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On the beta expansion of Salem numbers of degree 8

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Abstract

Boyd showed that the beta expansion of Salem numbers of degree 4 were always eventually periodic. Based on an heuristic argument, Boyd had conjectured that the same is true for Salem numbers of degree 6 but not for Salem numbers of degree 8. This paper examines Salem numbers of degree 8 and collects experimental evidence in support of Boyd's conjecture.

1. *Introduction and basic definitions*

The representations of numbers in a non-integer base $\beta > 1$ was pioneered by Rényi [**11**], where he introduced the beta expansion (called also greedy expansion) to represent any real number x of the interval $[0,1]$ in base β by a sequence of digits $x_1x_2x_3\ldots$ which can be computed by the following algorithm.

Greedy algorithm. Denote by $\lfloor y \rfloor$ and $\{y\}$ the integer part and the fractional part of a real number y, respectively.

Set $r_0 = x$ and for $i \geqslant 1$, $x_i = \lfloor \beta r_{i-1} \rfloor$, $r_i = \{\beta r_{i-1}\}$.

Or, similarly, using the beta transformation $T = T_\beta$ of the unit interval which is the mapping:

$$\begin{aligned} T : [0,1] &\longrightarrow [0,1) \\ x &\longmapsto \beta x \bmod(1) \end{aligned}$$

where for every $i \geqslant 1$, $x_i = \lfloor \beta T^{i-1}(x) \rfloor$.

The beta expansion of x is denoted by $d_\beta(x) = x_1x_2x_3\ldots$ and it satisfies many proprieties (see, for instance, [**5**, **9**]). Mainly, we have $\forall i \geqslant 1$, $x_i \in \mathcal{A}_\beta = \{0,1,\ldots,\lfloor\beta\rfloor\}$ and

$$x = \sum_{i=1}^{\infty} x_i \beta^{-i}.$$

In this paper, we are only concerned with the beta expansion of 1 in some particular base and we will use the following notation.

The β expansion of a given $\beta > 1$, is the sequence of integers $(c_1, c_2, \ldots)$ defined by

$$\forall n \geqslant 1, \quad c_n = \lfloor \beta\alpha_{n-1} \rfloor, \quad \text{where } \alpha_0 = 1 \text{ and } \alpha_n = T^n(1).$$

Parry [**9**] defined β to be a beta number (more recently also called Parry numbers), if the orbit $\{\alpha_n : n \geqslant 1\}$ is finite. In this case, there exist smallest $m \geqslant 1$ and $p \geqslant 1$ for which $\alpha_m = \alpha_{m+p}$, we denote $D = D(\beta) = \operatorname{card}\{\alpha_n\} = m + p - 1$.

In particular, if $\alpha_m = 0$ (so $c_n = 0$ for $n > m$), we write $d_\beta(1) = (c_1, c_2, \ldots, c_m)$ and β is called a simple beta number.

Otherwise, if $\alpha_m = \alpha_{m+p} \neq 0$, then $d_\beta(1)$ will be eventually periodic and we write $d_\beta(1) = (c_1, \ldots, c_m : c_{m+1}, \ldots, c_{m+p})$. The values m and p are known as the *preperiod length* and the *period length*, respectively.

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It is not difficult to check by induction that for all $n \geqslant 1$, $\alpha_n = P_n(\beta)$ where

$$P_n(x) = x^n - c_1 x^{n-1} \dots - c_n.$$

Consequently, if β is a beta number, then β is a root of a monic polynomial R with integral coefficients, called the companion polynomial of β, with

$$R(x) = \begin{cases} P_m(x) & \text{if } \alpha_m = 0, \\ P_{m+p}(x) - P_m(x) & \text{if } \alpha_m = \alpha_{m+p} \neq 0. \end{cases}$$

Hence, any beta number is, in particular, an algebraic integer and its minimal polynomial $P(x)$ divides the companion polynomial $R(x)$. It is useful to note that the degree of $P(x)$ is always less than or equal to the degree of $R(x)$, which is exactly equal to $D(\beta)$. In addition, in this case we will have $R(x) = P(x)Q(x)$, where the polynomial $Q(x)$ (possibly constant) is called the cofactor of the beta expansion.

Parry [9] showed that the roots of the companion polynomial $R(x)$ other than β (called beta-conjugates) lie in the disk $|z| < \min(2, \beta)$, and this was improved to $|z| \leqslant (\sqrt{5}+1)/2$ by Solmyak [14] and independently by Flatto, Lagarias, and Poonen [4].

We recall the following facts.

(1) A Perron number is a real algebraic integer $\beta > 1$ such that all of its Galois conjugates $\beta^{(i)}$ satisfy $|\beta^{(i)}| < \beta$.
(2) A Pisot number is a real algebraic integer $\beta > 1$ such that all of its Galois conjugates have modulus less than one (see [10]).
(3) A Salem number is a real algebraic integer $\beta > 1$ such that all of its Galois conjugates have modulus at most one, and at least one conjugate has modulus one (see [12]).

The last definition implies that a Salem number is an algebraic integer where its minimal polynomial is monic, reciprocal, and with even degree. Furthermore, its minimal polynomial has one root τ outside the unit disk, one root $1/\tau$ inside the unit disk, and all other roots on the unit disk. In the rest of the paper we will call such polynomial a 'Salem polynomial'.

