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Nagoya Math. J.
Vol. 39 (1970), 23–28

NOTE ON THE CHARACTERS OF SOLVABLE GROUPS

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§ 1.

Let $\mathfrak{G}$ be a solvable group of order g. Let p be a prime and let $g = p^a g'$ with $(p, g') = 1$. In [4] we have tried to find sufficient conditions for $\mathfrak{G}$ to possess an irreducible character of p-defect 0, that is, a character whose degree is divisible by p^a.

The following theorem (for arbitrary finite groups) is well-known ([1], (9F)).

I. If $\mathfrak{G}$ possesses an irreducible character of p-defect 0, then $\mathfrak{G}$ contains no non-trivial normal p-subgroup.

Now what actually was proved in the proof of the main theorem in [4] (Theorem 1) is the following theorem (cf. [5]).

II. Let $\mathfrak{G}$ contain no non-trivial normal p-subgroup. (1) If p is odd and is not a Mersenne prime, then there exist two Sylow p-subgroups $\mathfrak{P}_1$ and $\mathfrak{P}_2$ such that $\mathfrak{P}_1 \cap \mathfrak{P}_2 = \mathfrak{E}$. (2) (1) also holds for a Mersenne prime p, provided that the order of $\mathfrak{G}$ is odd. (3) (1) also holds for $p = 2$, provided that every odd prime divisor q of the order of $\mathfrak{G}$ is not a Fermat prime and is congruent to 1 mod 4. (4) (1) also holds for any prime p, provided that elements of order p of a Sylow p-subgroup together with the identity forms a subgroup.

Since then Green ([2]) has proved the following theorem (for arbitrary finite groups).

III. If $\mathfrak{G}$ possesses an irreducible character of p-defect 0, then there exist two Sylow p-subgroups $\mathfrak{P}_1$ and $\mathfrak{P}_2$ such that $\mathfrak{P}_1 \cap \mathfrak{P}_2 = \mathfrak{E}$. Namely, the conclusion of II, (1) holds without any restriction on p.

The following theorem (for arbitrary finite groups) is also well-known ([1], (6G)).

Received Nov. 15, 1968.
* This work was partially supported by NSF Grant GP–6539.

https://doi.org/10.1017/S0027763000013672 Published online by Cambridge University Press

IV. If $\mathfrak{G}$ possesses an irreducible character of p-defect 0, then $\mathfrak{G}$ contains an element G of p-defect 0, that is, an element G the order of whose centralizer is prime to p.

On the other hand, the following fact is noticed in [4] (Lemma 1).

V. Let $\mathfrak{N}$ be a normal subgroup of $\mathfrak{G}$, whose order is prime to p. If $\mathfrak{N}$ contains an element of p-defect 0 (in $\mathfrak{G}$), then $\mathfrak{G}$ possesses an irreducible character of p-defect 0.

Now the primary concern of [4] was the following proposition.

(♯) Under certain circumstances the non-existence of non-trivial normal p-subgroups implies the existence of an irreducible character of p-defect 0.

Unfortunately, in the formulation of Theorem 1 in [4] a strong condition, which is the main driving power of the induction argument in the proof of Theorem 1 in [4] is carelessly not stated.** That is the following condition.

($\mathfrak{P} \longrightarrow p$). Let $\mathfrak{P}$ be a fixed Sylow p-subgroup of $\mathfrak{G}$. Then every element of $\mathfrak{G}$, which is commutative with no element ($\neq E$) of $\mathfrak{P}$, has p-defect 0.

Now the purpose of this note is (1) to show that under the condition ($\mathfrak{P} \longrightarrow p$), together with the condition in II securing the conclusion of II, (1), (♯) always is true, (ii) to state some conditions on the group structure under which (♯) is true without assuming the condition ($\mathfrak{P} \longrightarrow p$), (iii) to discuss some examples which show the necessity of the condition ($\mathfrak{P} \longrightarrow p$), and (iv) to discuss the proof of a theorem to which Theorem 1 of [4] has been applied ([6]).

§ 2.

PROPOSITION 1. *Assume that $\mathfrak{G}$ satisfies the condition ($\mathfrak{P} \longrightarrow p$). Let G be an element of $\mathfrak{G}$ such that $\mathfrak{P} \cap \overset{-1}{G}\mathfrak{P}G = \mathfrak{E}$. Then G is an element of p-defect* 0.

Proof. Assume that G is not an element of p-defect 0. Then there exists an element $H (\neq E)$ of $\mathfrak{G}$ which is commutative with G and has order a power of p. By the condition ($\mathfrak{P} \longrightarrow p$) we may assume that H belongs to $\mathfrak{P}$. Then $\mathfrak{P} \cap G^{-1}\mathfrak{P}G$ contains H. This is a contradiction.

