CLIFFORD ALGEBRAS AND ISOTROPES

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Introduction. Isotropes play a distinguished rôle in the algebra of spinors. Let V be an even-dimensional real vector space equipped with an inner product B of arbitrary signature. An isotrope of (V, B) is a subspace of the complexification $V^{\mathbb{C}}$ on which $B^{\mathbb{C}}$ is identically zero. Denote by ρ the spin representation of the complex Clifford algebra $C(V^{\mathbb{C}}, B^{\mathbb{C}})$ on a space S of spinors.

An isotrope Z of (V, B) annihilates a nonzero subspace S^Z of S under ρ. Chevalley has shown that if Z is a maximal isotrope of (V, B) then S^Z is one-dimensional. Let W be a non-maximal isotrope of (V, B); $W \subset V^{\mathbb{C}}$ is a proper subspace of its orthogonal $W^{\perp} \subset V^{\mathbb{C}}$, and the quotient $W^{\perp}/W$ inherits an inner product B_W. We show that S^W is canonically a spin module for the Clifford algebra of $(W^{\perp}/W, B_W)$.

The groups $\mathrm{Spin}(V, B)$ and $\mathrm{Spin}^c(V, B)$, along with the Clifford groups $\Gamma(V, B)$ and $\Gamma^c(V, B)$, may be realized as groups of units in $C(V^{\mathbb{C}}, B^{\mathbb{C}})$ stabilizing $V \subset C(V^{\mathbb{C}}, B^{\mathbb{C}})$ under the twisted adjoint representation. We denote by $\Gamma^c(V, B; Z)$ the subgroup of $\Gamma^c(V, B)$ stabilizing $Z \subset V^{\mathbb{C}}$ in the vector representation of $\Gamma^c(V, B)$ as orthogonal transformations of (V, B). If Z is an isotrope of (V, B) then the spin representation of $\Gamma^c(V, B; Z)$ stabilizes $S^Z \subset S$.

If Z is a maximal isotrope of (V, B) then the spin action of $\Gamma^c(V, B; Z)$ on the complex line S^Z is given by a character; the identification of this character is implicit in Chevalley. Let $L \subset V$ be a real isotrope of (V, B) ; identify L with $L^{\mathbb{C}} \subset V^{\mathbb{C}}$. Orthogonal transformations of (V, B) stabilizing L induce orthogonal transformations of the real inner product space $(L^{\perp}/L, B_L)$; the resulting homomorphism of orthogonal groups may be called isotropic reduction. We show that this homomorphism lifts naturally to the level of Clifford groups and deduce that isotropic reduction of special orthogonal groups lifts to the level of spin groups.

Our results have clear implications for pseudoriemannian geometry and for mathematical physics; these we shall pursue elsewhere. Many of our results concerning the spin representation were inspired by their counterparts for its symplectic analogue, the metaplectic representation. However, there are notable differences: for instance, the symplectic counterpart of isotropic reduction does not lift to metaplectic double covers. See **[5]** for a detailed account of the metaplectic representation.

This paper is organized as follows. Section 1 contains a brief account of Clifford algebras and their various groups of units; this material is quite standard and may be found in **[1]**, **[2]**, **[3]**, **[4]**. In §2 we describe completely the space of spinors annihilated by an arbitrary isotrope. This description reveals properties of Clifford groups and spin groups, which we present in §3.

1. Clifford algebras. Let V be a real vector space equipped with a nonsingular symmetric bilinear form B. We assume that V is of even dimension $2m$ but make no assumption regarding the signature of B.

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The orthogonal group $O(V, B)$ is the group of all linear automorphisms g of V satisfying

$$x, y \in V \Rightarrow B(gx, gy) = B(x, y);$$

the special orthogonal group $\mathrm{SO}(V, B)$ is defined by

$$\mathrm{SO}(V, B) = \{g \in O(V, B) \mid \mathrm{Det}\, g = 1\}.$$

A Clifford map for (V, B) is a pair (A, ϕ) consisting of an algebra A together with a linear map $\phi: V \to A$ satisfying the Clifford property

$$v \in V \Rightarrow \phi(v)^2 = B(v, v)1.$$

Clifford maps for (V, B) form the objects of a category $\mathscr{C}(V, B)$; the morphisms from (A_1, ϕ_1) to (A_2, ϕ_2) are algebra maps $\chi: A_1 \to A_2$ such that $\chi \circ \phi_1 = \phi_2$.

