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A NOTE ON INHOMOGENEOUS DIOPHANTINE APPROXIMATION WITH A GENERAL ERROR FUNCTION

AI-HUA FAN

Department of Mathematics, Wuhan University, Wuhan, Hubei, 430072, P.R.China and LAMFA, CNRS UMR 6140, Université de Picardie, 80039 Amiens, France
e-mail: aihua.fan@u-picardie.fr

and JUN WU

Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei, 430074, P.R.China and LAMFA, CNRS UMR 6140, Université de Picardie, 80039 Amiens, France
e-mail: wujunyu@public.wh.hb.cn

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Abstract. Let α be an irrational number and $\varphi\colon \mathbb{N} \to \mathbb{R}^+$ be a decreasing sequence tending to zero. Consider the set

$$E_\varphi(\alpha) = \{\beta \in \mathbb{R} : \ \|n\alpha - \beta\| < \varphi(n) \text{ holds for infinitely many } n \in \mathbb{N}\},$$

where $\|\cdot\|$ denotes the distance to the nearest integer. We show that for general error function φ, the Hausdorff dimension of $E_\varphi(\alpha)$ depends not only on φ, but also heavily on α. However, recall that the Hausdorff dimension of $E_\varphi(\alpha)$ is independent of α when $\varphi(n) = n^{-\gamma}$ with $\gamma > 1$.

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1. Introduction. Let α be an irrational real number. Minkowski [**9**] showed that the inequality

$$\|n\alpha - \beta\| < \frac{1}{4n}$$

has infinitely many solutions for any β which is not in the orbit of α, i.e. $\beta \notin \mathbb{Z} + \alpha\mathbb{Z}$. In general the constant $\frac{1}{4}$ is the best possible (see [**4**]). In 1999, Bernik and Dodson [**1**, p. 105] considered the set of real numbers which are well approximated by $(\{n\alpha\})_{n\geq 1}$:

$$E_\gamma(\alpha) = \left\{\beta \in \mathbb{R} : \ \|n\alpha - \beta\| < \frac{1}{n^\gamma} \text{ holds for infinitely many } n \in \mathbb{N}\right\}$$

where $\gamma > 1$ and $\{\cdot\}$ denotes the fractional part. They proved that the Hausdorff dimension of $E_\gamma(\alpha)$ satisfies

$$\frac{1}{\omega\gamma} \leq \dim E_\gamma(\alpha) \leq \frac{1}{\gamma} \tag{1}$$

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where $\omega \geq 1$ is any positive number for which

$$\|n\alpha\| \geq \frac{1}{n^{\omega}} \qquad \text{for all sufficiently large integers } n. \tag{2}$$

This, together with Khintchine's theorem, see [**3**, Theorem 1.10], implies that for almost all real numbers α (with respect to the Lebesgue measure)

$$\dim E_{\gamma}(\alpha) = \frac{1}{\gamma}. \tag{3}$$

In 2003, Bugeaud [**2**], Schmeling and Troubetzsky [**10**] improved, independently, the above result due to Bernik and Dodson as follows.

THEOREM 1 (Bugeaud-Schmeling-Troubetzsky). *For any irrational number α,*

$$\dim E_{\gamma}(\alpha) = \frac{1}{\gamma}.$$

Schmeling and Troubetzsky [**10**] used Theorem 1 and an inhomogeneous version of Jarník-Besicovitch theorem to show some strong recurrence properties of the billiard flow in certain polygons.

In this note we would like to know what happens when $n^{-\gamma}$ is replaced by a general decreasing sequence. Let $\varphi\colon \mathbb{N} \to \mathbb{R}^{+}$ be a function satisfying the following hypothesis

$$\varphi(n) \downarrow 0 \quad \text{as} \quad n \to \infty, \qquad n\varphi(n) \leq \frac{1}{2} \quad \text{for large } n. \tag{H}$$

Let α be an irrational number. Consider the set

$$E_{\varphi}(\alpha) = \{\beta \in \mathbb{R} : \ \|n\alpha - \beta\| < \varphi(n) \text{ holds for infinitely many } n \in \mathbb{N}\}.$$

It may be proved that for almost all real numbers α,

$$\dim E_{\varphi}(\alpha) = \limsup_{n\to\infty} \frac{\log n}{-\log \varphi(n)}. \tag{4}$$

In fact, the lower bound of $\dim E_{\varphi}(\alpha)$ can be deduced from [**1**, Theorem 5.1] and the arguments in [**1**, p. 105]. The upper bound is a consequence of Theorem 1 (see also the formula (6) below). The formula (4) also holds when α is an irrational number with bounded partial quotients (see the remark at the end of the note).