Bertrand [1] and Schmidt [13] have independently shown that if β is a Pisot number, then $\mathrm{Per}(\beta) = [0,1) \cap \mathbb{Q}(\beta)$, where $\mathrm{Per}(\beta)$ is the set of periodic points under T_β in $[0,1)$. Consequently, all Pisot numbers are beta numbers.

Furthermore, Schmidt conjectured that the same statement is true also for the case of Salem numbers [13]. If indeed this conjecture is true, then all Salem numbers would also be beta numbers.

The structure of the beta expansion of Pisot numbers was extensively studied and hence it is very well known compared with the case of Salem numbers (see, for instance, [8, 15, 16]). Next we summarize some related known results.

Boyd [2] showed that a Salem number cannot be a simple beta number and that all Salem numbers of degree 4 are beta numbers. More precisely, he proved that for every Salem number β of degree 4, we have:

(1) the preperiod $m = 1$;
(2) the orbit size $D(\beta) \leqslant 2\,\mathrm{trace}(\beta) - 3$;
(3) the companion polynomial is reciprocal and the cofactor is cyclotomic.

Boyd [2, 3] also computed the beta expansion for all, but 86, of the Salem numbers of degree 6 and trace at most 15 (there are 11 836 such numbers). He showed, through experimental results, that there are many differences between the case of degree 4 and the case of degree 6, which are mainly as follows.

(1) The preperiod m is not always equal to one and it can take many other values.
(2) The orbit size $D(\beta)$ can be very large relative to the size of β.
(3) The companion polynomial is not always reciprocal and the cofactor can be non-cyclotomic.

More recently, Hare and Tweedle [6] have studied the beta expansion of some sequences of Salem numbers approaching certain Pisot numbers. In particular, they showed the existence of infinite families of Salem numbers having eventually periodic beta expansion.

A heuristic probabilistic model proposed by Boyd [3], based on the ideas of the geometry of numbers, predicts that almost all Salem numbers of degree 6 are beta numbers. It also predicts that there exists a strictly positive proportion of Salem numbers of any fixed degree at least eight, which are not beta numbers. Motivated by this latter prediction, we study, in this paper, the case of Salem numbers of degree 8.

We start in § 2 by determining the sufficient and necessary conditions for the coefficients of a reciprocal polynomial of degree 8, to become a Salem polynomial. We use this to determine numerically (using a Maple program) the set of all Salem numbers of degree 8 and of trace at most 15.

In § 3, we characterize the reciprocal algebraic integers β, of degree $2n$, which are beta numbers having minimal value of $(m, p) = (1, 2n-1)$. As a consequence we get a complete characterization of Salem numbers of degree 8 which are beta numbers having minimal values of $(m, p) = (1, 7)$. Based on the numerical results, the set of such Salem numbers of degree 8 is quite large. In addition we determine a characterization of Salem numbers having $(m, p) = (1, 8)$.

Finally, § 4 summarizes the results of computational experiments on the beta expansion of Salem numbers of degree 8, which are, in particular, in connection with Boyd's conjecture and which allow us to make a comparison between the case of degree 6 and the case of degree 8. In the end we look for other eventual factors on which the orbit size of Salem number depends.

2. *Characterization of Salem numbers of degree 8*

Let P be a reciprocal integer polynomial of degree 8,

$$P(x) = x^8 + ax^7 + bx^6 + cx^5 + dx^4 + cx^3 + bx^2 + ax + 1. \tag{2.1}$$

By definition, P is a minimal polynomial of a Salem number β if and only if P is irreducible and

$$P(x) = (x - \beta)\left(x - \frac{1}{\beta}\right) \prod_{i=1}^{3} (x - \beta_i)(x - \overline{\beta}_i),$$

where $\beta > 1$ and $|\beta_i| = 1$ for $i \in \{1, 2, 3\}$, thus the trace of β satisfies

$$\operatorname{trace}(\beta) = -a = \left(\beta + \frac{1}{\beta}\right) + \sum_{i=1}^{3} (\beta_i + \overline{\beta}_i).$$

However, the fact that P is irreducible and $|\beta_i| = 1$ gives us $\beta_i + \overline{\beta}_i > -2$ for $i \in \{1, 2, 3\}$. In addition, from $\beta > 1$ we get $\beta + 1/\beta > 2$. Therefore,

$$-a = \left(\beta + \frac{1}{\beta}\right) + \sum_{i=1}^{3} (\beta_i + \overline{\beta}_i) > 2 - 6 = -4,$$

which gives us the first necessary condition:

$$a \leqslant 3.$$

The basic idea behind our characterization is classic and it was used by Boyd [2, 3] to characterize Salem numbers of degree 4 and 6. However, the calculations for degree 8 are much more complicated. For this reason we give, here, just the necessary steps of the calculation without going into the details.