PROPOSITION 2. *Assume that $\mathfrak{G}$ satisfies the condition ($\mathfrak{P} \longrightarrow p$), and that $\mathfrak{G}$ contains no non-trivial normal p-subgroups, and that $\mathfrak{G}$ satisfies the condition in II*

** The author is indebted to Paul Fong for a valuable comment.

https://doi.org/10.1017/S0027763000013672 Published online by Cambridge University Press

securing the conclusion of II, (1). *Let* $\mathfrak{F}$ *be the Fitting subgroup of* $\mathfrak{G}$. *Then* $\mathfrak{F}$ *contains an element of p-defect* 0 (*in* $\mathfrak{G}$).

Proof. Consider the subgroup $\mathfrak{F}\mathfrak{P}$. By a theorem of Fitting ([3]) the centralizer of $\mathfrak{F}$ in $\mathfrak{G}$ is contained in $\mathfrak{F}$. Hence $\mathfrak{F}\mathfrak{P}$ contains no non-trivial normal p-subgroup (of $\mathfrak{F}\mathfrak{P}$). Therefore, by assumption, there exists an element G of $\mathfrak{F}\mathfrak{P}$ such that $\mathfrak{P}\cap \overset{-1}{G}\mathfrak{P}G=\mathfrak{E}$. By Proposition 1 G is then an element of p-defect 0.

THEOREM 1. *Assume that* $\mathfrak{G}$ *satisfies the condition* ($\mathfrak{P}\longrightarrow p$), *that* $\mathfrak{G}$ *contains no non-trivial normal p-subgroups, and that* $\mathfrak{G}$ *satisfies the condition in II securing the conclusion of II,* (1). *Then* $\mathfrak{G}$ *possesses an irreducible character of p-defect* 0.

Proof. By Proposition 2 it suffices to apply V to $\mathfrak{F}$ and $\mathfrak{G}$.

Here a sufficient condition for $\mathfrak{G}$ to secure the property ($\mathfrak{P}\longrightarrow p$) will be noticed.

PROPOSITION 3. If a Sylow p-complement $\mathfrak{H}$ of $\mathfrak{G}$ is abelian, then $\mathfrak{G}$ statisfies the condition ($\mathfrak{P}\longrightarrow p$).

Proof. Let G be an element of $\mathfrak{G}$ such that G has order prime to p and commutes with no element ($\neq E$) of $\mathfrak{P}$. Using P. Hall's theorem we may assume that G belongs to $\mathfrak{H}$. Let $H\neq E$ be an element of $\mathfrak{G}$ such that H has order a power of p and commutes with G. Then we may write $H=KL$, where K and L are elements of $\mathfrak{H}$ and $\mathfrak{P}$ respectively. Since $GH=HG$ and $GK=KG$, we obtain $GL=LG$. By assumption this implies that $L=E$ and $H=E$. This is a contradiction.

§ 3.

THEOREM 2. *Assume that* $\mathfrak{G}$ *contains no non-trivial normal p-subgroup, and that* $\mathfrak{G}$ *satisfies the condition in II securing the conclusion of II,* (1). *If* $\mathfrak{G}$ *has nilpotent length* 2, *then* $\mathfrak{G}$ *posesses an irreducible character of p-defect* 0.

Proof. By assumption there exists a nilpotent subgroup $\mathfrak{N}$ of $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{N}$ is also nilpotent. By assumption the order of $\mathfrak{N}$ is prime to p. Now we apply an induction argument with respect to the order of $\mathfrak{G}$. If $\mathfrak{G}/\mathfrak{N}$ is not a p-group, $\mathfrak{G}$ contains a proper normal subgroup $\mathfrak{H}$ whose index in $\mathfrak{G}$ is prime to p. By the induction hypothesis $\mathfrak{H}$ possesses an irreducible character ζ of p-defect 0. Let χ be an irreducible component of the

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character of $\mathfrak{G}$ induced by ζ. Then using Clifford's theorem ([3], p. 565), we see that χ has p-defect 0. Thus we can assume that $\mathfrak{G}/\mathfrak{N}$ is a p-group, that is, $\mathfrak{N}$ is a Sylow p-complement of $\mathfrak{G}$.

