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NORMAL FORMS FOR ELEMENTS OF $o(p, q)$ AND HAMILTONIANS WITH INTEGRALS LINEAR IN MOMENTA

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Abstract

We solve the problem of finding a simultaneous matrix normal form for an element of the Lie algebra $o(p, q)$ and the underlying indefinite inner product. The results are used to determine several classes of classical Hamiltonian dynamical systems which possess a first integral linear in the momentum variables.

1. Introduction

The purpose of this paper is an attempt to characterise Hamiltonian systems which have a first integral linear in the momentum variables. The Hamiltonians concerned are "classical", that is, the sum of kinetic and potential energy terms, but the metric although flat is not assumed to be positive definite. In order to effect this characterisation of the Hamiltonian systems it is necessary to be able to find a normal form for elements of the Lie algebra $o(p, q)$ of the generalised orthogonal group $O(p, q)$. This normal form problem is considerably more complicated than that of finding normal forms for elements of $o(n)$ (see [2]), essentially because the eigenspaces of an element M of $o(p, q)$ need not produce a basis for $\mathbb{R}^{p+q}$.

In Section 2, the normal form problem for elements of $o(p, q)$ is solved completely and by methods which require nothing other than standard techniques from elementary linear algebra. It is shown that a given element M of $o(p, q)$ can always be put into real Jordan normal form in such a way that, at the same time, the inner product g used to define $o(p, q)$ by the requirement that the matrix gM be skew-symmetric, is also in normal form.

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However, it is not usually the case that g will be in diagonal form with 1's and -1's on the diagonal. The approach pursued here is to be contrasted with the theoretical investigations of Burgoyne and Cushman [1], who considered the problem of characterising the orbits of the adjoint representation of the classical Lie algebras.

In Section 3, the normal form theory is applied to obtain several classes of classical Hamiltonians with linear integrals of motion. In particular, we obtain all such Hamiltonians corresponding to Euclidean and Lorentzian metrics.

The notion, such as it is, is explained in the paper itself. We write $\operatorname{diag}(A, B, \ldots, C)$ for a matrix in block diagonal form for which the blocks are $A, B, \ldots, C$.

As a final remark in the Introduction, we should like to point out that we are currently engaged on the project of using the theory of Section 2 to normalise the holonomy algebra of the Levi-Civita connection of a pseudo-Riemannian metric. The results of that investigation will be reported elsewhere.

2. Normal forms for elements of $o(p, q)$

We let g be the inner product with matrix $\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$ relative to the standard basis for $\mathbb{R}^{p+q}$. This representation of g is for the purpose of exposition only and will subsequently be changed. We shall also denote the g-inner product of two vectors $x, y \in \mathbb{R}^{p+q}$ by $\langle x, y\rangle$ $(= g(x, y))$. A $(p+q) \times (p+q)$ matrix M belongs to the Lie algebra $o(p, q)$ if and only if

$$\langle Mx, y\rangle + \langle x, My\rangle = 0 \tag{2.1}$$

for all $x, y \in \mathbb{R}^{p+q}$. Since we are assuming for the moment that the matrix of g is $\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$, (2.1) is equivalent to M being of the form

$$M = \begin{bmatrix} M & Q \\ Q^{\mathsf{T}} & N \end{bmatrix}, \tag{2.2}$$

where M and N are skew $p \times p$ and $q \times q$ matrices, respectively, and Q is $p \times q$. The adjoint representation is given by

$$\operatorname{ad}(A)M = A^{-1}MA \tag{2.3}$$

where $A \in O(p, q)$ and $M \in o(p, q)$. It is a simple matter to check that the form of M given by (2.2) remains invariant under the adjoint action.

Our normal form problem consists of finding a simple matrix description for a given element M of $o(p, q)$ under the adjoint action. However, we shall also allow ourselves to change the representation of the inner product g by making a linear transformation, T say, of $\mathbb{R}^{p+q}$. Under such a transformation, M and g change according to $T^{-1}MT$ and $T^{\mathsf{T}}gT$, of course.

We shall establish the following lemmas.

LEMMA 2.1. (i) *If λ is an eigenvalue of M, then so too is $-\lambda$.*

(ii) *If λ and μ are eigenvalues of M with associated eigenvectors x and y, respectively, then x and y are g-orthogonal if $\lambda+\mu$ is nonzero.*

(iii) *If λ is not zero, any eigenvector of M with eigenvalue λ is null.*

PROOF. Let x be an eigenvector of M with eigenvalue λ, that is,

$$Mx = \lambda x. \tag{2.4}$$

Equations (2.1) and (2.4) imply that for all $y \in \mathbb{R}^{p+q}$,

$$\langle x, (M+\lambda I)y\rangle = 0. \tag{2.5}$$

Now if $-\lambda$ were not an eigenvalue of M, then $M+\lambda I$ would be invertible and then x would be orthogonal to all $y \in \mathbb{R}^{p+q}$. In turn it would follow that x were zero, a contradiction.

(ii) We have in addition to (2.4)

$$Mx = \mu y. \tag{2.6}$$

Equations (2.1), (2.4) and (2.6) imply that

$$(\lambda+\mu)\langle x, y\rangle = 0 \tag{2.7}$$

and the result follows.

(iii) Suppose that (2.4) holds. Then setting $y = x$ in (2.1) gives

$$\lambda\langle x, x\rangle = 0 \tag{2.8}$$

and the result is now clear.

LEMMA 2.2. *If the subspace W is M-invariant then so too is $W^{\perp}$.*

PROOF. The proof is standard noting that it requires only nondegeneracy of g and not positive-definitiveness.

LEMMA 2.3. *The dimensions of the kernels of $M-\lambda I$ and $M+\lambda I$ are equal.*

PROOF. Note first of all that (2.1) implies that the adjoint M^* of M is given by $-M$. Thus the adjoint of $M-\lambda I$ is $-(M+\lambda I)$ and

$$\operatorname{Ker}(M-\lambda I) = [\operatorname{Im}(M-\lambda I)^*]^{\perp} = [\operatorname{Im}(-(M+\lambda I))]^{\perp}.$$

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Now $\dim[\operatorname{Im}(-(M+\lambda I))]^{\perp} = \dim(\operatorname{Ker}(M+\lambda I))$ and the result follows.