First, we determine the polynomial U of degree 4 that satisfies

$$P(x) = x^4 U\left(x + \frac{1}{x}\right) \tag{2.2}$$

and we find that

$$U(x) = x^4 + ax^3 + (b-4)x^2 + (c-3a)x + d - 2b + 2. \tag{2.3}$$

Note that the zeros of U are the numbers $\beta + 1/\beta > 2$ and $-2 < \beta_i + 1/\beta_i < 2$ for $i \in \{1, 2, 3\}$. Note also that P can only be factored into factors of even degree (reciprocals), since the roots β and $1/\beta$ must belong to the same factor. Otherwise, there would be a factor where the product of whose roots is in absolute value less than one, which is clearly impossible. Thus, P is irreducible if and only if U is irreducible.

Consequently, a polynomial P of the form in (2.1) defines a Salem number of degree 8 if and only if U is an irreducible polynomial that has four distinct real roots, where three are in the interval $(-2, 2)$ and one root is greater than two. To simplify the formalization, the value $\sqrt{9a^2 - 24b + 96}$ is denoted by the symbol δ and we denote $\Delta(U)$ the discriminant of the polynomial satisfying (2.3).

PROPOSITION 2.1. *The polynomial P defined in (2.1) is a Salem polynomial if and only if P is irreducible and the following relations are true:*

(i) $\Delta(U) > 0$;

(ii) $|d/2 + b + 1| < -(a + c)$ *AND* $c + 9a < 4b + 16$;

(iii) $|3a + \delta| < 24$;

(iv) $c + 9a \leqslant -4b + 16$ *OR* $(c + 9a > -4b + 16$ *AND* $-3a + \delta < 24)$.

Proof. We start by showing that all of these conditions are necessary.

First, in order for U to have four distinct real roots, it is necessary that the discriminant of the polynomial U satisfy $\Delta(U) > 0$.

Next, to obtain a polynomial U without roots in the interval $(-\infty, -2)$ and having a root in the interval $(2, \infty)$, it is necessary in our case that $U(-2) > 0$, $U'(-2) < 0$, and $U(2) < 0$, which is equivalent to

$$\left|\frac{d}{2} + b + 1\right| < -(a + c)$$

and

$$c + 9a < 4b + 16.$$

Now, if the polynomial U satisfies the conditions above, it will have four distinct real roots $y_1 < y_2 < y_3 < y_4$ and consequently the polynomial U'' will have two distinct real roots which we denote $r_1 < r_2$. As we need that $-2 < y_1 < y_2 < y_3 < 2$, we must have $-2 < r_1 < 2$, which is equivalent to

$$|3a + \delta| < 24.$$

Finally, we need y_4 to be the only root in the interval $(2, \infty)$, which means $y_3 < 2 < y_4$. By considering all possible cases, it is not difficult to check that the condition to add is

$$U'(2) \leqslant 0 \quad \text{OR} \quad (U'(2) > 0 \text{ AND } r_2 < 2).$$

which is equivalent to

$$c + 9a \leqslant -4b + 16 \quad \text{OR} \quad (c + 9a > -4b + 16 \text{ AND } -3a + \delta < 24).$$

Conversely, under all of the previous conditions, the polynomial U has four distinct real roots $y_1 < y_2 < y_3 < y_4$ where $y_4 > 2$ and with at least a root in $(-2, 2)$. Hence, to end the proof, it will be sufficient to show that the case where U has two roots less than -2 and the

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case where has three roots greater than two do not hold. For the first case, U' has a root less than -2 and by the condition $U'(-2) < 0$, U' has another root less than -2 and this leads immediately to a contradiction because U'' has no root less than -2. For the second case, if we denote by $x_1 < x_2 < x_3$ the roots of U', we will have $x_1 < r1 < 2$ and $2 < x_2 < x_3$ which gives $r_2 > 2$ and $U'(2) > 0$; by (4) we obtain a contradiction. □

Note that, without the condition of irreducibility, Proposition (2.1) always gives a polynomial where its dominant root is a Salem number. But this Salem number can be of degree 8, 6, 4 or also a reciprocal Pisot number (which is considered as a degenerate Salem number). To get a Salem number of degree 8 only, we must assure the irreducibility of P. For that we can use the following point.

REMARK 2.2. Any polynomial P, of the form in (2.1) and satisfying the four conditions of Proposition 2.1, has always one root β outside the unit disk, one root $1/\beta$ inside the unit disk, and all other roots are on the unit disk. So, in case P is reducible, it becomes necessarily divisible by an integer polynomial Q such that all of its roots are of modulus one (since P is reciprocal and the roots β and $1/\beta$ must belong to the same factor). But this means, by the famous Kronecker's theorem [7], that Q is a product of cyclomatic polynomials of degree at most six, which are of the form:

$$x-1, \quad x+1,$$
$$x^2+1, \quad x^2-x+1, \quad x^2+x+1,$$
$$x^4-x^3+x^2-x+1, \quad x^4+x^3+x^2+x+1, \quad x^4-x^2+1,\ x^4+1, \quad x^4+x^2+1,$$
$$x^6+x^5+x^4+x^3+x^2+x+1, \quad x^6-x^3+1, \quad x^6+x^3+1 \text{ and } x^6-x^5+x^4-x^3+x^2-x+1.$$

