ since $\tau_{\geq -m}$ is right adjoint. In view of this, the chain complexes $A^{\mathcal{I}}(X)(\mathbf{m})$ are analogous to truncations of $A_{\text{PL}}(X)$.

LEMMA 5.3. *The maps in (5.2) are absolute level equivalences between absolute \mathcal{I} -fibrant objects when evaluated in simplicial degree p .*

Proof. Let \mathbf{m} be an object in \mathcal{I} . By Lemma 4.16 the map $S^0 = U^{\mathcal{I}}(\mathbf{m}) \rightarrow A_p^{\mathcal{I}}(\mathbf{m})$ is a quasi-isomorphism between cofibrant and fibrant objects and thus even a chain homotopy equivalence. The map $S^0 \rightarrow C(\Delta^p) = C_p$ is a quasi-isomorphism by the known computation of $H^*(\Delta^p; k)$. Applying $F_0^{\mathcal{I}}$, it provides an absolute level equivalence $U^{\mathcal{I}} \rightarrow C_p^{\mathcal{I}}$. By (5.1), we can decompose $U^{\mathcal{I}}(\mathbf{m}) \rightarrow B_p^{\mathcal{I}}(\mathbf{m})$ as

$$S^0 \rightarrow C_p \xrightarrow{\cong} S^0 \otimes C_p \rightarrow A_p^{\mathcal{I}} \otimes C_p.$$

We already showed that the first map is a quasi-isomorphism. The last one is a quasi-isomorphism since $- \otimes C_p$ preserves chain homotopy equivalences. The \mathcal{I} -chain complexes $A_p^{\mathcal{I}}, B_p^{\mathcal{I}}$, and $C_p^{\mathcal{I}}$ are absolute \mathcal{I} -fibrant for each $p \geq 0$ since they are absolute level equivalent to $U^{\mathcal{I}}$ and $U^{\mathcal{I}} = \text{const}_{\mathcal{I}} S^0$ is absolute \mathcal{I} -fibrant. \square

LEMMA 5.4. *For all $q \in \mathbb{Z}$ and all positive objects \mathbf{m} in \mathcal{I} , the simplicial k -module $B_{\bullet}^{\mathcal{I}}(\mathbf{m})_q$ is contractible to 0.*

Proof. From (5.1) we get an isomorphism $B_{\bullet}^{\mathcal{I}}(\mathbf{m})_q \cong \bigoplus_{r+s=q} A_{\bullet}^{\mathcal{I}}(\mathbf{m})_r \otimes C_{\bullet,s}$ of simplicial k -modules. Since $C_{\bullet,s}$ is contractible for every s by Lemma 3.10(ii), so are the tensor products and thus also the sum. \square

LEMMA 5.5. *Let $D_{\bullet}: \Delta^{\text{op}} \rightarrow \text{Ch}_k^{\mathcal{I}}$ be a simplicial object in \mathcal{I} -chain complexes such that for all positive objects \mathbf{m} in \mathcal{I} and all integers q with $q > -|\mathbf{m}|$, the simplicial k -module $D_{\bullet}(\mathbf{m})_q$ is contractible to 0. Then for all $p \geq 0$, the boundary inclusion $\partial \Delta^p \rightarrow \Delta^p$ induces a descending level fibration $D(\Delta^p) \rightarrow D(\partial \Delta^p)$.*

Proof. A map in $\text{Ch}_k^{\mathcal{I}}$ is a descending level fibration if and only if it has the right lifting property against the maps $(U \rightarrow V) = F_{\mathbf{m}}^{\mathcal{I}}(0 \rightarrow D^q)$ with \mathbf{m} positive and $q > -|\mathbf{m}|$. By the adjunction (3.1), the lifting property for $U \rightarrow V$ and $D(\Delta^p) \rightarrow D(\partial \Delta^p)$ is equivalent to the lifting property