In view of Lemma 1(i), we may classify the eigenvalues of M into the following types: 0, $\pm\alpha$; $\pm i\beta$; $\pm\gamma \pm i\delta$, where α, β, γ, δ are nonzero and real. In the sequel, we shall consistently use the letters α, β, γ, δ as above to distinguish the four different kinds of eigenvalue (and consequently eigenspace) and append subscripts when we wish to consider several eigenvalues of the same type simultaneously. It will also be convenient to introduce the notion of *generalised eigenspace* V_λ, which extends the usual notion of eigenspace in two ways. First of all, for a given eigenvalue λ of M, we consider kernels of powers of $M - \lambda I$ just as in Jordan canonical form theory. For fixed (real) λ these kernels form an increasing sequence of subspaces of $\mathbb{R}^{p+q}$, which eventually stabilise. (Indeed the index of stabilisation is just the power of $x - \lambda$ occurring in the minimum polynomial of M with indeterminate x.)

The second way in which the notion of eigenspace is extended depends on the nature of the eigenvalue λ itself. For $\lambda = 0$ we take V_0 to be the kernel of M^{r_0}, where r_0 is the index of stabilisation in the sense defined above. For $\lambda = \pm\alpha$ we take for V_α the direct sum of the subspaces $\operatorname{Ker}(M-\alpha)^{r_\alpha}$ and $\operatorname{Ker}(M+\alpha)^{r_\alpha}$, where r_α is the index of stabilisation corresponding to both α and $-\alpha$. (The fact that these indices are equal follows from an easy modification of Lemma 2.3.) For $\lambda = \pm i\beta$ to define V_β we proceed as for $\lambda = \pm\alpha$ except that it is to be understood that we take the *real* subspace determined by $\operatorname{Ker}(M - i\beta)^{r_\beta}$ and $\operatorname{Ker}(M + i\beta)^{r_\beta}$. (For every element x of $\operatorname{Ker}(M - i\beta)^{r_\beta}$ the conjugate $\overline{x}$ belongs to $\operatorname{Ker}(M + i\beta)^{r_\beta}$. Thus $x + \overline{x}$ and $i(x - \overline{x})$ span a real two-dimensional subspace.) Finally, for $\lambda = \pm\gamma \pm i\delta$ we take the direct sum of the kernels of $(M - i(\pm\gamma \pm \delta))^{r_{\gamma,\delta}}$ regarded as real subspaces whose dimension is divisible by four. Here $r_{\gamma,\delta}$ is the index of stabilisation corresponding to each of the four values $\lambda = \pm\gamma \pm i\delta$.

The following lemma greatly simplifies the task of finding a normal form for M.

LEMMA 2.4. *The generalised eigenspaces of M are g-orthogonal.*

PROOF. Suppose that λ and μ are eigenvalues of M and that $\lambda + \mu$ is nonzero. Suppose further that the index of stabilisation of $M - \lambda$ and $M - \mu$ are r and s, respectively. Assume also that $(M-\lambda)^r v = 0$ and $(M-\mu)^s w = 0$, so that v and w belong to the generalised eigenspaces corresponding to λ and μ, respectively. We have to show that $\langle v, w\rangle = 0$ and we use induction as follows.

We suppose there exists a positive integer l such that for $l \le \rho + \sigma \le r + s$

$$\langle (M-\lambda)^\rho v, (M - \mu^\sigma) w\rangle = 0. \tag{2.9}$$

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We show that (2.9) is also true for $l-1 \le \rho+\sigma \le r+s$. Proceeding in this way, we can reduce to the case where (2.9) holds for $\rho=\sigma=0$, which gives the desired conclusion.

From (2.1) we have the following identity:

$$\langle M(M-\lambda)^\rho v\,,\,(M-\mu)^\sigma w\rangle + \langle (M-\lambda)^\rho v\,,\,M(M-\mu)^\sigma w\rangle = 0 \tag{2.10}$$

$$\begin{aligned}\Rightarrow \langle (M-\lambda)^{\rho+1} v\,,\,(M-\mu)^\sigma w\rangle + \lambda\langle (M-\lambda)^{\rho+1} v\,,\,(M-\mu)^\sigma w\rangle \\ + \langle (M-\lambda)^\rho v\,,\,(M-\mu)^{\sigma+1} w\rangle + \mu\langle (M-\lambda)^\rho v\,,\,(M-\mu)^\sigma w\rangle = 0.\end{aligned} \tag{2.11}$$

Now by the induction hypothesis, the first and third terms in (2.11) vanish for $l \le \rho+\sigma+1 \le r+s$ and since $\lambda+\mu$ is nonzero, it follows that (2.9) holds for $l-1 \le \rho+\sigma \le r+s$.

We note also that $\mathbb{R}^{p+q}$ is a direct sum of the generalised eigenspaces of M as follows from the usual Jordan canonical form theory. In the case where $\rho = n$ and $q=0$, that is, M is a skew-symmetric matrix, the eigenspaces of M span $\mathbb{R}^n$ and there is no need to consider generalised eigenspaces. The same is not true for $M \in o(p\,,q)$ in general, however; see, for example, the matrices belonging to $o(2\,,2)$ numbered V and VI in Appendix 2.

According to Lemma 2.4, in order to find a normal form for M, we need to study the generalised eigenspaces of M. It will, however, suffice to consider subspaces of these generalised eigenspaces which are *irreducible* in the sense that they cannot be split as a direct sum of M-invariant subspaces. In effect, we are breaking M into irreducible Jordan block components and then combining them according to the nature of the eigenvalues so as to obtain maximal, M-invariant, irreducible, real subspaces. These generalised Jordan blocks give a block decomposition for the matrix of M on each generalised eigenspace and thus determine the entire normal form for M.

Let W be a maximal, M-invariant, irreducible real subspace. Then W has a subspace spanned by (possibly complex) eigenvectors of M, which we call the eigensubspace of W and denote by V. We claim that V is a null subspace, in the sense that the restriction of the inner product g to V is zero. Indeed, suppose that V were not null. Then we could find a subspace V_1 of V on which g were nondegenerate. Howver, in that case W would be the direct sum of V_1 and its $g|W$-orthogonal space $V_1^\perp$, which would contradict the irreducibility of W.