Therefore, to get irreducible polynomials only, we must eliminate the above cases, which is equivalent to

$$d \neq -2(1+a+b+c), \quad -2(1-a+b-c), \quad 1-2a+b+c, \quad 2b-2, \quad 1+2a+b-c,$$
$$(c,d) \neq (0,1-b), \quad (a,2), \quad (-2a,b+1), \quad (1+a-b,b-a), \quad (a+b-1,a+b),$$
$$(b,c,d) \neq (1,-1,-a), \quad (1,1,a), \quad (a+1,a+1,a+1), \quad (1-a,a-1,1-a).$$

3. *Particular cases*

Regarding the beta expansion of Salem numbers of degree 8, we find that there is relatively a large proportion for which $(m,p) = (1,7)$. Note that the particularity in this case is the size of the orbit $\{T^n(1)\}_{n\in\mathbb{N}}$, which is $D = m + p = 8$. It is minimal and the companion polynomial coincides with the minimal polynomial (that is the cofactor of beta expansion is equal to one). All of the above, motivates us to ask the question of when a Salem number of degree $2n$ is a beta number satisfying $(m,p) = (1, 2n-1)$.

We try here to give a partial response to this question with a more general statement. Let us first recall Parry's criterion.

LEMMA 3.1 (Parry). *Let $(c_1, c_2, \dots)$ be a sequence of non-negative integers which is different from $1(0)^w$ and satisfying the following inequalities $c_1 > 0$ and $c_k \leqslant c_1$ for $k \geqslant 1$. The unique solution $\beta > 1$ of*

$$x = c_1 + c_2x^{-1} + c_3x^{-2} + \dots$$

has $c_1c_2c_3\dots$ as the beta expansion of one if and only if

$$\forall k \geqslant 1, \quad \sigma^k(c_1, c_2, \dots) <_{\text{lex}} (c_1, c_2, \dots)$$

where $\sigma(c_1, c_2, \dots) = (c_2, c_3, \dots)$.

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Proof. See Corollary 1 of Theorem 3 of [9]. □

LEMMA 3.2. *Every beta number is a Perron number without a proper real conjugate greater than one.*

Proof. It is a direct consequence of Proposition 7.2.21 and Remark 7.2.23 of [5, p. 218]. □

NOTATION. For simplicity, if a sequence $s = (c_j)_{j \geqslant 0}$ is periodic with period ℓ and pre-period n, we write

$$s = c_1, \ldots, c_n : c_{n+1}, \ldots, c_{n+\ell}$$

where $c_{n+i+k\ell} = c_{n+i}$, for all $k, i \in \mathbb{N}$.

THEOREM 3.3. *Let P be the following irreducible reciprocal integer polynomial:*

$$P(x) = x^{2n} + a_1 x^{2n-1} + \ldots + a_{n-1} x^{n+1} + a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + 1.$$

Here P has a unique root β in $(1, \infty)$ and β is a beta number with $(m, p) = (1, 2n-1)$ if and only if:

(i) $a_1 \leqslant -1$;
(ii) $a_i \leqslant 0$ *for* $2 \leqslant i \leqslant n$;
(iii) $\sigma^k(s) <_{\text{lex}} s$ *for all* $k \geqslant 1$, *where*

$$s = -a_1 : -a_2, \ldots, -a_n, \ldots, -a_2, -a_1 - 1, -a_1 - 1. \tag{3.1}$$

If β is a such beta number, then

$$d_\beta(1) = -a_1 : -a_2, \ldots, -a_n, \ldots, -a_2, -a_1 - 1, -a_1 - 1.$$

Proof. First, suppose that the polynomial P has a root $\beta > 1$ which is a beta number with $(m, p) = (1, 2n-1)$ and let $s = d_\beta(1) = c_1 : c_2, \ldots, c_n, \ldots, c_{2n-2}, c_{2n-1}, c_{2n}$ its beta expansion. Then β is a root of the companion polynomial which is

$$R(x) = P_{m+p}(x) - P_m(x) = P_{2n}(x) - P_1(x).$$

Note that R is monic and an integer polynomial, which has the same degree of P the minimal polynomial of β. So we will have $R = P$ which implies that

$$c_1 : c_2, \ldots, c_n, \ldots, c_{2n-2}, c_{2n-1}, c_{2n} = -a_1 : -a_2, \ldots, -a_n, \ldots, -a_2, -a_1 - 1, -a_1 - 1.$$

The condition $c_i \geqslant 0$ implies that for all i such that $2 \leqslant i \leqslant n$, we have $a_i \leqslant 0$, which proves point (2). It also implies that $-a_1 - 1 \geqslant 0$, which proves point (1). Finally, the condition $(c_1, c_2, \ldots) >_{\text{lex}} (c_k, c_{k+1}, \ldots)$ for all $k > 1$ gives us point (3) directly.

Conversely, we set

$$\begin{aligned} s &= c_1 \colon c_2, \ldots, c_n, \ldots, c_{2n-2}, c_{2n-1}, c_{2n} \\ &= a_1 \colon -a_2, \ldots, -a_n, \ldots, -a_2, -a_1 - 1, -a_1 - 1. \end{aligned} \tag{3.2}$$

It is clear that the conditions of Parry's criterion follow, for the sequence s, from the inequalities (1), (2), and (3) of Theorem 3.3. Thus, according to Lemma 3.1, the unique solution $\beta > 1$ of

$$x = c_1 + c_2 x^{-1} + c_3 x^{-2} + \ldots$$

has as beta expansion $s = d_\beta(1) = c_1 : c_2, \ldots, c_{2n}$. But this implies in particular that $T(1) = T^{2n}(1)$ and then β will be a root of the polynomial $P_{2n}(x) - P_1(x)$ which is exactly P according to (3.2).