A Clifford algebra for (V, B) is a universally repelling object in $\mathscr{C}(V, B)$; such exists and is uniquely determined up to a unique isomorphism. Choose and fix a Clifford algebra (C, ψ) for (V, B). The Clifford map $\psi: V \to C$ is injective; this allows us to identify V with $\psi V \subset C$. For emphasis we shall write C as $C(V, B)$, or simply $C(V)$; thus $V \subset C(V, B)$.

By universality, there exists a unique automorphism γ of $C(V, B)$ whose restriction to V is minus the identity. γ is an involution: $\gamma^2 = I$; as a consequence, $C(V, B)$ decomposes as the linear direct sum

$$C(V, B) = C_0(V, B) \oplus C_1(V, B)$$

with $\gamma = I$ on $C_0(V, B)$ and $\gamma = -I$ on $C_1(V, B)$. We refer to the elements of $C_0(V, B)$ as even and to the elements of $C_1(V, B)$ as odd. Again by universality, there exists a unique antiautomorphism α of $C(V, B)$ whose restriction to V is $-I$; we remark that $\alpha\gamma = \gamma\alpha$.

Let $G(V, B)$ be the group of units in $C(V, B)$. The twisted adjoint representation of $G(V, B)$ on $C(V, B)$ is given by

$$u \,.\, x = \gamma(u)xu^{-1}$$

for $u \in G(V, B)$ and $x \in C(V, B)$. The inclusion of V in $C(V, B)$ picks out a subgroup $\Gamma(V, B)$ of $G(V, B)$ as follows:

$$\Gamma(V, B) = \{u \in G(V, B) \mid v \in V \Rightarrow \gamma(u)vu^{-1} \in V\};$$

we refer to $\Gamma(V, B)$ as the Clifford group of (V, B). A surjective homomorphism

$$\sigma: \Gamma(V, B) \to O(V, B)$$

is defined by the prescription

$$\sigma(u)v = \gamma(u)vu^{-1}$$

for $u \in \Gamma(V, B)$ and $v \in V$; the kernel of σ consists precisely of the nonzero scalars $\mathbb{R}^{\bullet} \subset \Gamma(V, B)$. If $u \in \Gamma(V, B)$ then $\alpha(u) \in \Gamma(V, B)$ and $\sigma(\alpha(u)u) = I$; thus $\alpha(u)u \in \mathbb{R}^{\bullet}$.

The resulting map

$$\eta:\Gamma(V, B)\to\mathbb{R}^{\cdot}:u\mapsto\alpha(u)u$$

is a character of the Clifford group. $\mathrm{Spin}(V, B)$ is the group of all even elements u in $\Gamma(V, B)$ such that $\eta(u)=\pm 1$; σ restricts to an epimorphism from $\mathrm{Spin}(V, B)$ to $\mathrm{SO}(V, B)$ having kernel $\{\pm 1\}$.

All of the above applies to the complex case in suitably modified form. The complexified algebra $C(V, B)^{\mathbb{C}}$ is naturally isomorphic to the Clifford algebra $C(V^{\mathbb{C}}, B^{\mathbb{C}})$ of the complexification $(V^{\mathbb{C}}, B^{\mathbb{C}})$. We have the Clifford group

$$\Gamma(V^{\mathbb{C}}, B^{\mathbb{C}})=\{u\in G(V^{\mathbb{C}}, B^{\mathbb{C}})\mid\gamma(u)V^{\mathbb{C}}u^{-1}=V^{\mathbb{C}}\}$$

and a character

$$\eta:\Gamma(V^{\mathbb{C}}, B^{\mathbb{C}})\to\mathbb{C}^{\cdot}:u\mapsto\alpha(u)u.$$