Let $\Phi(\mathfrak{N})$ be the Frattini subgroup of $\mathfrak{N}$. If $\Phi(\mathfrak{N}) \neq \mathfrak{E}$, then consider $\mathfrak{G}/\Phi(\mathfrak{N})$. If $\mathfrak{G}/\Phi(\mathfrak{N})$ contains a non-trivial normal p-subgroup $\mathfrak{Q}\Phi(\mathfrak{N})/\Phi(\mathfrak{N})$, where $\mathfrak{Q} \neq \mathfrak{E}$ is a p-subgroup of $\mathfrak{G}$, then using Sylow's theorem get $\mathfrak{G} = N(\mathfrak{Q})\Phi(\mathfrak{N})$, where $N(\mathfrak{Q})$ is the normalizer of $\mathfrak{Q}$ in $\mathfrak{G}$. This is a contradiction. Thus we may assume that $\Phi(\mathfrak{N}) = \mathfrak{E}$, which implies that $\mathfrak{N}$ is abelian.

By Proposition 3 and Theorem 1 we now get Theorem 2.

THEOREM 3. *Assume that $\mathfrak{G}$ contains no non-trivial normal p-subgroup. If $\mathfrak{G}$ is metabelian, then $\mathfrak{G}$ possesses an irreducible character of p-defect* 0.

Proof. Take the commutator subgroup of $\mathfrak{G}$ as $\mathfrak{N}$ in the proof of Theorem 2. Then we can reduce $\mathfrak{G}$ to the case where the Sylow p-complements and Sylow p-subgroups are abelian.

THEOREM 4. *Let $\mathfrak{G}$ contain no non-trivial normal p-subgroup. If $\mathfrak{P}$ is cyclic, then $\mathfrak{G}$ possesses an irreducible character of p-defect* 0.

Proof. Let $\mathfrak{K}$ be a minimal normal subgroup of $\mathfrak{G}$. By assumption $\mathfrak{K}$ has order prime to p. If $\mathfrak{G}/\mathfrak{K}$ contains no non-trivial normal p-subgroup, then applying an induction argument to $\mathfrak{G}/\mathfrak{K}$, we see that $\mathfrak{G}/\mathfrak{K}$, and hence $\mathfrak{G}$, possesses an irreducible character of p-defect 0. So let $\mathfrak{Q}\mathfrak{K}/\mathfrak{K}$ be a normal subgroup of $\mathfrak{G}/\mathfrak{K}$ of order p, where $\mathfrak{Q}$ is a subgroup of $\mathfrak{G}$ order p. By Sylow's theorem $\mathfrak{Q}\mathfrak{K}$ contains all elements of $\mathfrak{G}$ of order p. Let G be an element ($\neq E$) of $\mathfrak{K}$. If G is not of p-defect 0, then the centralizer of G contains some non-trivial, and hence all elements of $\mathfrak{Q}$. Thus G belongs to the center of $\mathfrak{Q}\mathfrak{K}$. Since $\mathfrak{K}$ is minimal, $\mathfrak{Q}\mathfrak{K} = \mathfrak{Q} \times \mathfrak{K}$. Therefore $\mathfrak{Q}$ is normal in $\mathfrak{G}$. This is a contradiction. Thus G has p-defect 0. Now by V we get Theorem 4.

§ 4.

It is not difficult to construct groups which do not satisfy the condition ($\mathfrak{P} \longrightarrow p$). The following examples show, however, that Theorems 2, 3 and are also best possible.

Let p, q and r be distinct prime numbers such that $\dfrac{q^p - 1}{q - 1} = pr$.

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Examples of such triplets are $\{p, q, r\} = \{2, 5, 3\}$, $\{3, 13, 61\}$, $\{5, 11, 2331\}, \ldots$

First we notice some properties of such triplets.

(i) $q^p \equiv 1 \pmod r$ and $q \not\equiv 1 \pmod r$.

(ii) $q \equiv 1 \pmod p$.

(iii) $r \equiv 1 \pmod p$.

Proof. (i) If $q \equiv 1 \pmod r$, then $\frac{q^p - 1}{q - 1} = q^{p-1} + \cdots + q + 1 \equiv p \equiv 0 \pmod r$. This is a contradiction. (ii) By Fermat's theorem $q^{p-1} \equiv 1 \pmod p$. Since $q^p \equiv 1 \pmod p$, we get that $q \equiv 1 \pmod p$. (iii) By Fermat's theorem $q^{r-1} \equiv 1 \pmod r$. Thus by (i) we obtain $r \equiv 1 \pmod p$.