However P is monic and irreducible, so it is the minimal polynomial of β. Consequently, by Lemma 3.2, β must be a Perron number without a real conjugate strictly greater than one, which implies that P has only one root β in the interval $(1,\infty)$ which is a beta number having $(m,p)=(1,2n-1)$ and

$$d_\beta(1) = -a_1(-a_2,\ldots,-a_n,\ldots,-a_2,-a_1-1,-a_1-1)^w. \qquad \square$$

REMARK 3.4. The assumed periodicity of s allows us to check condition (3) of Theorem 3.3 only for a finite number of cases. More precisely, condition (3) holds only if for all $1 \leqslant k \leqslant 2n-2$, $\sigma^k(s) <_{\text{lex}} s$.

The next corollary gives a very easy sufficient condition which can be more useful in practice.

COROLLARY 3.5. *Let $n \geqslant 2$ and let P be the following irreducible, reciprocal, and integer polynomial:*

$$P(x) = x^{2n} + a_1x^{2n-1} + \ldots + a_{n-1}x^{n+1} + a_nx^n + a_{n-1}x^{n+1} + \ldots + a_1x + 1.$$

If $a_1 < a_i \leqslant 0$ for $2 \leqslant i \leqslant n$, then P has a unique root β in $(1,\infty)$ and β is a beta number having $(m,p)=(1,2n-1)$.

Proof. It is clear that all conditions of Theorem 3.3 are satisfied. $\square$

REMARK 3.6. In Theorem 3.3 or Corollary 3.5, β is not necessarily a Salem number since it can have Galois conjugate in $\{z \in \mathbb{C}; |z| > 1\}$ which is forbidden for Salem number. For instance, the polynomial $x^8 - 8x^7 - 6x^6 - 7x^4 - 6x^2 - 8x + 1$ satisfies the conditions of Corollary 3.5 and then we know immediately that it will have a unique root β in the interval $(1,\infty)$ which will be a beta number with $(m,p)=(1,7)$. But here β is not a Salem number since it has a Galois conjugate $\gamma \simeq -1.1320$.

Theorem 3.3 generalizes the theorem given by Boyd [**3**, p. 866] for Salem numbers of degree 6 and also the case (ii) of Theorem 1 [**2**, p. 61] for Salem numbers of degree 4. It can also be applied to all Salem numbers of degree $2n$.

As an example, if we apply the Theorem 3.3 to Salem numbers of degree 8 we get the following corollary.

COROLLARY 3.7. *Let β be a Salem number of degree 8 whose minimal polynomial is given by (2.1). Here β is a beta number having $(m,p)=(1,7)$ if and only if:*

(i) *$a \leqslant -1$;*
(ii) *$a \leqslant \min(b,c,d) \leqslant \max(b,c,d) \leqslant 0$;*
with the further conditions:
(iii) *if $a = c$, then $b \leqslant d$;*
(iv) *if $a = d$, then $b \leqslant c$.*

Proof. According to Theorem 3.3, we have that β is a beta number having $(m,p)=(1,7)$ if and only if $a \leqslant -1$, $\max(b,c,d) \leqslant 0$ and for all $k \geqslant 1$, $\sigma^k(s) <_{\text{lex}} s$, where $s = c_1 : c_2,\ldots,c_8 = -a : -b,-c,-d,-c,-b,-a-1,-a-1$. This proves the corollary and the further conditions follow from the definition of lexicographic order. $\square$

Using Parry's criterion again, we give in the next theorem a characterization of Salem numbers of degree 8, which are beta numbers having $(m,p)=(1,8)$ (the cofactor of its beta expansion will be $x+1$).

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THEOREM 3.8. *Let β be a Salem number of degree 8 whose minimal polynomial is given by* (2.1). *Then, β is a beta number having $(m,p)=(1,8)$, if and only if:*

(i) $a \leqslant -2$;

(ii) $a+1 \leqslant \min(a+b, b+c, c+d) \leqslant \max(a+b, b+c, c+d) \leqslant 0$;
with the further conditions:

(iii) *if $a+1=b+c$, then $a+b \leqslant c+d$;*

(iv) *if $a+1=c+d$, then $b=1$.*

Proof. First, suppose that $(m,p)=(1,8)$, then the beta expansion of β is

$$d_\beta(1) = c_1 : c_2, \ldots, c_9.$$

This implies that the companion polynomial of β is

$$R(x) = P_9(x) - P_1(x) = x^9 - c_1x^8 - \ldots - c_7x^2 - (c_8+1)x - (c_9 - c_1).$$

Since β is a root of the integer polynomial R, the minimal polynomial P must divide R. According to the degrees and coefficients of P and R, the quotient will be of the form $x+e$ where $e \in \mathbb{Z}^*$ (if $e=0$ we will have $c_8 = -2 < 0$, which is impossible since c_k is always a positive integer).