We shall assume that a basis for $\mathbb{R}^{p+q}$ has been chosen, relative to which the matrix of M is in "real Jordan canonical form." With the convention for eigenvalues introduced above, this means that for $0, \pm\alpha, \pm i\beta, \pm\gamma \pm i\delta$, the canonical Jordan blocks have the following forms, respectively:

$$\begin{bmatrix} 0 & I & 0 & 0 & & & 0 \\ 0 & 0 & I & 0 & & & \\ 0 & 0 & 0 & I & & & \\ & & & & & & \\ & & & & I & 0 & 0 \\ & & & & 0 & I & 0 \\ & & & & 0 & 0 & I \\ 0 & & & & 0 & 0 & 0 \end{bmatrix} \tag{2.12$_0$}$$

$$\begin{bmatrix} \alpha I & 0 & I & 0 & & & & 0 \\ 0 & -\alpha I & 0 & I & & & & \\ 0 & 0 & \alpha I & 0 & & & & \\ 0 & 0 & 0 & -\alpha I & & & & \\ & & & & & & & \\ & & & & I & 0 & 0 & 0 \\ & & & & 0 & I & 0 & 0 \\ & & & & \alpha I & 0 & I & 0 \\ & & & & 0 & -\alpha I & 0 & I \\ & & & & 0 & 0 & \alpha I & 0 \\ 0 & & & & 0 & 0 & 0 & -\alpha I \end{bmatrix} \tag{2.12$_\alpha$}$$

$$\begin{bmatrix} 0 & \beta I & I & 0 & & & & 0 \\ -\beta I & 0 & 0 & I & & & & \\ 0 & 0 & 0 & \beta I & & & & \\ & & & & & & & \\ & & & & I & 0 & 0 & 0 \\ & & & & 0 & I & 0 & 0 \\ & & & & 0 & \beta I & I & 0 \\ & & & & -\beta I & 0 & 0 & I \\ & & & & 0 & 0 & 0 & \beta I \\ 0 & & & & 0 & 0 & -\beta I & 0 \end{bmatrix} \tag{2.12$_\beta$}$$

$$\begin{bmatrix} \gamma I+\delta J & 0 & I & 0 & & 0 \\ 0 & -(\gamma I+\delta J) & 0 & I & & \\ 0 & 0 & \gamma I+\delta J & 0 & & \\ 0 & 0 & 0 & -(\gamma I+\delta J) & & \\ & & & & & \\ & & & & I & 0 \\ & & & & 0 & I \\ & & & & \gamma I+\delta J & 0 \\ 0 & & & & 0 & -(\gamma I+\delta J) \end{bmatrix} \qquad (2.12_{\gamma,\delta})$$

In the matrices (2.12) we have been deliberately vague about the size of the blocks. However, we can certainly assume from Jordan canonical form theory, that the following is the case: if the block on the (i, j)th entry is of size $i_r \times i_s$, then $i' \geq i$ and $j' \geq j$ imply that $i'_{r'} \leq i_r$ and $j'_{s'} \leq j_s$. In fact we shall see that in order for the matrix of gM to be skew-symmetric and g to be nondegenerate, the blocks in each case in (2.12) are square and of the same size. We shall write the matrix for g (or better the restriction of g to the maximal, M-invariant, irreducible real subspaces corresponding to (2.12)) in the following form

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & & C_{1k} \\ C_{21} & C_{22} & & & \\ C_{31} & & & & \\ & & & & \\ C_{k1} & & & & C_{kk} \end{bmatrix} \qquad (2.13)$$

where C_{11} is zero in (2.12_0) and $C_{11}, C_{12}, C_{21}, C_{22}$ are zero in the other cases in (2.12), $C_{33}, \ldots, C_{kk}$ are symmetric and ${}^{\mathsf{T}}C_{mj}$ is equal to C_{jm} for j different from m.

We consider next the case where M is given by $(2.12_{\gamma,\delta})$. Here we assume that (2.13) has the same block decomposition as $(2.12_{\gamma,\delta})$. The analysis will be facilitated by means of several lemmas. The matrices in the lemmas are all $2n \times 2n$ and J denotes the canonical complex structure $\left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ on $\mathbb{R}^{2n}$. We prove only the nontrivial parts of the lemmas.

LEMMA 2.5. (i) *A $2n \times 2n$ matrix M commutes with J if and only if M is "complex," that is, $M = \left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right]$ for some $n \times n$ matrices A, B.*

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(ii) *A* $2n \times 2n$ *matrix* M *anti-commutes with* J $(MJ + JM = 0)$ *if and only if* $M = \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$ *for some* $n \times n$ *matrices* A, B.

(iii) *If the* $2n \times 2n$ *matrix* M *satisfies the condition*

$$2\gamma M + \delta(MJ - JM) = 0, \tag{2.14}$$

where γ *is not zero, then* M *is zero.*

PROOF OF (iii). Pre-multiplying and post-multiplying (2.14) by J and adding the results imply, as α is not zero, that

$$MJ + JM = 0.$$

Equation (2.14) and the latter condition give

$$(\gamma I - \delta J)M = 0.$$

The result now follows because if γ is not zero, $\gamma I - \delta J$ is invertible. It will be useful to refer to a matrix which anti-commutes with J as "anti-complex."

LEMMA 2.6. *Given a* $2n \times 2n$ *matrix* A *which is invertible and anti-complex, there exists complex* P *such that*

$$AP = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

PROOF. If A is anti-complex, $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ A is complex and equal to P^{-1}, say, which gives as required,

$$AP = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

LEMMA 2.7. (i) *Suppose* C *is a matrix such that* $CJ - JC$ *is complex. Then* C *is complex.*

(ii) *Suppose* C *is a matrix such that* $CJ + JC$ *is anti-complex. Then* C *is anti-complex.*

PROOF. (i)

$$J(CJ - JC) - (CJ - JC)J = 0$$
$$\Rightarrow 2JCJ + 2C = 0 \qquad \Rightarrow JC - CJ = 0.$$

(ii) The proof of (ii) is similar to that of (i).

We now reconsider the matrix of gM, where g and M are given by (2.13) and $(2.12_{\gamma,\delta})$, respectively. In order for gM to be skew-symmetric it

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is necessary and sufficient that the following conditions hold for $1 \leq j, m \leq k$:

$$(-1)^j C_{mj}(\gamma I + \delta J) + (-1)^m (\gamma I - \delta J) C_{mj} = C_{mj-2} + C_{m-2j}. \tag{2.15}$$

In (2.15) j and m may be supposed to assume all values between 1 and k and where those C's occur which are not defined, such as C_{10} and $C_{2,-1}$ these C's may be taken to be zero.

Suppose that both j and m are even in (2.15). Then one finds that

$$2\gamma C_{mj} + \delta(C_{mj}J - JC_{mj}) = C_{mj-2} + C_{m-2j}. \tag{2.16}$$

It follows, by induction on the value of $j + m$ and from Lemma 2.5, that whenever m and j are both even, C_{mj} is zero. The induction starts because C_{04}, C_{22} and C_{40} are all zero. Similarly, whenever m and j are both odd, C_{mj} is also zero.

Suppose next that j is even and m is odd in (2.15) so that

$$\delta(C_{mj}J + JC_{mj}) = C_{mj-2} + C_{m-2j}. \tag{2.17}$$

By induction and Lemma 2.7 it follows that each C_{mj} with j even and m odd is anti-complex. Similarly, C_{mj} with j odd and m even is anti-complex. Furthermore we have that

$$C_{mj-2} + C_{m-2j} = 0 \tag{2.18}$$

whenever the sum of j and m is odd and satisfies $3 \leq j + m \leq 2k - 2$.