for $\partial\Delta^p \rightarrow \Delta^p$ and $K_D(V) \rightarrow K_D(U)$. Inspecting the definition of K_D , it follows that asking the latter lifting property for all $p \geq 0$ is equivalent to asking the map of simplicial sets $\text{Ch}_k^{\mathcal{I}}(V, D_{\bullet}) \rightarrow \text{Ch}_k^{\mathcal{I}}(U, D_{\bullet})$ to be an acyclic Kan fibration. Since $(F_{\mathbf{m}}^{\mathcal{I}}, \text{Ev}_{\mathbf{m}})$ is an adjunction and since morphisms in Ch_k out of D^q correspond to level q elements, the assumption that $U \rightarrow V$ is $F_{\mathbf{m}}^{\mathcal{I}}(0 \rightarrow D^q)$ implies that $\text{Ch}_k^{\mathcal{I}}(V, D_{\bullet}) \rightarrow \text{Ch}_k^{\mathcal{I}}(U, D_{\bullet})$ is isomorphic to $D_{\bullet}(\mathbf{m})_q \rightarrow 0$. The source of this map is contractible by assumption and a Kan complex because it is the underlying simplicial set of a simplicial k -module. Hence $D_{\bullet}(\mathbf{m})_q \rightarrow 0$ is an acyclic Kan fibration. \square

Remark 5.6. When $D_{\bullet}(\mathbf{m})_q$ is not contractible, $D(\Delta^p)_q \rightarrow D(\partial\Delta^p)_q$ fails to be surjective. In view of Remark 3.8, this shows for example that $A^{\mathcal{I}}(\Delta^p) \rightarrow A^{\mathcal{I}}(\partial\Delta^p)$ is not an absolute level fibration.

PROPOSITION 5.7. *Let $D_{\bullet} \rightarrow D'_{\bullet}$ be a natural transformation of functors $\Delta^{\text{op}} \rightarrow \text{Ch}_k^{\mathcal{I}}$. Suppose that for all $p \geq 0$, the map $D_p \rightarrow D'_p$ is an absolute level equivalence between absolute \mathcal{I} -fibrant objects and that for all positive objects \mathbf{m} and all $q > -|\mathbf{m}|$, the simplicial k -modules $D_{\bullet}(\mathbf{m})_q$ and $D'_{\bullet}(\mathbf{m})_q$ are contractible. Then for every simplicial set X , the map $D(X) \rightarrow D'(X)$ is a descending level equivalence between descending \mathcal{I} -fibrant objects.*

Proof. As usual, this is proved by cell induction. Let us first assume that for all $p \geq 0$, the map $D(\partial\Delta^p) \rightarrow D'(\partial\Delta^p)$ is a descending level equivalence between descending \mathcal{I} -fibrant objects. Any simplicial set X can be written as a cell complex $X = \text{colim}_{\lambda < \kappa} X_{\lambda}$ built from attaching cells of the form $\partial\Delta^p \rightarrow \Delta^p$. The functor D takes the inclusion $\partial\Delta^p \rightarrow \Delta^p$ to a descending level fibration by Lemma 5.5. Since we assume that $D(\partial\Delta^p)$ and $D_p \cong D(\Delta^p)$ are descending \mathcal{I} -fibrant, it follows from Corollary 4.9 that $D(\Delta^p) \rightarrow D(\partial\Delta^p)$ is a descending \mathcal{I} -fibration. The same holds for D' . Since both D and D' take colimits to limits by Lemma 3.3, the cogluing lemma in the descending level model structure and the fact that base change preserves \mathcal{I} -fibrations shows that $D(X) \rightarrow D'(X)$ arises as a limit of pointwise descending level equivalences between inverse systems of descending \mathcal{I} -fibrations. Hence it is a descending level equivalence between descending \mathcal{I} -fibrant objects.

Since $\partial\Delta^p$ only has non-degenerate simplices in dimensions strictly less than p , an analogous induction over the dimension of $\partial\Delta^p$ shows the remaining claim about $D(\partial\Delta^p) \rightarrow D'(\partial\Delta^p)$. \square

Proof of Theorem 5.1. Combining Lemma 3.7, Corollary 3.11(ii), Lemmas 5.4, and 5.3, the two maps $A^{\mathcal{I}} \rightarrow B^{\mathcal{I}}$ and $C^{\mathcal{I}} \rightarrow B^{\mathcal{I}}$ satisfy the hypotheses of Proposition 5.7. \square

We can now also prove Theorem 1.5 from the introduction.