By the result of Solomyak [**14**], since the integer $-e$ is a root of R other than β, we will have $|e| < (1+\sqrt{5})/2$ and hence $e = \pm 1$. On the other hand, by Parry's criterion $e = c_1 - c_9 \geqslant 0$ which means $e=1$ and so $R(x) = (x+1)P(x)$. By identification we obtain

$$\begin{aligned} c_1 : c_2, \ldots, c_9 \; = \; -(a+1) \; : \; &-(a+b), -(b+c), -(c+d), -(c+d), \\ &-(b+c), -(a+b), -(a+2), -(a+2). \end{aligned}$$

Now, using the facts that $c_k \geqslant 0$ and $(c_k, c_{k+1}, \ldots) <_{\text{lex}} c_1, c_2, \ldots$ for all $k \geqslant 1$, we can easily show that

$$a \leqslant -2$$

and that

$$a+1 \leqslant \min(a+b, b+c, c+d) \leqslant \max(a+b, b+c, c+d) \leqslant 0.$$

For the particular cases, they follow from the definition of lexicographic order. We need that $(c_3, c_4, \ldots) <_{\text{lex}} (c_1, c_2, \ldots)$ so that if $a+1=b+c$, then we will have $c_1 = c_3$. Thus, we must have $c_2 \geqslant c_4$, which is equivalent to $a+b \leqslant c+d$.

We also need that $(c_4, c_5, \ldots) <_{\text{lex}} (c_1, c_2, \ldots)$, so that if $a+1=c+d$, then we will have $c_1 = c_4$. Thus, we must have $c_2 \geqslant c_5$ meaning $a+b \leqslant c+d$. But in this case we will have $a+1 \leqslant a+b \leqslant c+d = a+1$, which gives $b=1$.

Conversely, it is easy to verify from the inequalities in the statement of Theorem 3.8 that Parry's criterion follows for

$$\begin{aligned} s &= c_1 : c_2, \ldots, c_9 \\ &= -(a+1) \; : \; -(a+b), -(b+c), -(c+d), -(c+d), -(b+c), \\ &\qquad -(a+b), -(a+2), -(a+2) \end{aligned} \tag{3.3}$$

Indeed, the assumed periodicity of the sequence s involves checking only a finite number of cases, we give here just two examples and the remaining cases are similar.

(1) To verify that $(c_3, c_4, \ldots) <_{\text{lex}} (c_1, c_2, \ldots)$, we note that $a+1 \leqslant b+c$ gives $c_1 \geqslant c_3$, and that $c_3 = c_1$ holds only if $a+1=b+c$, which implies, according to condition (3) of Theorem 3.8, that $a+b \leqslant c+d$ and then $c_2 \geqslant c_4$.

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Next, note that according to condition (2) of Theorem 3.8, we have $a + 1 \leqslant a + b$, which implies $1 \leqslant b$. In addition, the fact that $a+1 = b+c$ implies that $c \leqslant a$. If we also have $c_2 = c_4$ meaning $a + b = c + d$, then we get $b \leqslant d$. Finally, we have two subcases:

(i) if $b < d$, then we have $b + c < c + d$, which means $c_3 > c_5$;

(ii) if $b = d$, then we have $a = c$, which implies $c_4 = c_6$ and $c_5 = c_7$, but $c_6 > c_8$.

(2) To verify that $(c_4, c_5, \ldots) <_{\text{lex}} (c_1, c_2, \ldots)$, we note that $c_1 \geqslant c_4$ follows from $a+1 \leqslant c+d$, and that $c_4 = c_1$ holds only if $a + 1 = c + d$. According to condition(3) of Theorem 3.8, this means that $b = 1$ and $a + b = a + 1 \leqslant c + d$, and hence we have $c_2 \geqslant c_5$. Also, if $c_2 = c_5$, then we have $c_3 = c_6$ (always true here) and $c_4 = c_7$, but $c_5 > c_8$.

Therefore, the sequence s, defined in (3.3), is the beta expansion for some real number $\beta' > 1$ with $m = 1$ and p a divisor of eight.

From other part, the relation (3.3) implies $T(1) = T^9(1)$ and so β' is a root of the polynomial $P_9(x) - P_1(x)$ which is exactly $(x + 1)P(x)$ according to (3.3). However, the fact that $\beta' > 1$ implies that β' is a root of the irreducible polynomial P which is of degree 8 and consequently we must have $m + p \geqslant 8$. Hence, we get $\beta' = \beta$ and $(m, p) = (1, 8)$ immediately. □

REMARK 3.9. Corollary 3.7 and Theorem 3.8 explain why we get numerically more and more Salem numbers which are beta numbers having $(m, p) = (1, 7)$ and $(m, p) = (1, 8)$ for large values of $-a = \text{trace}(\beta)$.

4. *Numerical results*

The aim of this section is mainly to provide some information and examples for the beta expansion of Salem numbers of degree 8 and to test the predictions of Boyd's model for such numbers.

First, using the characterization given in Theorem 2.1 and Remark 2.2, we can determine numerically all Salem numbers of degree 8 of the same trace (same coefficient a). Since, for each fixed value of a there are only a finite number of possible values for the coefficients b, c and d. Table 1 gives an idea about the distribution of such numbers according to their trace.