We denote by $\Gamma^c(V, B)$ the group of all elements u in $G(V^{\mathbb{C}}, B^{\mathbb{C}})$ preserving $V\subset C(V^{\mathbb{C}}, B^{\mathbb{C}})$ in the sense $\gamma(u)Vu^{-1}=V$; we have a central short exact sequence

$$1\longrightarrow\mathbb{C}^{\cdot}\longrightarrow\Gamma^c(V, B)\xrightarrow{\sigma}O(V, B)\longrightarrow 1$$

and a character

$$\eta:\Gamma^c(V, B)\to\mathbb{C}^{\cdot}.$$

Note that we can identify $\Gamma(V, B)$ with the subgroup of $\Gamma^c(V, B)$ fixed pointwise under the natural conjugation in $C(V^{\mathbb{C}}, B^{\mathbb{C}})$; note also that η restricts to the squaring map on $\mathbb{C}^{\cdot}\subset\Gamma^c(V, B)$. We denote by $\mathrm{Spin}^c(V, B)$ the group of all even elements u in $\Gamma^c(V, B)$ such that $|\eta(u)|=1$; we have a central short exact sequence

$$1\longrightarrow U(1)\longrightarrow\mathrm{Spin}^c(V, B)\xrightarrow{\sigma}\mathrm{SO}(V, B)\longrightarrow 1$$

and a unitary character

$$\eta:\mathrm{Spin}^c(V, B)\to U(1).$$

The complex Clifford algebra $C(V^{\mathbb{C}}, B^{\mathbb{C}})$ is isomorphic to a full matrix algebra. Fix a minimal left ideal S in $C(V^{\mathbb{C}}, B^{\mathbb{C}})$; left multiplication defines an algebra isomorphism

$$\rho:C(V^{\mathbb{C}}, B^{\mathbb{C}})\to\mathrm{End}\,S,$$

which we call the spin representation on the space S of spinors. Note that the Clifford property of $C(V^{\mathbb{C}}, B^{\mathbb{C}})$ implies the following anticommutation relations for ρ: if $x, y\in V^{\mathbb{C}}$ then

$$\rho(x)\rho(y)+\rho(y)\rho(x)=2B^{\mathbb{C}}(x, y)I.$$

2. Isotropes and spinors. A complex isotrope of (V, B) is a (complex) subspace $Z\subset V^{\mathbb{C}}$ such that

$$x, y\in Z\Rightarrow B^{\mathbb{C}}(x, y)=0;$$

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real isotropes are defined similarly, as subspaces of V on which B vanishes identically. Maximal complex isotropes of (V, B) have dimension m; the dimension of a maximal real isotrope is $\min\{p, q\}$ when B has signature (p, q).

Fix a space S of spinors for (V, B) with corresponding spin representation ρ. If Z is a subspace of $V^{\mathbb{C}}$ then we set

$$S^Z = \{f \in S \mid v \in Z \Rightarrow \rho(v)f = 0\}.$$

From the anticommutation relations for ρ it is clear that S^Z is zero unless Z is an isotrope. In this section we describe the space S^Z for any isotrope Z of (V, B).

The case of a maximal complex isotrope is familiar.

THEOREM 2.1. *If Z is a maximal complex isotrope of (V, B) then S^Z is a complex line.*

Proof. See pp. 71–72 of [2].

Now suppose W to be a non-maximal complex isotrope of (V, B); W is then a proper subspace of its orthogonal

$$W^\perp = \{v \in V^{\mathbb{C}} \mid w \in W \Rightarrow B^{\mathbb{C}}(v, w) = 0\}.$$

The quotient space $W^\perp/W$ naturally inherits a nonsingular symmetric bilinear form B_W defined by

$$B_W(x + W, y + W) = B(x, y)$$

whenever $x, y \in W^\perp$. The assignment

$$Z \mapsto Z_W = Z/W$$

is a bijection from the space of all isotropes Z of (V, B) such that $W \subset Z$ to the space of all isotropes of $(W^\perp/W, B_W)$; moreover, Z_W is maximal iff Z is maximal.