Next we exploit the fact that $(2.12_{\gamma,\delta})$ is invariant under a change of basis corresponding to a matrix of the form

$$\begin{bmatrix} I & 0 & a_1 & 0 & a_2 & 0 & a_3 & 0 & & & 0 \\ 0 & I & 0 & b_1 & 0 & b_2 & 0 & b_3 & & & \\ 0 & 0 & I & 0 & a_1 & 0 & a_2 & 0 & & & \\ 0 & 0 & 0 & I & 0 & b_1 & 0 & b_2 & & & \\ 0 & 0 & 0 & 0 & I & 0 & a_1 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & I & 0 & b_1 & & & \\ & & & & & & & & & & \\ & & & & & & & & & a_1 & 0 \\ & & & & & & & & & 0 & b_1 \\ & & & & & & & & & I & 0 \\ 0 & & & & & & & & & 0 & I \end{bmatrix} \tag{2.19}$$

where the a's and b's are complex but otherwise arbitrary. We find that under (2.19) the $C_{2s+1,k'}$s transform as follows:

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$$\overline{C}_{2s+1,k} = a_s^{\mathsf{T}} C_{1k} + a_{s-1}^{\mathsf{T}}(-C_{1k}b_1 + C_{3k}) + a_{s-2}^{\mathsf{T}}(C_{1k}b_2 - C_{3k}b_1 + C_{5k}) + \cdots + a_1^{\mathsf{T}}((-1)^{s-1}C_{1k}b_{s-1} \cdots - C_{2s-3,k}b_1 + C_{2s-1,k}) + (-1)^s C_{1k}b_s + \cdots - C_{2s-5,k}b_3 + C_{2s-3,k}b_{2} - C_{2s-1,k}b_1 + C_{2s+1,k}. \tag{2.20}$$

In view of (2.20), (2.13) may be reduced to a block diagonal form with blocks running from the upper right to the lower left hand corners. Because of (2.18) we only have to reduce to zero the entries in the last column other than C_{1k}. Now we could, for example, set all the b's to zero and determine the a's inductively by requiring $\overline{C}_{2s+1,k}$ to be zero. This is possible because a_s can be solved for since C_{1k} is nonsingular.

We have reduced the marix (2.13) to one in which the only nonzero blocks are those on the diagonal from the upper right to lower left hand corners. We still have at our disposal a transformation corresponding to a matrix of the form

$$\operatorname{diag}(P, Q, P, Q, \dots, P, Q) \tag{2.21}$$

where P and Q are non-singular and complex and there are $k/2$ such P's and Q's. Such a transformation leaves invariant the matrix $(2.12_{\gamma,\delta})$ and induces the transformation $P^{\mathsf{T}}C_{1k}Q$ on C_{1k}. Using Lemma 2.6 we can find P and Q such that $P^{\mathsf{T}}C_{1k}Q$ is the form $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. This gives the normal form for (2.13) starting from $(2.12_{\gamma,\delta})$ and the details are given in Appendix 1.

We examine next the case where M is given by (2.12_α). The analysis is very similar to that just performed for $(2.12_{\gamma,\delta})$ with δ equal to zero. However, the blocks in (2.12_α) may be $n \times n$ with n odd and the block decomposition in (2.13) is now assumed to be aligned with (2.12_α). From the discussion given for $(2.12_{\gamma,\delta})$, we may conclude in the present context the following: C_{mj} is zero whenever j and m have the same parity. Furthermore, whenever j and m have opposite parity and $3 \le j+m \le 2k-2$,

$$C_{mj-2} + C_{m-2j} = 0. \tag{2.22}$$

Again (2.12_α) is invariant under transformations which correspond to matrices which formally resemble (2.19) and (2.21) except that the blocks involved no longer need to be complex. The reduction of (2.13) to normal form then proceeds much as in the previous case except that the fundamental block C_{1k} may now be reduced to I. The details are to be found in Appendix 1.

We examine now the situation where the matrix of M is given by (2.13_β). Since k is even, we define l to be one-half the value of k and recast (2.12_β)

as follows

$$\begin{bmatrix} \beta J & I & 0 & & & 0 \\ 0 & \beta J & I & & & \\ 0 & 0 & \beta J & & & \\ & & & & & \\ & & & & \beta J & I \\ 0 & & & & 0 & \beta J \end{bmatrix} \tag{2.23}$$

We now assume that (2.13) gives an $l \times l$ block decomposition for the matrix of g aligned with (2.23) and for which C_{11} is zero. The conditions required to ensure that the matrix of gM is skew-symmetric are

$$\beta(C_{mj}J - JC_{mj}) + C_{mj-1} + C_{m-1j} = 0 \tag{2.24}$$

for $1 \leq j, m \leq l$ and the C_{mj}'s undefined in (2.24) such as C_{01} are understood to be zero. From (2.24), Lemma 2.7 and induction, it follows that each C_{mj} in (2.13) is complex and also that

$$C_{mj-1} + C_{m-1j} = 0 \tag{2.25}$$

for $2 \leq j + m \leq 2l - 2$. In turn, (2.25) implies that C_{mj} is zero for $2 \leq j + m \leq l$.

Note that (2.12_β) is invariant under a change of basis corresponding to a matrix of the form

$$\begin{bmatrix} I & a_1 & a_2 & a_3 & & a_{l-2} & a_{l-1} \\ 0 & I & a_1 & a_2 & & a_{l-3} & a_{l-2} \\ 0 & 0 & I & a_1 & & & \\ 0 & 0 & 0 & I & & & \\ & & & & & & \\ & & & & & a_1 & a_2 \\ & & & & & I & a_1 \\ 0 & & & & & 0 & I \end{bmatrix}, \tag{2.26}$$

where $a_1, \ldots, a_{l-1}$ are complex but otherwise arbitrary. By making such a change of basis, we can determine the a's inductively so that (2.13) is in block diagonal form much in the same way as for $(2.12_{\gamma,\delta})$. The normalisation procedure then reduces to finding a normal form for A $(= C_{1l})$ under the action of $P^{\mathrm{T}}AP$ where P is in $\mathrm{GL}(n, C)$. This is because (2.12_β) remains invariant under a change of basis corresponding to an $l \times l$ block matrix of the form

$$\operatorname{diag}(P, P, P, \ldots, P). \tag{2.27}$$

The matrix A is complex and distinct cases arise according to whether l is even or odd. When l is even, A is skew-symmetric and may be brought into

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symplectic normal form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. When l is odd, however, A is symmetric and may be brought into diagonal form with 1's and -1's on the diagonal. The details of the normal form are given in Appendix 1.