Section 4.1 describes the results of the computation of the beta expansion for the 26 715 Salem numbers of degree 8 and trace at most eight. A special attention is given to the distribution of the pairs (m, p) and the nature of the companion polynomial. This allows us to make a comparison between beta expansion of Salem numbers of degree 8 and of degree 6 as was described in [**2**, **3**] and then to list some differences between the two cases.

Finally in § 4.2 we test the predictions given by Boyd's model and we look for more eventual factor on which the size of orbit of $D(\beta)$ depends.

TABLE 1. *Distribution of Salem numbers of degree 8 with respect to their trace.*

trace(β)	Salem numbers	trace(β)	Salem numbers
−1	1	9	14 810
0	14	10	20 387
1	76	11	27 257
2	251	12	35 551
3	651	13	45 387
4	1387	14	56 922
5	2598	15	70 281
6	4400	–	–
7	6955	–	–
8	10 382	–	–
Total for trace(β) $\leqslant 8$	26 715	–	–

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4.1. *General description*

By examining the numerical results of the computation of the beta expansion of the set of 26 715 Salem numbers of degree 8 and of trace at most 8, the following observations can be made.

(1) The majority of Salem numbers are beta numbers with $m = 1$. Indeed, among the 26 715 Salem numbers, there are 20 938 beta numbers for which $D = m+p < 1000$ (about 78%) and 17 023 for which $m = 1$ (about 64%).
(2) The smallest Salem number of degree 8 that has a very large orbit size (or perhaps an infinite orbit) is $\beta_0 \approx 2.03$ of the minimal polynomial $x^8 - x^7 - 3x^5 - x^4 - 3x^3 - x + 1$, which is numerically checked to be $D > 7.10^7$.
(3) There is a very large diversity in the values taken by the pair (m, p). But we can say that the most common factor is that each pair (m, p) appears many times in regularity with some relations between coefficients of the minimal polynomial P. This is quite noticeable, in particular, for small values of the type $(1, 7), (1, 8), (1, 9), (1, 10), (1, 11)$, and so on.
(4) We note also that, for the most part, the preperiod m is less than the period p. But, there are also many cases where the preperiod is very large compared with the period. For example, the Salem number defined by the polynomial $x^8 - 2x^7 - 5x^6 - 3x^5 - 3x^3 - 5x^2 - 2x + 1$ has $(m, p) = (2525, 64)$.

REMARK 4.1. We make the following remarks.

(1) These results are, in great part, similar to Salem numbers of degree 6 as described in [**3**]. However, Table 2 shows that the proportion of Salem numbers of degree 8, having $\max(m, p) > 1000$, is very large compared with the case of degree 6.
(2) Another eventual difference is the following: Boyd mentioned that for all Salem numbers of degree 6 that he has checked, the companion polynomial is reciprocal if $m = 1$ and non-reciprocal if $m > 1$, see [**2**]. Although this also remains true for most Salem numbers of degree 8, which we have investigated, we note that there are few exceptions. Indeed, we find 14 counterexamples in the interval $[1, 5]$. As an example, consider the Salem number $\beta \simeq 4.2$ of minimal polynomial $x^8 - 3x^7 - 4x^6 - 5x^5 - 3x^4 - 5x^3 - 4x^2 - 3x + 1$. It is a beta number having $(m, p) = (1, 12)$, yet its companion polynomial $R(x)$ is not reciprocal:

$$R(x) = x^{13} - 4x^{12} - x^{11} - 2x^9 - 2x^8 - 3x^7 - x^6 - 3x^5 - 3x^4 - 4x^3 - 4x + 1.$$

Table 3 presents the distribution of Salem numbers of degree 8 depending on the trace of β, and the percentage of those having $D(\beta) \leqslant 1000$. It can be seen that this percentage increases with larger values of trace(β).

TABLE 2. *Comparison between Salem numbers of degree* 6 *and* 8.

Type	Degree 6 and trace $\leqslant 15$	Degree 8 and trace $\leqslant 8$
Salem numbers	11 836	26 715
$\max(m, p) > 1000$	199	5746
Percentage	1.68%	21.5%

TABLE 3. *Percentage of Salem numbers which have $D(\beta) \leqslant 1000$ and with trace up to* 15.

trace(β)	9	10	11	12	13	14	15
Salem numbers	14 810	20 873	27 387	35 551	45 387	56 922	70 281
$D(\beta) \leqslant 1000$	11 991	16 699	22 424	29 436	37 738	47 596	58 904
Percentage	80.96%	81.91%	82.26%	82.79%	83.14%	83.61%	83.81%

4.2. *Study of interaction between* $C(\beta)$, trace(β), *and* $D(\beta)$

We recall that the heuristic argument given by Boyd [3] suggests that the expected magnitude of $D(\beta) = m + p$ of a Salem number of degree d, is linked directly to the magnitude of

$$C(\beta) = \frac{\beta^{d-1}(\pi/6)^{(d-2)/2}}{|\mathrm{disk}(\beta)|^{1/2}}.$$

More precisely, according to Boyd [3, pp. 873–874], the probability that the orbit size of the beta expansion for a Salem number of degree 8 ($d = 8$) is less than or equal to an integer n is

$$p_n = \Pr\{D(\beta) \leqslant n\} = 1 - \prod_{k=1}^{n}\left(1 - \frac{1}{C(\beta)k^2}\right). \tag{4.1}$$

Consequently, one expect that Salem numbers of degree 8 having small values of $C(\beta)$ should have small orbits and vice versa.