From the anticommutation relations for ρ it follows that if $v \in W^\perp$ then $\rho(v)$ stabilizes $S^W \subset S$; by definition, if $w \in W$ then $\rho(w)$ is zero on S^W. The induced map from $W^\perp/W$ to $\mathrm{End}(S^W)$ has the Clifford property and so extends to an algebra map from $C(W^\perp/W, B_W)$ to $\mathrm{End}(S^W)$.

Fix a space S_W of spinors for $(W^\perp/W, B_W)$; we then have the spin representation ρ_W of $C(W^\perp/W, B_W)$ on S_W. Our description of S^W is embodied in the statement that S^W and S_W are equivalent as $C(W^\perp/W, B_W)$-modules.

THEOREM 2.2. *There exists an isomorphism*

$$\theta_W : S^W \to S_W$$

intertwining the respective representations of $C(W^\perp/W, B_W)$.

Proof. Let N be a maximal (complex) isotrope of (V, B) with $W \subset N$ and choose a maximal isotrope P such that $V^{\mathbb{C}} = N \oplus P$. Let f_P be the product of the elements in a basis for P. According to [2], we may take S to be the left ideal in $C(V^{\mathbb{C}}, B^{\mathbb{C}})$ generated by f_P, and the map $x \mapsto xf_P$ defines a linear isomorphism from the exterior algebra $\Lambda(N) = C(N)$

to S; moreover, $\rho(y)(xf_P) = (y \wedge x)f_P$ for $x \in \Lambda(N)$ and $y \in N$. It follows from Section 5.26 of [4] that S^W consists precisely of those spinors xf_P with x in the ideal of $\Lambda(N)$ generated by the product f_W of elements in a basis for W. Let $Q = W^\perp \cap P$; the quotient $M = (Q+W)/W$ is then a maximal isotrope of $(W^\perp/W, B_W)$ complementary to the maximal isotrope $N_W = N/W$, and we may take S_W to be the left ideal $\Lambda(N/W)f_M \subset C(W^\perp/W, B_W)$. Let $q: \Lambda(N) \to \Lambda(N/W)$ be the canonical map induced from the quotient map $N \to N/W$. An isomorphism θ_W is now defined by

$$\theta_W(xf_Wf_P) = q(x)f_M$$

for $x \in \Lambda(N)$. That θ_W intertwines the representations of $C(W^\perp/W, B_W)$ is verified by a routine argument based on explicit formulae for ρ such as those presented in Section 2.2 of [2]; we omit the details.

REMARK 2.3. θ_W is unique up to multiplication by a nonzero complex number; this is so since θ_W intertwines the representations of $C(W^\perp/W, B_W)$ and the spin representation is irreducible.

REMARK 2.4. If Z is an isotrope of (V, B) such that $W \subset Z$ then θ_W restricts to an isomorphism from $S^Z \subset S^W$ to $(S_W)^{Z_W}$.

3. Isotropes and Clifford groups. Let Z be an isotrope of (V, B). If $g \in O(V, B)$ then

$$g \,.\, Z = \{g^{\mathbb{C}}(v) \mid v \in Z\}$$

is an isotrope of (V, B). We write $O(V, B; Z)$ for the stabilizer of Z under this action of $O(V, B)$ on the space of all isotropes of (V, B); similarly

$$\mathrm{SO}(V, B; Z) = \{g \in \mathrm{SO}(V, B) \mid g \,.\, Z = Z\}.$$

We denote by $\Gamma^c(V, B; Z)$ the full preimage of $O(V, B; Z)$ in $\Gamma^c(V, B)$ under σ; thus

$$\Gamma^c(V, B; Z) = \{u \in \Gamma^c(V, B) \mid \gamma(u)Zu^{-1} = Z\}.$$

In like manner we define subgroups $\Gamma(V, B; Z) \subset \Gamma(V, B)$, $\mathrm{Spin}^c(V, B; Z) \subset \mathrm{Spin}^c(V, B)$ and $\mathrm{Spin}(V, B; Z) \subset \mathrm{Spin}(V, B)$. It is clear that the spin representation of $\Gamma^c(V, B; Z)$ stabilizes $S^Z \subset S$ and so defines a representation of $\Gamma^c(V, B; Z)$ on S^Z; likewise for the spin representations of $\Gamma(V, B; Z)$, $\mathrm{Spin}^c(V, B; Z)$ and $\mathrm{Spin}(V, B; Z)$.