All these results show that the formula (4) doesn't depend on α in all cases studied. However, as we shall prove, in general the Hausdorff dimension of $E_{\varphi}(\alpha)$ depends not only on φ, but also heavily on α. The formula (4) is not always true.

THEOREM 2. *There exist an irrational number α_0 and a function $\varphi\colon \mathbb{N} \to \mathbb{R}^{+}$ satisfying* (H) *such that*

$$\limsup_{n\to\infty} \frac{\log n}{-\log \varphi(n)} > 0, \quad \textit{but} \quad \dim E_{\varphi}(\alpha_0) = \liminf_{n\to\infty} \frac{\log n}{-\log \varphi(n)} = 0.$$

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2. Proof of Theorem 2. Define

$$l(\varphi) = \limsup_{n\to\infty} \frac{\log n}{-\log \varphi(n)}, \quad u(\varphi) = \liminf_{n\to\infty} \frac{\log n}{-\log \varphi(n)}. \tag{5}$$

These quantities $l(\varphi)$ and $u(\varphi)$ are closely related to the upper and lower orders at infinity of $1/\varphi$, used by Dodson [**6**] (see also Dodson [**7**] and Dickinson [**5**]) to generalize the Jarník-Besicovitch theorem. We remark that it is easy to deduce from Theorem 1 that for any irrational number α,

$$u(\varphi) \leq \dim E_\varphi(\alpha) \leq l(\varphi). \tag{6}$$

For a given irrational number $\alpha \in (0,1)$, let $[0; a_1, a_2, \cdots, a_n, \cdots]$ be the simple continued fraction expansion of α. The convergents are obtained via finite truncations

$$\frac{p_n}{q_n} := [0; a_1, a_2, \cdots, a_n].$$

With the convention $p_{-1} = q_0 = 1$, $q_{-1} = p_0 = 0$, we have the well known recursive relations

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } n \geq 1, \tag{7}$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \geq 1. \tag{8}$$

We are now going to construct our desired number $\alpha_0 \in (0,1)$ and sequence $(\varphi(n))_{n\geq 1}$ satisfying (H).

Construct $\alpha_0 = [0; a_1, a_2, \ldots, a_n, \ldots]$ by choosing a_n in the following recursive way,

$$a_1 = 1, \quad a_{n+1} = 2^{3q_n} \quad \text{for any } n \geq 1$$

where the q_n are recursively determined by (8). For any $k \geq 1$, write

$$N_k = q_{k-1} + q_k + a_{k+1}^{1/3} \cdot q_k - 1.$$

Let $1 < \gamma < 2$. Define

$$\varphi(n) = \frac{1}{N_k^\gamma}, \text{ if } N_{k-1} < n \leq N_k.$$

It is easy to check that

$$l(\varphi) = \frac{1}{\gamma}, \qquad u(\varphi) = 0.$$

It remains to prove $\dim E_\varphi(\alpha_0) = 0$ for the above defined α_0 and φ. At first we prove that $\dim(E_\varphi(\alpha_0) \bigcap [0,1)) = 0$. Obviously we can regard $\mathbb{R}/\mathbb{Z}$ as $[0,1)$. For any $k \geq 1$, we consider the finite sequence of points $\{n\alpha_0\}$ with $q_{k-1} + q_k \leq n < q_k + q_{k+1}$. The distribution of these points is well described by Three Distance Theorem (see Halton [**8**] or Slater [**11**]). Since $q_k + q_{k+1} = q_{k-1} + q_k + a_{k+1} \cdot q_k$ (see (8)), any n satisfying $q_{k-1} + q_k \leq n < q_k + q_{k+1}$ can be written as

$$n = q_{k-1} + q_k + t \cdot q_k + m, \quad \text{with} \quad 0 \leq t < a_{k+1} \quad \text{and} \quad 0 \leq m < q_k.$$