Proof of Theorem 1.5. The adjunction $(A^{\mathcal{I}}, \langle - \rangle_{\mathcal{I}})$ arises from $A_{\bullet}^{\mathcal{I}}$ by applying Construction 3.2. Lemmas 3.7 and 5.5 show that $A^{\mathcal{I}}$ sends cofibrations to descending level fibrations and thus to descending \mathcal{I} -fibrations. Corollary 5.2 implies that $A^{\mathcal{I}}$ sends weak homotopy equivalences to descending level equivalences and thus to \mathcal{I} -equivalences. The rest is an immediate consequence of the self-duality of model structures with respect to the passage to opposite categories and the adjunction isomorphisms (3.1). \square

5.8 The relation to polynomial forms

In order to relate $A^{\mathcal{I}}$ to the functor A_{PL} used in rational homotopy theory, we first identify the functor $A_{\text{PL}, \bullet}$ described in (1.3) as a two-sided bar construction. Arguing as in §3.4 and using

the notation D_r^0 introduced there, we get an isomorphism

$$B_p(S^0, \mathbb{C}D^0, S^0) \cong \mathbb{C}(D_{r_1(p)}^0 \oplus \cdots \oplus D_{r_p(p)}^0)$$

where \mathbb{C} denotes the free commutative dga on a chain complex. The assignments

$$r_j(p) \mapsto \sum_{0 \leq l \leq j-1} t_l(p) \quad \text{and} \quad t_j(p) \mapsto r_{j+1}(p) - r_j(p)$$

define inverse isomorphisms between $B_p(S^0, \mathbb{C}D^0, S^0)$ and $A_{\text{PL},p}$, and these isomorphisms are compatible with the structure maps described in § 1.3.

By adjunction, the canonical map $D^0 \rightarrow (\text{const}_{\mathcal{I}} \mathbb{C}D^0)(\mathbf{1})$ induces a map of commutative \mathcal{I} -dgas $\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D^0) \rightarrow \text{const}_{\mathcal{I}} \mathbb{C}D^0$. Using the above description of $A_{\text{PL},\bullet}$ as a two-sided bar construction, this map in turn induces a map $A_{\bullet}^{\mathcal{I}} \rightarrow \text{const}_{\mathcal{I}} A_{\text{PL},\bullet}$ in $\text{Ch}_k^{\mathcal{I}}[\mathcal{C}]$ and thus a natural map $A^{\mathcal{I}}(X) \rightarrow \text{const}_{\mathcal{I}} A_{\text{PL}}(X)$ on the extensions to simplicial sets.

THEOREM 5.9. *Let k be a field of characteristic 0. Then $A^{\mathcal{I}}(X) \rightarrow \text{const}_{\mathcal{I}} A_{\text{PL}}(X)$ is a descending level equivalence. It induces a quasi-isomorphism $A^{\mathcal{I}}(X)_{h\mathcal{I}} \rightarrow A_{\text{PL}}(X)$ that is an \mathcal{E} -algebra map if we view the cdga $A_{\text{PL}}(X)$ as an \mathcal{E} -algebra by restricting along the canonical operad map from \mathcal{E} to the commutativity operad.*

Proof. In characteristic zero the homology groups of $(D^0)^{\otimes n}/\Sigma_n$ are isomorphic to the coinvariants $H_*(D^0)^{\otimes n}/\Sigma_n$ and the latter term is trivial for $n \geq 1$ because D^0 is acyclic. Therefore $\mathbb{C}F_{\mathbf{1}}^{\mathcal{I}}(D^0) \rightarrow \text{const}_{\mathcal{I}} \mathbb{C}D^0$ is an absolute level equivalence. The claim about general X follows from Proposition 5.7 and the contractibility property of $A_{\text{PL},\bullet}$ established in [BG76, Proposition 1.1]. Applying $(-)_h\mathcal{I}$ to this descending level equivalence and composing with the natural quasi-isomorphism (4.1) gives the quasi-isomorphism $A^{\mathcal{I}}(X)_{h\mathcal{I}} \rightarrow A_{\text{PL}}(X)$. To see that it is an \mathcal{E} -algebra map, we note that it follows from the definitions that (4.1) is an \mathcal{E} -algebra map when evaluated on a cdga. □