Let $GF(q)$ and $GF(q^p)$ (containing $GF(q)$) denote the fields of q and q^p elements respectively. Let σ be an element of order p in the Galois group of $GF(q^p)$ over $GF(q)$. By (i) $GF(q^p)$ contains a primitive r-th root of unity ε. Then

$$A = \begin{pmatrix} \varepsilon & & & \\ & \varepsilon^\sigma & & \\ & & \ddots & \\ & & & \varepsilon^{\sigma^{p-1}} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 1 \\ 1 & 0 & 0 \cdots 0 \end{pmatrix}$$

generate a non-cyclic group of order rp. Since the trace of every matrix of $\langle A, B \rangle$ lies in $GF(q)$, there exists a non-singular matrix V with entries in $GF(q^p)$ such that $A^* = V^{-1}AV$ and $B^* = V^{-1}BV$ have entries in $GF(q)$ ([3], p. 545). By (ii) $GF(q)$ contains a primitive p-th root of unity τ. Then A^*, B^* and $C = \begin{pmatrix} \tau & & \\ & \ddots & \\ & & \tau \end{pmatrix}$ generate a group of order $p^2 r$, which is the direct product of $\langle A^*, B^* \rangle$ and $\langle C \rangle$. Let $\mathfrak{G}$ be the split extension of the p-dimensional vector space $\mathfrak{V}$ over $GF(q)$ by $\langle A^*, B^*, C \rangle$.

$\mathfrak{G}$ is an A-group of order $p^2 q^p r$. $\mathfrak{G}$ has the nilpotent length 3, and the second commutator subgroup of $\mathfrak{G}$ equals $\mathfrak{V}$ which is abelian. $\mathfrak{G}$ contains no non-trivial normal p-subgroup.

Now we show that $\mathfrak{G}$ does not possess an irreducible character of p-defect 0. By IV it is enough to show that $\mathfrak{G}$ does not contain an element of p-defect 0.

Since the Sylow p-subgroups of $\mathfrak{G}$ are not cyclic, there exists an element $V \neq E$ of $\mathfrak{V}$ which is commutative with an element of order p ([3],

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p. 502). Since C belongs to the normalizer of $\langle V\rangle$ and does not commute with V, the normalizer of $\langle V\rangle$ has order p^2q^p. If there exists an element V_0 of $\mathfrak{V}$ of p-defect 0, then the normalizer of $\langle V_0\rangle$ has order pq^p. Since the number of subgroups of order q of $\mathfrak{V}$ equals $\frac{q^p-1}{q-1}=pr$, every subgroup of order q of $\mathfrak{V}$ must be conjugate to $\langle V_0\rangle$. But certainly $\langle V\rangle$ is not conjugate to $\langle V_0\rangle$. This shows that there exists no element of $\mathfrak{V}$ of p-defect 0.

§ 5.

Theorem 1 of [4] has been applied to prove the following fact ([6], Proposition 2). If $\mathfrak{G}$ is an A-group and if G is an element of $\mathfrak{G}$ not belonging to the Fitting subgroup $\mathfrak{F}$ of $\mathfrak{G}$, then there exists an irreducible character χ of $\mathfrak{G}$ such that $\chi(G)=0$. We can prove this as follows.

We use an induction argument with respect to the order of the group. Let $\mathfrak{M}$ be a minimal normal subgroup of $\mathfrak{G}$ and let $\boldsymbol{F}(\mathfrak{M})/\mathfrak{M}$ be the Fitting subgroup of $\mathfrak{G}/\mathfrak{M}$. If G does not belong to $\boldsymbol{F}(\mathfrak{M})$, then we can apply the induction hypothesis to $G\mathfrak{M}$ and $\mathfrak{G}/\mathfrak{M}$. Hence we may assume that G belongs to $\boldsymbol{F}(\mathfrak{M})$, which implies that $\boldsymbol{F}(\mathfrak{M})\neq\mathfrak{F}$. Thus $\boldsymbol{F}(\mathfrak{M})$ has nilpotent length 2.

Now let p be a prime divisor of the order of $G\mathfrak{F}$, and let $\mathfrak{P}$ be a Sylow p-subgroup of $\mathfrak{F}$. Then $\frac{\boldsymbol{F}(\mathfrak{M})}{\mathfrak{P}}$ contains no non-trivial p-normal subgroup. By Theorem 2 $\frac{\boldsymbol{F}(\mathfrak{M})}{\mathfrak{P}}$ possesses an irreducible character ζ of p-defect 0. Let χ be an irreducible component of the character of $\frac{\mathfrak{G}}{\mathfrak{P}}$ induced by ζ. Then by a theorem of Clifford ([3], p. 565) we see that $\chi\,|\,\boldsymbol{F}(\mathfrak{M})/\mathfrak{P}$ decomposes into irreducible characters of $\boldsymbol{F}(\mathfrak{M})/\mathfrak{P}$ of p-defect 0. Then we get $\chi(G)=0$ ([1], $6E$).

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