It remains to consider the case where M is given by (2.12_0). We assume that the matrix of g given by (2.13) is aligned with (2.12_0) and that C_{11} is zero. In this case, the analysis is very similar to that for (2.12_β) with β set equal to zero, except that the blocks in (2.13) are not necessarily complex. Again (2.12_0) is invariant under transformations corresponding to matrices of the form (2.26) and (2.27) except that the a's, b's and P need not be complex. The normal form for M reduces to finding the normal form for C_{1k} under the transformation $P^{\mathrm{T}} C_{1k} P$, where C_{1k} is skew-symmetric for k even and symmetric for k odd. The details of the normal form are given in Appendix 1.

Before stating a theorem which summarises the results obtained thus far in this section, we make one further point concerning the normal forms given in Appendix 1. To each maximal, M-invariant, irreducible subspace we attach a signature, which is the difference between the number of positive and negative signs when each of the inner products are written in diagonal form. Of course in terms of building up the possible normal forms for M and g, one may take the signatures given in Appendix 1 or their opposites corresponding to replacing g by $-g$. Actually, it is only in cases $(2.12_0, k$ odd) and $(2.12_\beta, 1/2k$ odd) that the signature of g can be nonzero. In this cases, the signature of g is expressed in terms of the signature of the fundamental block A.

THEOREM 2.8. *Given an element M of $o(p, q)$, there exists a basis of $\mathbb{R}^{p+1}$ relative to which the matrix representing M is block diagonal (real Jordan canonical form), each block being one of the types occurring in* (2.12). *Furthermore, the matrix of g has the same block decomposition as M with each block given by one of the types in Appendix* 1.

To conclude this section we use the theory developed above to obtain normal forms for elements of several Lie algebras. The results will be quoted in Section 3. Consider first of all the case of $o(n)$, that is, $p = n$, $q = o$. In this case the signature of g must be n and the only possible blocks in the normal form for $M \in o(n)$ are of type $(2.12_0$, k odd) and $(2.12_\beta$, $1/2k$ odd). It is then easy to see that the only way to obtain signature n for g is to have, up to change of basis of course, g correspond to the standard Euclidean metric and M be given by, for some $0 \le r \le [n/2]$ (the integral part of $n/2$)

$$\operatorname{diag}(\beta_1 J, \beta_2 J, \ldots, \beta_r J, 0). \tag{2.28}$$

In (2.28) J is the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and "0" the $(n-2r)\times(n-2r)$ zero matrix. Notice that we obtain as a corollary that the eigenvalues of M are zero or pure imaginary.

We next consider the Lie algebra $o(p, 1)$. It will be convenient to discuss separately the cases where p is even, say $p = 2n$, and p is odd, say $p = 2n+1$. In these cases the signature of g is $2n-1$ and $2n$, respectively. Immediately we can assert that the only possible blocks in the normal form for $M \in o(2n, 1)$ or $o(2n+1, 1)$ are of type (2.12_0, k odd), (2.12_α) and (2.12_β, $1/2k$ odd). For the other block types would lead to 4-dimensional subspaces with signature zero which would make it impossible for g to be Lorentzian. In the case of $o(2n, 1)$ one finds that the signature $2n-1$ can be achieved if and only if M can be brought into one of the following three forms where $0 \le r \le n$ in (2.29) and $0 \le r \le n-1$ in (2.30) and (2.31):

$$\operatorname{diag}(\beta_1 J, \beta_2 J, \dots, \beta_r J, 0) \tag{2.29}$$
$$(0,\ \text{the } (2(n-r)+1)\times(2(n-r)+1) \text{ zero matrix}),$$

$$\operatorname{diag}\left(\beta_1 J, \beta_2 J, \dots, \beta_r J, \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}, 0\right) \tag{2.30}$$
$$(0,\ \text{the } (2(n-r-1)+1)\times(2(n-r-1)+1) \text{ zero matrix}),$$

$$\operatorname{diag}\left(\beta_1 J, \beta_2 J, \dots, \beta_r J, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, 0\right) \tag{2.31}$$
$$(0,\ \text{the } 2(n-r-1)\times(2n-r-1) \text{ zero matrix}).$$

Note that the matrix for g corresponding to (2.31) is

$$\operatorname{diag}(1, 2, \dots, 1, 1, -1)$$

so that the normal form given in Appendix 1, to be specific (2.12_0) with k equal to 3 and A equal to 1, has been modified slightly so as to make the matrix for g diagonal.

The corresponding analysis for $M \in o(2n+1, 1)$ leads to the conclusion that the matrix for g can be put into the form $\operatorname{diag}(1, 1, \dots, 1, 1, -1)$ and that the matrix for M can be brought into forms similar to (2.29), (2.30) and (2.31), except that "0" is now the $2(n-r+1)\times 2(n-r+1)$, $2(n-r)\times 2(n-r)$ and $(2(n-r)-1)\times(2(n-r)-1)$ matrix, respectively. In the first and second of these cases r may assume all integral values between 0 and n inclusive and in the third case between 0 and $n-1$ inclusive.

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3. Classical Hamiltonians with linear integrals of motion

We suppose throughout this section that we are given a Hamiltonian function $h: T^*M \to \mathbb{R}$, where T^*M is the cotangent bundle of some smooth m-dimensional manifold M. We suppose further that M carries a flat, non-degenerate but not necessarily positive definite metric g and denote the dual or contravariant cometric by G. Letting Π be the natural submersion map from T^*M to M, the Hamiltonian h is given by

$$h(p) = 1/2G(p, p) + (\Pi^* V)(p), \tag{3.1}$$

where p is a typical point of T^*M and $V: M \to \mathbb{R}$ is the potential energy function.

Two such Hamiltonians h_1 and h_2 with

$$h_1(p) = 1/2G_1(p, p) + (\Pi^* V_1)(p), \tag{3.2}$$

$$h_2(p) = 1/2G_2(p, p) + (\Pi^* V_2)(p) \tag{3.3}$$

will be said to be equivalent if there exists a diffeomorphism ϕ of M such that

$$\phi_* G_1 = G_2, \tag{3.4}$$

$$\phi^* V_2 = V_1. \tag{3.5}$$

Since the metrics concerned are flat, such a diffeomorphism ϕ can be assumed locally to be of the form

$$\overline{x} = Ax + b \tag{3.6}$$

where $A \in O(p, q)$, $b \in \mathbb{R}^m$, $p + q = m$ and the metrics have signature (p, q).