6. Comparison of E_{∞} structures

Let \mathcal{E} be the Barratt–Eccles operad introduced in Definition 2.11. We now define $A: \text{sSet}^{\text{op}} \rightarrow \text{Ch}_k[\mathcal{E}]$ to be the composite $A = (A^{\mathcal{I}})_{h\mathcal{I}}$ of the functor $A^{\mathcal{I}}$ from the previous section and the functor $(-)_h\mathcal{I}: \text{Ch}_k^{\mathcal{I}}[\mathcal{C}] \rightarrow \text{Ch}_k[\mathcal{E}]$ resulting from Theorem 2.13. The following proposition shows that A is a *cochain theory* in the sense of [Man02].

PROPOSITION 6.1. *The functor $A: \text{sSet}^{\text{op}} \rightarrow \text{Ch}_k[\mathcal{E}]$ has the following properties.*

- (i) *It sends weak equivalence of simplicial sets to quasi-isomorphisms.*
- (ii) *For a sub-simplicial set $Y \subseteq X$, the induced map from $\text{hofib}(A(X/Y) \rightarrow A(*))$ to $\text{hofib}(A(X) \rightarrow A(Y))$ is a quasi-isomorphism.*
- (iii) *For a family $(X_j)_{j \in J}$ of simplicial sets indexed by a set J , the canonical map $A(\coprod_{j \in J} X_j) \rightarrow \prod_{j \in J} A(X_j)$ is a quasi-isomorphism.*
- (iv) *It satisfies $H_0(A(*)) \cong k$ and $H_n(A(*)) \cong 0$ if $n \neq 0$.*

Proof. Part (i) follows from Corollary 5.2, part (iv) is an immediate consequence of Theorem 5.1, and part (iii) follows from Lemma 4.12 because $A^{\mathcal{I}}$ takes coproducts in sSet to products of fibrant objects in $\text{Ch}_k^{\mathcal{I}}$.

For part (ii), we view X/Y as the pushout of $* \leftarrow Y \rightarrow X$. The functor $A^{\mathcal{I}}$ sends this pushout to a pullback diagram displayed as the front face in the following cube.

$$\begin{array}{ccccc}
 & & A^{\mathcal{I}}(X/Y)' & \longrightarrow & A^{\mathcal{I}}(X)' \\
 & \nearrow \sim & \downarrow & & \downarrow \\
 A^{\mathcal{I}}(X/Y) & \longrightarrow & A^{\mathcal{I}}(X) & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & \nearrow \sim & A^{\mathcal{I}}(*)' & \longrightarrow & A^{\mathcal{I}}(Y)' \\
 A^{\mathcal{I}}(*) & \longrightarrow & A^{\mathcal{I}}(Y) & & \\
 & & \downarrow & & \downarrow \\
 & & A^{\mathcal{I}}(Y) & &
 \end{array}$$