If Z is a maximal complex isotrope of (V, B) then S^Z is one-dimensional; the spin action of $\Gamma^c(V, B; Z)$ on S^Z is thus given by a character

$$\rho_Z: \Gamma^c(V, B; Z) \to \mathbb{C}^{\bullet}$$

with

$$\rho(u)(f) = \rho_Z(u)f$$

for $u \in \Gamma^c(V, B; Z)$ and $f \in S^Z$. If $u \in \Gamma^c(V, B; Z)$ then $g = \sigma(u)$ stabilizes $Z \subset V^{\mathbb{C}}$ and the prescription

$$\mathrm{Det}_Z u = \mathrm{Det}_{\mathbb{C}}(g^{\mathbb{C}} \mid Z)$$

defines a character Det_Z of $\Gamma^c(V, B; Z)$.

THEOREM 3.1. *If Z is a maximal complex isotrope of (V, B) then*

$$(\rho_Z)^2 = \eta \,.\, \mathrm{Det}_Z.$$

Proof. A routine consequence of III.2.6 on p. 80 of **[2]** modulo differences in convention.

As an immediate corollary of this theorem, a canonical splitting

$$s : \mathrm{SO}(V, B; Z) \to \mathrm{Spin}^c(V, B; Z)$$

of the short exact sequence

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Spin}^c(V, B; Z) \xrightarrow{\sigma} \mathrm{SO}(V, B; Z) \longrightarrow 1$$

is defined as follows: if $g \in \mathrm{SO}(V, B; Z)$ then $s(g)$ is the unique element u of $\mathrm{Spin}^c(V, B)$ such that $\sigma(u) = g$ and $\rho_Z(u) = |\mathrm{Det}_{\mathbb{C}}(g^{\mathbb{C}} \mid Z)|^{1/2}$.

Now suppose L to be a real isotrope of (V, B) such that the complex isotrope $W = L^{\mathbb{C}}$ is non-maximal; it will be notationally convenient to confuse L and W.

If $g \in O(V, B; L)$ then g stabilizes the orthogonal $L^\perp \subset V$ and descends to an element of $O(L^\perp/L, B_L)$ given by

$$g_L(v + L) = gv + L$$

for $v \in L^\perp$; the resulting homomorphism

$$\nu_L : O(V, B; L) \to O(L^\perp/L, B_L) : g \mapsto g_L$$

is surjective. Since B identifies $V/L^\perp$ with the dual L^*, if $g \in \mathrm{SO}(V, B; L)$ then $g_L \in \mathrm{SO}(L^\perp/L, B_L)$; ν_L maps $\mathrm{SO}(V, B; L)$ onto $\mathrm{SO}(L^\perp/L, B_L)$.

Let $u \in \Gamma^c(V, B; L)$. The spin action of u stabilizes $S^L \subset S$ and so induces an automorphism $\theta_L \circ \rho(u) \circ \theta_L^{-1}$ of S_L; see Theorem 2.2 and Remark 2.3. Denote by u_L the corresponding element of $C(L^\perp/L, B_L)^{\mathbb{C}}$; thus

$$\rho_L(u_L) = \theta_L \circ \rho(u) \circ \theta_L^{-1}.$$

Since θ_L intertwines the representations of $C(L^\perp/L, B_L)^{\mathbb{C}}$, it follows that u_L lies in $\Gamma^c(L^\perp/L, B_L)$ and that $\sigma_L(u_L) = \nu_L(\sigma(u))$, where σ_L is the standard homomorphism from $\Gamma^c(L^\perp/L, B_L)$ onto $O(L^\perp/L, B_L)$.