We shall now investigate under what conditions a Hamiltonian h of the form (3.1) has a first integral, k say, which is linear in the momentum variables p. By Noether's theorem, such as integral can be identified with a certain vector field K on M, the function $k(p)$ being given by the pairing of K and p, for any point $p \in T^*M$. The necessary and sufficient conditions for K to give rise to a first integral are:

$$L_K G = 0 \tag{3.7}$$

and

$$KV = 0. \tag{3.8}$$

(In (3.7) L denotes the Lie derivative operator and (3.8) represents the directional derivative of V along K.)

Equation (3.7) states that K is a Killing vector field of the cometric G and hence the corresponding metric g. Since g is flat, it may be identified locally with the standard indefinite metric of signature (p, q) on $\mathbb{R}^{p+q}$ (see [3]). Using standard coordinates (x^i) on $\mathbb{R}^{p+q}$, we may write an arbitrary Killing field K as

$$K = a_i^j x^i \frac{\partial}{\partial x^j} + \mu^i \frac{\partial}{\partial x^i} \tag{3.9}$$

where $a \in o(p, q)$, $\mu \in \mathbb{R}$ and the summation convention on repeated indices applies. The matrix a in (3.9) corresponds to a general, infinitesimal pseudo-rotation and μ to an arbitrary translation.

Consider next the effect of applying the isometric transformation (3.6) to K as given by (3.9). If K is transformed into $\overline{K}$ where

$$\overline{K} = \overline{a}_i^j \overline{x}^i \frac{\partial}{\partial \overline{x}^j} + \overline{\mu}^i \frac{\partial}{\partial \overline{x}^i} \tag{3.10}$$

then a and $\overline{a}$ and μ and $\overline{\mu}$, respectively, are related by

$$a = A^{-1} \overline{a} A, \tag{3.11}$$

$$\mu = A^{-1} \overline{a} b + A^{-1} \overline{u}. \tag{3.12}$$

(The products on the right-hand side of (3.11) and (3.12) are matrix products and b and μ are properly regarded as column vectors.)

Equation (3.11) reveals the connection between the problem at hand and the normal form theory of Section 2. In principle, one can classify Hamiltonians of the form (3.1) having a first integral linear in momenta by bringing the matrix a, and the inner product g into normal form. It is obviously impractical to effect this classification in complete generality owing to the very large number of normal forms. Accordingly, we shall content ourselves by discussing several important cases.

The first of these cases is where $p = m$ and $q = 0$, so that g is a Euclidean metric. Referring to (3.11) we can reduce a to the normal form given by (2.28) of Section 2. Thereafter we set $A = I$ in (3.6) and (3.12) becomes

$$\mu = \overline{a} b + \overline{\mu}. \tag{3.13}$$

From (3.13) it follows that we may translate to zero each μ^i in (3.9) which corresponds to a nonzero eigenvalue in the normal form of a. In short, K

can be normalised to the form:

$$K = \beta_1\left(s^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right) + \beta_2\left(x^3\frac{\partial}{\partial x^4} - x^4\frac{\partial}{\partial x^3}\right) + \cdots + \beta_r\left(x^{2r-1}\frac{\partial}{\partial x^{2r}} - x^{2r}\frac{\partial}{\partial x^{2r-1}}\right) + \mu_{2r+1}\frac{\partial}{\partial x^{2r+1}} + \cdots + \mu_m\frac{\partial}{\partial x^m}. \tag{3.14}$$

Next, we reconsider (3.12) and write A as $\begin{bmatrix} I_{2r} & 0 \\ 0 & B\end{bmatrix}$ where $B \in O(m-2r)$. By choosing B appropriately, we can rotate the constant coefficient part of (3.14) so that on introducing the abbreviation K_{ij} to denote $x^i\partial/\partial x^j - x^j\partial/\partial x^i$

$$K = \sum_{i=1}^{r}\beta_i K_{2i-1,2i} + \mu\frac{\partial}{\partial x^{2r+1}}. \tag{3.15}$$

With K given by (3.15) the partial differential equation (3.8) may be integrated in closed form by the method of characteristics, yielding the solution as an arbitrary smooth function of $m-1$ arguments. We summarise these results in the following theorem, in which $\arctan(x^i/x^j)$ is abbreviated to a_{ij}. Similarly in Theorem 3.2, $\operatorname{arctanh}(x_i/x_j)$ is denoted by b_{ij}.

THEOREM 3.1. *Let h be a Hamiltonian of the form* (3.1) *with G a flat Euclidean metric. Then the Hamiltonian flow determined by h has a first integral I of degree one in momenta if and only if h can be represented locally as $h = \frac{1}{2}\delta^{ij}p_ip_j + V$, where V is an arbitrary smooth function of the following $m-1$ arguments (r being some integer satisfying $0 \le r \le [m/2]$, $[m/2]$ denoting the integral part of $m/2$):*

$$(x^{2i-1})^2 + (x^{2i})^2 \quad (1 \le i \le r), \qquad \beta_j a_{2j-2,2j-3} - \beta_{j-1}a_{2j,2j-1} \quad (2 \le j \le r),$$

$$\sum_{i=1}^{r}(\beta_i x^{2r+1} - \mu a_{2i,2i-1}), \qquad x_k \quad (2r+2 \le k \le m).$$

Furthermore in that case I is given by $\sum_{i=1}^{r}\beta_i(x^{2i-1}p_{2i} - x^{2i}p_{2i-1}) + \mu p_{2r+1}$.

The second of the cases that we consider is where p is $m-1$ and q is 1, so that g is a flat Lorentz metric. It is convenient to separate further the cases where p is even and p is odd, so we assume first of all that p is $2n$. We proceed much in the same way as we did in the Euclidean case, this time making use of the normal forms (2.29), (2.30) and (2.31). Thus we find that the analogue of (3.15) either formally resembles (3.15) but on a space with the extra variable x^{2n+1}, or is given by

$$K = \sum_{i=1}^{r} \beta_i K_{2i-1,2i} + \alpha \left(x^{2r+1} \frac{\partial}{\partial x^{2r+2}} + x^{2r+2} \frac{\partial}{\partial x^{2r+1}} \right) + \mu \frac{\partial}{\partial x^{2r+3}} \tag{3.16}$$

or

$$K = \sum_{i=i}^{r} \beta_i K_{2i-1,2i} + x^{2r+2} \frac{\partial}{\partial x^{2r+1}} + (x^{2r+3} - x^{2r+1}) \frac{\partial}{\partial x^{2r+2}} + x^{2r+2} \frac{\partial}{\partial x^{2r+3}} + \mu \frac{\partial}{\partial x^{2r+4}}. \tag{3.17}$$

In each of these three cases (3.6) can be integrated explicitly and leads to the following result.