To test this prediction, we distribute the sets of all Salem numbers of degree 8 and trace at most 8, of trace 11 and 15 into subsets according to the value of $C(\beta)$ and we illustrate the results of the computation in Tables 4, 5, and 6, respectively.

As an example, according to (4.1), the Salem number $\beta \simeq 12.59$ of minimal polynomial $P(x) = x^8 - 8x^7 - 48x^6 - 112x^5 - 145x^4 - 112x^3 - 48x^2 - 8x + 1$ has the largest probability, among all Salem numbers of degree 8 and trace at most eight, to have a very large orbit size. This is because it has the largest value of $C(\beta)$, namely ($C(\beta) \simeq 145$). In this case, we checked numerically that $D(\beta) > 7.10^7$.

The aforementioned results clearly support the connection between $C(\beta)$ and $D(\beta)$. But we note that there are many exceptions, which indicates that $C(\beta)$ cannot be the only factor that determines the size of the orbit. We cite here just two examples.

TABLE 4. *Salem numbers of degree* 8 *and trace at most eight.*

	$C(\beta)$ range				
	$(0, 0.001)$	$[0.001, 0.01)$	$[0.01, 0.1)$	$[0.1, 1)$	$[1, \infty)$
Salem numbers	4175	18 295	3572	605	68
$D(\beta) = m + p \leqslant 1000$	4051	15 139	1653	91	4
Percentage	97.03%	82.74%	46.27%	15.04%	5.88%

TABLE 5. *Salem numbers of degree* 8 *and trace* 11.

	$C(\beta)$ range				
	$(0, 0.001)$	$[0.001, 0.01)$	$[0.01, 0.1)$	$[0.1, 1)$	$[1, \infty)$
Salem numbers	3797	18 761	3967	651	81
$D(\beta) = m + p \leqslant 1000$	3721	16 309	2241	146	7
Percentage	97.99%	86.93%	56.49%	22.42%	8.64%

TABLE 6. *Salem numbers of degree* 8 *and trace* 15.

	$C(\beta)$ range				
	$(0, 0.001)$	$[0.001, 0.01)$	$[0.01, 0.1)$	$[0.1, 1)$	$[1, \infty)$
Salem numbers	9427	48 709	10 668	1774	266
$D(\beta) = m + p \leqslant 1000$	9266	42 709	6458	436	35
Percentage	98.29%	88.70%	60.53%	24.57%	13.15%

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(1) The Salem number defined by the polynomial

$$x^8 - 7x^7 - 7x^6 + x^5 + 4x^4 + x^3 - 7x^2 - 7x + 1$$

satisfies $D(\beta) = m + p > 9.10^7$, although $C(\beta) \simeq 0.0009$, which is relatively very small.

(2) Inversely, the Salem number defined by the polynomial

$$x^8 - 11x^7 + 36x^6 - 67x^5 + 81x^4 - 67x^3 + 36x^2 - 11x + 1$$

is an example for which the constant $C(\beta)$ is relatively very large but $D(\beta) = m + p$ is quite small. For this example we have $C(\beta) = 9.4787$ and $(m, p) = (1, 66)$, that is $D(\beta) = 67$.

Remark 4.2. We think that it is important to note that Tables 4, 5, and 6 show that the increase in percentages holds for all classes of $C(\beta)$. This indicates that the orbit size $D(\beta)$, effectively depends on trace(β) and not only on $C(\beta)$.

Moreover, the results of Table 7 suggest that the orbit size $D(\beta)$ also depends on whether trace(β) $> \beta$ or trace(β) $< \beta$,and then on the distribution of Galois conjugates of β on the unit circle.

5. *Conclusion*

In this work, we was unable to do a definitive verification or refutation of Boyd's conjecture, but we think that this paper helps understand more the structure of the beta expansion of Salem numbers, and make a contribution to the efforts for answering the question if Salem number can be a non-beta number or not.

In fact, all of the experimental results in this paper support the predictions of Boyd's model and show that there are some differences between the cases of Salem numbers of degree 6 and of degree 8. This implies that Salem numbers, of degree 8 that are not beta numbers, would eventuality exist. Moreover, it confirms the fact that, to look for some eventual examples of such numbers, it would be sufficient to consider only the set of Salem numbers which are not very large but having large values of $C(\beta)$.

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Table 7. *Another eventual factor on which $D(\beta)$ depends.*

	Salem numbers	$\beta <$ trace(β)	$\beta >$ trace(β)
	Total	6642	12 119
$0.001 \leqslant C(\beta) < 0.01$	$D(\beta) \leqslant 1000$	6131	10 178
	Percentage	92.30%	83.98%
	Total	1097	2861
$0.01 \leqslant C(\beta) < 0.1$	$D(\beta) \leqslant 1000$	769	1463
	Percentage	70.10%	51.13%
	Total	143	508
$0.1 \leqslant C(\beta) < 1$	$D(\beta) \leqslant 1000$	57	89
	Percentage	39.86%	17.52%
	Total	15	66
C(β) > 1	$D(\beta) \leqslant 1000$	6	1
	Percentage	40%	1.51%

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