In this way we construct a surjective homomorphism

$$\nu_L^c : \Gamma^c(V, B; L) \to \Gamma^c(L^\perp/L, B_L) : u \mapsto u_L$$

which lifts ν_L:

$$\sigma_L \circ \nu_L^c = \nu_L \circ \sigma.$$

We claim that if $u \in \Gamma^c(V, B; L)$ then

$$\eta_L(u_L) = \eta(u)\mathrm{Det}_L\, u,$$

where η_L is the standard character of $\Gamma^c(L^\perp/L, B_L)$.

Consider first the special case of an element $u \in \Gamma^c(V, B; L)$ for which $\sigma_L(u_L) = I$. Let Z be a maximal complex isotrope of (V, B) with $L^{\mathbb{C}} \subset Z$; since Z_L is stabilized by $\sigma_L(u_L) = I$, it follows that Z is stabilized by $\sigma(u)$. According to Theorem 3.1, we have

$$\rho_Z(u)^2 = \eta(u)\mathrm{Det}_Z(u);$$

similarly

$$\rho_{Z_L}(u_L)^2 = \eta_L(u_L)\mathrm{Det}_{Z_L}(u_L).$$

From Remark 2.4, we see that

$$\rho_Z(u) = \rho_{Z_L}(u_L);$$

elementary linear algebra reveals

$$\mathrm{Det}_Z(u) = \mathrm{Det}_{Z_L}(u_L)\mathrm{Det}_L\, u.$$

It now follows that

$$\eta_L(u_L) = \eta(u)\mathrm{Det}_L\, u$$

in this special case.

In order to complete the proof of our claim, it suffices to show that if $h \in O(L^\perp/L, B_L)$ then

$$\eta_L(u_L) = \eta(u)\mathrm{Det}_L\, u$$

for some $u \in \Gamma^c(V, B; L)$ such that $\sigma_L(u_L) = h$; we do this as follows.

Some preparation is necessary. Let R be a real isotrope paired to L via B, so that

$$V = L \oplus (L^\perp \cap R^\perp) \oplus R$$

and B is nonsingular on $L^\perp \cap R^\perp$; see I.3.2 on p. 13 of [**2**]. The quotient map $L^\perp \to L^\perp/L$ restricts to an isometric isomorphism $L^\perp \cap R^\perp \to L^\perp/L$ and so induces an isomorphism of Clifford algebras $C(L^\perp \cap R^\perp) \to C(L^\perp/L, B_L)$. Since V is the orthogonal sum of $L \oplus R$ and $L^\perp \cap R^\perp$, $C(V, B)$ is the graded tensor product $C(L \oplus R) \hat{\otimes} C(L^\perp \cap R^\perp)$; see Section 10.7 of [**4**]. Let $S(L \oplus R)$ and $S(L^\perp \cap R^\perp)$ be minimal left ideals in $C(L \oplus R)^{\mathbb{C}}$ and $C(L^\perp \cap R^\perp)^{\mathbb{C}}$ respectively. We may then take S to be $S(L \oplus R) \otimes S(L^\perp \cap R^\perp)$ and S_L to be the image of $S(L^\perp \cap R^\perp)$ in $C(L^\perp/L, B_L)^{\mathbb{C}}$; in this case, S^L is $S(L \oplus R)^L \otimes S(L^\perp \cap R^\perp)$ and θ_L corresponds to a choice of basis vector in the complex line $S(L \oplus R)^L$.

Now choose any $t \in \Gamma^c(L^\perp/L, B_L)$ with $\sigma_L(t) = h$; we may regard t as an element of $\Gamma^c(L^\perp \cap R^\perp)$. By considering the spin action on $S^L = S(L \oplus R)^L \otimes S(L^\perp \cap R^\perp)$, it is clear that $u = 1 \hat{\otimes} t$ is then an element of $\Gamma^c(V, B; L)$ with $u_L = t$ and $\sigma(u) \mid L = I$. From $\eta(u) = \eta_L(t)$, we now deduce that

$$\eta_L(u_L) = \eta(u)\mathrm{Det}_L\, u$$

as required.

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Our claim is thus completely justified; we formalize it as follows.

THEOREM 3.2. *If* $u \in \Gamma^c(V, B; L)$ *then* $\eta_L(u_L) = \eta(u)\mathrm{Det}_L\, u$.