THEOREM 3.2. *A Hamiltonian of the form* (3.1) *with* G *the flat Lorentz metric* $\mathrm{diag}(1, 1, \ldots, 1, -1)$ *of signature* $2n-1$ *has a first integral* I *of degree one in momenta if* V *is of the form given in Theorem* 3.1 *with* m *equal to* $2n+1$, *or else* V *is an arbitrary smooth function of the following two sets of* $2n$ *variables corresponding to* (3.16) *and* (3.17), *respectively*:

(i) $(x^{2i-1})^2 + (x^{2i})^2 \quad (1 \le i \le r)$,
$\beta_j a_{2j-2,2j-3} - \beta_{j-1} a_{2j,2j-1} \quad (2 \le j \le r)$,
$\alpha a_{2r,2r-1} - \beta_r b_{2r+2,2n+1}$, $\quad (x^{2r+1})^2 - (x^{2r+2})^2$,
$\sum_{i=1}^{r} (\beta_i x^{2r+3} - \mu a_{2i,2i-1}) + \alpha x^{2r+3} - \mu b_{2r+2,2r+1}$,
$x^k \quad (2r+4 \le k \le 2n+1)$. (*Here* α *is not zero and* $0 \le r \le n-1$.)

(ii) $(x^{2i-1})^2 + (x^{2i})^2 \quad (1 \le i \le r)$,
$\beta_j a_{2j-2,2j-3} - \beta_{j-1} a_{2j,2j-1} \quad (2 \le j \le r)$, $\quad x^{2r+1} - x^{2r+3}$,
$(x^{2r+1})^2 + (x^{2r+2})^2 - (x^{2r+3})^2$, $\quad \sum_{i=1}^{r} (\beta_i x^{2r+2} + a_{2i,2i-1})$
$(x^{2r+1} - x^{2r+3}) + \mu x^{2r+2} + x^{2r+4}(x^{2r+1} - x^{2r+3})$,
$\sum_{i=1}^{r} (\beta_i x^{2r+4} - \mu a_{2i,2i-1})$, $\quad x^k \quad (2r+5 \le k \le 2n+1)$.
(*Here* $0 \le r \le n-1$).

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We consider next the situation where g is Lorentzian with signature $2n$. Pursuing the same sort of analysis as before leads to the following conclusion.

Theorem 3.3. *A Hamiltonian of the form* (3.1) *with* G *the flat Lorentz metric* $\operatorname{diag}(1, 1, \ldots, 1, -1)$ *of signature* $2n$ *has a first integral* I *of degree one in momenta iff* V *is of one of the following three types:*

(i) *as given in Theorem* 3.1 *with* m *equal to* $2n+2$ *and* $0 \le r \le n$.

(ii) *as given in Theorem* 3.2(i) *with* $0 \le r \le n$ *and* V *depending arbitrarily on* x^{2n+2}.

(iii) *as given in Theorem* 3.2(ii) *with in addition* V *depending arbitrarily on* x^{2n+2}.

Theorem 3.3 completes our discussion of the case in which G in (3.1) is Lorentzian. The other class of examples we shall consider is where G is a cometric with signature $(2, 2)$. Again we shall use (3.11) and (3.12) and (3.6) with $A \in o(2, 2)$. There are seven main classes which are numbered I through VII. The analysis for these classes proceeds in a manner very similar to the Euclidean and Lorentzian cases already considered. Accordingly we simply summarise the results for $o(2, 2)$ in the following theorem.

Theorem 3.4. *A Hamiltonian* h *of the form* (3.1) *with* G *a flat metric of signature* $(2, 2)$ *has a first integral* I *of degree one in momenta iff* h *can be represented locally as one of the following seven types:*

I. $I = \alpha_1(x^1p_1 - x^2p_2) + \alpha_2(x^3p_3 - x^4p_4) + \mu p_3$,

(i) $$\left.\begin{matrix}\alpha_1\alpha_2 \neq 0\\ \mu = 0\end{matrix}\right\} h = p_1p_2 + p_3p_4 + V\left(x^1x^2, x^3x^4, \alpha_2 \operatorname{arctanh}\left(\frac{x^1 - x^2}{x^1 + x^2}\right) - \alpha_1 \operatorname{arctanh}\frac{x^3 - x^4}{x^3 + x^4}\right),$$

(ii) $$\left.\begin{matrix}\alpha_1 \neq 0\\ \alpha_2 = 0\end{matrix}\right\} h = p_1p_2 + p_3p_4 + V\left(x^1x^2, \alpha_1 x^2 - \mu \operatorname{arctanh}\left(\frac{x^1 - x^2}{x^1 + x^2}\right), x^4\right),$$

(iii) $$\left.\begin{matrix}\alpha_1 = \alpha_2 = 0\\ \mu \neq 0\end{matrix}\right\} h = p_1p_2 + p_3p_4 + V(x^1, x^2, x^4).$$

II. $I = \beta_1(x^1p_2 - x^2p_1) + \beta_2(x^3p_4 - x^4p_3) + \mu p_3$,

(i) $\left.\begin{matrix}\beta_1\beta_2 \neq 0\\ \mu = 0\end{matrix}\right\} h = \frac{1}{2}(p_1^2 + p_2^2 - p_3^2 - p_4^2)$

$$+ V((x^1)^2 + (x^2)^2, (x^3)^2 + (x^4)^2, \beta_2 a_{21} - \beta_1 a_{43},$$

(ii) $\left.\begin{matrix}\beta_1 \neq 0\\ \beta = 0\end{matrix}\right) h = \frac{1}{2}(p_1^2 + p_2^2 - p_3^2 - p_4^2) + V((x^1)^2 + (x^2)^2),$

$\beta_1 x^3 - \mu a_{21}, \quad x^4,$

(iii) $\left.\begin{matrix}\beta_1 = \beta_2 = 0\\ \mu \neq 0\end{matrix}\right\} h = \frac{1}{2}(p_1^2 + p_2^2 - p_3^2 - p_4^2) + V(x^1, x^2, x^4).$

III.

$$I = (\gamma x^1 + \delta x^2)p_1 + (\gamma x^2 - \delta x^1)p_2 - (\gamma x^3 + \delta x^4)p_3 + (\gamma x^3 - \delta x^4)p_4,$$
$$h = p_1p_4 + p_2p_3 + V(\delta \ln((x^1)^2 + (x^2)^2)) - 2\gamma a_{21},$$
$$\delta(\ln((x^3)^2 + (x^4)^2)) - 2\gamma a_{34}, x^1x^3 - x^2x^4).$$

IV.