Here the vertical maps on the front are descending level fibrations between descending \mathcal{I} -fibrant objects by Theorem 1.5. The bottom face is obtained by applying a fibrant replacement in the absolute \mathcal{I} -model structure to the map $A^{\mathcal{I}}(*) \rightarrow A^{\mathcal{I}}(Y)$, and the inwards pointing arrows on the bottom are descending level equivalences by Corollary 4.10. The right hand face is obtained by factoring $A^{\mathcal{I}}(X) \rightarrow A^{\mathcal{I}}(Y)'$ as an acyclic cofibration $A^{\mathcal{I}}(X) \rightarrow A^{\mathcal{I}}(X)'$ followed by a fibration $A^{\mathcal{I}}(X)' \rightarrow A^{\mathcal{I}}(Y)'$ in the absolute \mathcal{I} -model structure. Then $A^{\mathcal{I}}(X) \rightarrow A^{\mathcal{I}}(X)'$ is also a descending level equivalence by Corollary 4.10. The last term $A^{\mathcal{I}}(X/Y)'$ is obtained by requiring the back face to be a pullback. Right properness of the descending level model structure implies that $A^{\mathcal{I}}(X/Y) \rightarrow A^{\mathcal{I}}(X/Y)'$ is a descending level equivalence. It then follows from Lemma 4.11 that the square obtained by applying $(-)_h\mathcal{I}$ to the front face is quasi-isomorphic to the square obtained by evaluating the back face at the object $\mathbf{0}$. The latter square is homotopy cartesian by construction. □

Let \mathcal{E}^{cof} be a cofibrant E_∞ operad in the sense of [Man02, Definition 4.2]. Then there exists an operad map $\mathcal{E}^{\text{cof}} \rightarrow \mathcal{E}$ to the Barratt–Eccles operad [Man02, Lemma 4.5], and by restricting along $\mathcal{E}^{\text{cof}} \rightarrow \mathcal{E}$ we may view A as a functor to \mathcal{E}^{cof} -algebras. On the other hand, the cosimplicial normalization functor for the category $\text{Ch}_k[\mathcal{E}^{\text{cof}}]$ provided by [Man02, Theorem 5.8] allows one to lift the ordinary cochain functor $C: \text{sSet}^{\text{op}} \rightarrow \text{Ch}_k$ to a functor with values in $\text{Ch}_k[\mathcal{E}^{\text{cof}}]$ (compare [Man02, §1]). We are now in a situation where [Man02, Main Theorem] applies.

THEOREM 6.2. *The functor $A: \text{sSet}^{\text{op}} \rightarrow \text{Ch}_k[\mathcal{E}^{\text{cof}}]$ is naturally quasi-isomorphic to the singular cochain functor $C: \text{sSet}^{\text{op}} \rightarrow \text{Ch}_k[\mathcal{E}^{\text{cof}}]$.*

Remark 6.3. It is well known how to express the cup- i products on the singular cohomology of spaces using the Barratt–Eccles operad, see for instance [BF04, Theorem 2.1.1]. This way the \mathcal{E} -algebra structure on $A(X) = A^{\mathcal{I}}(X)_{h\mathcal{I}}$ gives rise to cup- i products, and the previous theorem shows that they are equivalent to the usual cup- i products on the cochain algebra.

Theorem 6.2 also allows us to express Mandell’s theorem [Man06] using $A^{\mathcal{I}}$.

Proof of Theorem 1.2. Let X and Y be two finite type nilpotent spaces. By Proposition 4.15, the commutative \mathcal{I} -dgas $A^{\mathcal{I}}(X; \mathbb{Z})$ and $A^{\mathcal{I}}(Y; \mathbb{Z})$ are \mathcal{I} -equivalent in $\text{Ch}_{\mathbb{Z}}^{\mathcal{I}}[\mathcal{C}]$ if and only if $A^{\mathcal{I}}(X; \mathbb{Z})_{h\mathcal{I}}$ and $A^{\mathcal{I}}(Y; \mathbb{Z})_{h\mathcal{I}}$ are quasi-isomorphic in $\text{Ch}_{\mathbb{Z}}[\mathcal{E}]$, which is in turn equivalent to being quasi-isomorphic in $\text{Ch}_{\mathbb{Z}}[\mathcal{E}^{\text{cof}}]$. By Theorem 6.2, this holds if and only if $C^*(X; \mathbb{Z})$ and $C^*(Y; \mathbb{Z})$ are quasi-isomorphic in $\text{Ch}_{\mathbb{Z}}[\mathcal{E}^{\text{cof}}]$. By [Man06, Main Theorem], this is the case if and only if X and Y are weakly equivalent. □

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