Let κ be the natural conjugation in $C(V, B)^{\mathbb{C}}$. The conjugate $\kappa S = \bar{S}$ of S is a minimal left ideal in $C(V^{\mathbb{C}}, B^{\mathbb{C}})$; denote by $\bar{\rho}$ the spin representation on $\bar{S}$. If $u \in C(V^{\mathbb{C}}, B^{\mathbb{C}})$ then clearly

$$\bar{\rho}(\bar{u}) \circ \kappa = \kappa \circ \rho(u).$$

Similar considerations apply to the spin representation $\overline{\rho_L}$ of $C(L^{\perp}/L, B_L)^{\mathbb{C}}$ on $\kappa_L S_L = \overline{S_L}$. Since L is real, κ maps S^L to $\bar{S}^L$; as intertwining operator $\bar{S}^L \to \overline{S_L}$ we may take $\bar{\theta}_L = \kappa_L \circ \theta_L \circ \kappa^{-1}$. If $u \in \Gamma^c(V, B; L)$ then

$$\begin{aligned}
\overline{\rho_L}(\overline{u_L}) &= \kappa_L \circ \rho_L(u_L) \circ \kappa_L^{-1} \\
&= \kappa_L \circ \theta_L \circ \rho(u) \circ \theta_L^{-1} \circ \kappa_L^{-1} \\
&= \bar{\theta}_L \circ \kappa \circ \rho(u) \circ \kappa^{-1} \circ \bar{\theta}_L^{-1} \\
&= \bar{\theta}_L \circ \bar{\rho}(\bar{u}) \circ \bar{\theta}_L^{-1} \\
&= \overline{\rho_L}(\bar{u}_L)
\end{aligned}$$

whence

$$\overline{u_L} = \bar{u}_L.$$

Now suppose u to lie in $\Gamma(V, B; L)$; thus $u \in \Gamma^c(V, B; L)$ and $\bar{u} = u$. From above it follows that $u_L \in \Gamma^c(L^{\perp}/L, B_L)$ satisfies $\overline{u_L} = u_L$; thus $u_L \in \Gamma(L^{\perp}/L, B_L)$. We have established the following.

THEOREM 3.3. *If* $u \in \Gamma(V, B; L)$ *then* $u_L \in \Gamma(L^{\perp}/L, B_L)$.

A surjective homomorphism

$$\hat{v}_L : \Gamma^c(V, B; L) \to \Gamma^c(L^{\perp}/L, B_L)$$

is defined by the prescription

$$\hat{v}_L(u) = |\mathrm{Det}_L\, u|^{-1/2}\, v_L^c(u)$$

for $u \in \Gamma^c(V, B; L)$. In view of Theorem 3.2, we have the relation

$$\eta_L(\hat{v}_L(u)) = \eta(u)\mathrm{sign\,Det}_L\, u$$

whenever $u \in \Gamma^c(V, B; L)$; in view of Theorem 3.3, $\hat{v}_L$ maps $\Gamma(V, B; L)$ to $\Gamma(L^{\perp}/L, B_L)$. These observations have the following noteworthy consequence.

THEOREM 3.4. $\hat{v}_L$ *restricts to a lift* $\mathrm{Spin}(V, B; L) \to \mathrm{Spin}(L^{\perp}/L, B_L)$ *of* v_L *to the level of Spin groups.*

Proof. Let $u \in \mathrm{Spin}(V, B; L)$; then $u \in \Gamma(V, B; L)$ is even and $\eta(u) = \pm 1$. Since $\mathrm{sign\,Det}_L\, u = \pm 1$, it follows that $\eta_L(\hat{v}_L(u)) = \pm 1$; being an even element of $\Gamma(L^{\perp}/L, B_L)$, $\hat{v}_L(u)$ therefore lies in $\mathrm{Spin}(L^{\perp}/L, B_L)$.

REMARK 3.5. The lift of ν_L to Spin groups is not unique when L is nonzero. If L is maximal as a real isotrope then there are two distinct lifts; otherwise there are four.

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