$$I = x^3p_1 - x^2p_2 + x^1p_3 + \mu p_4,$$
$$h = \tfrac{1}{2}(2p_1p_3 - p_2^2 + p_4^2) + V((x^1)^2, -(x^3)^2, (x^1 + x^3)x^2, \mu x^2 + x^4).$$

V.

$$I = (\alpha x^1 + x^3)p_1 - (\alpha x^2 + x^4)p_2 + \alpha x^3 p_3 - \alpha x^4 p_4 + \mu p_3,$$
$$h = p_1p_3 + p_2p_4 + V\left(\frac{\alpha x^1}{x^3} - \ln(x^3), \frac{\alpha x^2}{x^4} - \ln(x^4), \frac{x^1}{x^3} + \frac{x^2}{x^4}\right).$$

VI.

$$I = (\beta x^2 + x^3)p_1 - (\beta x^1 + x^4)p_2 + \beta x^4 p_3 - \beta x^3 p_4,$$
$$h = p_1p_4 - p_2p_3 + V(2\beta(x^1x^4 - x^2x^3) + (x^4)^2 - (x^3)^2,$$
$$(x^3)^2 + (x^4)^2; \quad \beta(x^1x^3 + x^2x^4) + x^3x^4).$$

VII.

$$I = x^3p_1 - x^4p_2 + \mu p_3,$$
$$h = p_1p_4 - p_2p_3 + V(x^4, \mu x^2 + x^3x^4, 2\mu x^1 - (x^3)^2).$$

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Appendix 1

Normal forms for g corresponding to maximal, M-invariant, irreducible subspace (all matrices $(k \times k)$

(2.12_0) *k even signature zero*

$$\begin{bmatrix} 0 & 0 & & & & 0 & 0 & 0 & J \\ 0 & & & & & 0 & 0 & -J & 0 \\ & & & & & 0 & J & 0 & 0 \\ & & & & & -J & 0 & 0 & 0 \\ & & & & & & & & \\ 0 & 0 & 0 & J & & & & & \\ 0 & 0 & -J & 0 & & & & & \\ 0 & J & 0 & 0 & & & & & 0 \\ -J & 0 & 0 & 0 & & & & 0 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

(2.12_0) *k odd*

signature $= \begin{cases} -\mathrm{sig}(A), & \frac{k-1}{2} \text{ odd} \\ \mathrm{sig}(A), & \frac{k-1}{2} \text{ even} \end{cases}$

$$\begin{bmatrix} 0 & 0 & & & 0 & 0 & A \\ 0 & & & & 0 & -A & 0 \\ & & & & A & 0 & 0 \\ & & & & & & \\ 0 & 0 & A & & & & \\ 0 & -A & 0 & & & & 0 \\ A & 0 & 0 & & & 0 & 0 \end{bmatrix}$$

$$A = \mathrm{diag}(1, 1, 1, \dots, -1, -1, \dots, -1)$$

(2.12_α)
(k necessarily even)
signature zero

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & & & 0 & 0 & 0 & B \\ 0 & & & & & 0 & 0 & B & 0 \\ & & & & & 0 & B & 0 & 0 \\ & & & & & B & 0 & 0 & 0 \\ & & & & & & & & \\ 0 & 0 & 0 & B & & & & & \\ 0 & 0 & B & 0 & & & & & \\ 0 & B & 0 & 0 & & & & & 0 \\ B & 0 & 0 & 0 & & & & 0 & 0 \end{bmatrix}$$

(2.12$_\beta$)
1/2 k even
signature zero

$$\begin{bmatrix}
0 & 0 & & & 0 & 0 & 0 & I \\
0 & & & & 0 & 0 & -I & 0 \\
 & & & & 0 & -I & 0 & 0 \\
 & & & & I & 0 & 0 & 0 \\
 & & & & & & & \\
0 & 0 & 0 & I & & & & \\
0 & 0 & -I & 0 & & & & \\
0 & -I & 0 & 0 & & & & 0 \\
I & 0 & 0 & 0 & & & 0 & 0
\end{bmatrix}$$

1/2 k odd

$$\text{signature} = \begin{cases} 2\,\mathrm{sig}(A), & \frac{k-2}{4} \text{ even} \\ -2\,sig(A), & \frac{k-2}{4} \text{ odd} \end{cases}$$

$$\begin{bmatrix}
0 & 0 & & & 0 & 0 & A & 0 \\
0 & & & & 0 & 0 & 0 & A \\
 & & & & -A & 0 & 0 & 0 \\
 & & & & 0 & -A & 0 & 0 \\
 & & & & & & & \\
0 & 0 & -A & 0 & & & & \\
0 & 0 & 0 & -A & & & & \\
A & 0 & 0 & 0 & & & & 0 \\
0 & A & 0 & 0 & & & 0 & 0
\end{bmatrix}$$

$$A = (\mathrm{diag}(1, 1, \ldots, -1, -1, \ldots, -1)$$

(2.12$_{\gamma,\delta}$)
(1/2k necessarily even)
signature zero

$$\begin{bmatrix}
0 & 0 & & & & I & \\
0 & & & & I & & \\
 & & & I & & & \\
 & & & & & & \\
 & & & & & & 0 \\
I & & & & & 0 & 0
\end{bmatrix}$$

Appendix 2

Normal forms for elements of $o(2, 2)$

Here we list the normal forms for elements of the Lie algebra $o(2, 2)$. The list is complete in the sense that every $M \in o(2, 2)$ belongs to precisely one of the seven types.

	Inner Productg	*Canonical Form of M*	*Eignevalues*
I.	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & -\alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & -\alpha_2 \end{bmatrix}$	$\pm\alpha_1,\ \pm\alpha_2$ $\alpha_1 \geq 0,\ \alpha_2 \geq 0$
II.	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & \beta_1 & 0 & 0 \\ -\beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 \\ 0 & 0 & -\beta_2 & 0 \end{bmatrix}$	$\pm i\beta,\ \pm i\beta$ $\beta_1 \geq 0,\ \beta_2 \geq 0$
III.	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \gamma & \delta & 0 & 0 \\ -\delta & \gamma & 0 & 0 \\ 0 & 0 & -\gamma & -\delta \\ 0 & 0 & \delta & -\gamma \end{bmatrix}$	$\pm\gamma \pm i\delta$ $\gamma > 0,\ \delta > 0$
IV.	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	0 with multiplicity 4
V.	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \alpha & 0 & 1 & 0 \\ 0 & -\alpha & 0 & -1 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}$	$\pm\alpha$ each with multiplicity 2, $\alpha > 0$
VI.	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0- & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \beta & 1 & 0 \\ -\beta & 0 & 0 & 1 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{bmatrix}$	$\pm i\beta$ each with multiplicity 2, $\beta > 0$
VII.	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	0 with multiplicity 4

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