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For any fixed $0 \le m < q_k$, we call $\{(q_{k-1} + q_k + t \cdot q_k + m)\alpha_0,\ 0 \le t < a_{k+1}\}$ the mth subsequence. Thus the finite sequence $\{n\alpha_0,\ q_{k-1} + q_k \le n < q_k + q_{k+1}\}$ can be decomposed into q_k subsequences of length a_{k+1} corresponding to $m = 0, 1, \ldots, q_k - 1$. For any fixed $0 \le m < q_k$, consider the set of points

$$A_k(m) := \left\{\{(q_{k-1} + q_k + i \cdot q_k + m)\alpha_0\},\ i = 0, 1, \cdots, a_{k+1}^{1/3} - 1\right\}$$

which consists of the first $a_{k+1}^{1/3}$ points in the mth subsequence and $\bigcup_{m=0}^{q_k-1} A_k(m)$ coincides with $\{n\alpha_0\}$ for which $q_{k-1} + q_k \le n \le N_k$. Observe that the distance of two consecutive points in $A_k(m)$ satisfies

$$\begin{aligned}&\|(q_{k-1} + q_k + (i+1) \cdot q_k + m)\alpha_0 - (q_{k-1} + q_k + i \cdot q_k + m)\alpha_0\| \\ &\quad = \|q_k\alpha_0\| < \frac{1}{q_{k+1}} < \frac{1}{a_{k+1} \cdot q_k}.\end{aligned}$$

It follows that for any two points $x, y \in A_k(m)$ we have

$$\|x - y\| < \frac{1}{a_{k+1}^{2/3} \cdot q_k}.$$

When k is large enough, we have $\frac{1}{a_{k+1}^{2/3} \cdot q_k} \le N_k^{-\gamma}$. Thus for large k we have

$$\bigcup_{i=0}^{a_{k+1}^{1/3}-1} B\big(\{(q_{k-1} + q_k + i \cdot q_k + m)\alpha_0\}, N_k^{-\gamma}\big) \subset B\big(\{(q_{k-1} + q_k + m)\alpha_0\}, 3N_k^{-\gamma}\big),$$

where $B(x, r)$ denotes the ball with centre x and radius r. Consequently, for any integer $n \ge 1$ the set $E_\varphi(\alpha_0) \bigcap [0, 1)$ is contained in

$$\bigcup_{k=n}^{\infty} \left(\bigcup_{m=0}^{q_k-1} B\big(\{(q_{k-1} + q_k + m)\alpha_0\}, 3N_k^{-\gamma}\big) \bigcup \bigcup_{p=N_k+1}^{q_k+q_{k+1}-1} B\big(\{p\alpha_0\}, N_{k+1}^{-\gamma}\big) \right).$$

Then, for any $s > 0$, by the definitions of α_0 and φ, we can estimate the s-dimensional Hausdorff measure as follows:

$$\begin{aligned}\mathcal{H}^s(E_\varphi(\alpha_0) \cap [0, 1)) &\le \liminf_{n\to\infty} \sum_{k=n}^{\infty} \left[q_k \big(6N_k^{-\gamma}\big)^s + \big(a_{k+1} - a_{k+1}^{1/3}\big) \cdot q_k \cdot \big(2N_{k+1}^{-\gamma}\big)^s \right] \\ &\le \liminf_{n\to\infty} \sum_{k=n}^{\infty} \left[q_k \big(6N_k^{-\gamma}\big)^s + a_{k+1} \cdot q_k \cdot \big(2N_{k+1}^{-\gamma}\big)^s \right] = 0.\end{aligned}$$

Thus $\dim(E_\varphi(\alpha_0) \cap [0, 1)) = 0$. Since $E_\varphi(\alpha_0)$ is invariant under translations by $\mathbb{Z}$, so the full theorem follows from the conclusion in the unit interval. □

Using the similar idea as that in Bugeaud [**2**], Schmeling and Troubetzsky [**10**], we can get the following result (the details are omitted).

THEOREM 3. *If α is an irrational number with bounded partial quotients and φ: $\mathbb{N} \to \mathbb{R}^+$ is a function satisfying (H), we have*

$$\dim E_\varphi(\alpha) = u(\varphi).$$

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