ON SELF-ADJOINT FACTORIZATION OF OPERATORS

HEYDAR RADJAVI

The main result of this paper is that every normal operator on an infinite-dimensional (complex) Hilbert space $\mathscr{H}$ is the product of four self-adjoint operators; our Theorem 4 is an actually stronger result. A large class of normal operators will be given which cannot be expressed as the product of three self-adjoint operators.

This work was motivated by a well-known result of Halmos and Kakutani (**3**) that every unitary operator on $\mathscr{H}$ is the product of four symmetries, i.e., operators that are self-adjoint and unitary.

1. By "operator" we shall mean bounded linear operator. The space $\mathscr{H}$ will be infinite-dimensional (separable or non-separable) unless otherwise specified. We shall denote the class of self-adjoint operators on $\mathscr{H}$ by $\mathscr{M}$ and that of symmetries by $\mathscr{J}$. The following usual notation for the product of two classes $\mathscr{X}$ and $\mathscr{Y}$ will be used: $\mathscr{X}\mathscr{Y} = \{A: A = XY, X \in \mathscr{X}, Y \in \mathscr{Y}\}; \mathscr{X}^2 = \mathscr{X}\mathscr{X}$. We observe that $\mathscr{M}\mathscr{J} = \mathscr{J}\mathscr{M}$, since if $M \in \mathscr{M}$ and $J \in \mathscr{J}$, then $JMJ \in \mathscr{M}$ and $MJ = J(JMJ)$. Hence A is the product of m hermitian operators and n symmetries in any order if and only if $A \in \mathscr{M}^m\mathscr{J}^n$.

We shall implicitly use many results of the spectral theory for normal operators, but it is convenient to fix some notation: Let $E(\cdot)$ denote the resolution of the identity for the normal operator A on $\mathscr{H}$. Let

$$S = \{z: r_1 < |z| \leqq r_2\} \quad \text{and} \quad T = \{z: |z| = r\}.$$

Then the restrictions of A to $E(S)\mathscr{H}$ and to $E(T)\mathscr{H}$ will be denoted by $\phi(A, r_1, r_2)$ and $\phi(A, r)$, respectively.

The *rank* of an operator is the dimension of the closure of its range. The spectrum of A will be denoted by $\sigma(A)$.

2. We start with the following result.

LEMMA 1. *Let A be an operator on $\mathscr{H}$ and let $A = \sum_{j=1}^{\infty} \oplus A_j$, where each A_j is an invertible operator on $\mathscr{H}_j$ and where the $\mathscr{H}_j$ all have the same (finite or infinite) dimension. Assume that an integer n exists such that either*

$$||A_{j+1}|| \cdot ||A_j^{-1}|| \leqq 1$$

for all $j \geqq n$ or $||A_{j+1}^{-1}|| \cdot ||A_j|| \leqq 1$ for all $j \geqq n$. Then $A \in \mathscr{M}^2\mathscr{J}^2$. Furthermore, if A is invertible and if either of the above inequalities holds for all $j \geqq 1$,

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then operators B and C in $\mathscr{M}\mathscr{J}$ can be chosen such that $A = BC$, $||B|| \leqq ||A||$, and $||C|| \leqq ||A|| \cdot ||A^{-1}||$.

Proof. It is convenient to employ the notation of operator matrices: for any pair of natural numbers j and k, A_j can be considered as an operator on $\mathscr{H}_k$; it will cause no confusion to denote this operator by A_j also. Now define an operator B_j on $\mathscr{H}_j$ as follows:

$$B_1 = A_1, \qquad B_2 = A_1^*,$$
$$B_{2k+1} = (A_{2k+1}A_{2k-1}\ldots A_1)(A_{2k}^*A_{2k-2}^*\ldots A_2^*)^{-1},$$
$$B_{2k+2} = B_{2k+1}^*, \qquad k = 1, 2, \ldots.$$

Define C_j on $\mathscr{H}_j$ by $C_j = B_j^{-1}A_j$, $j = 1, 2, \ldots$; observe that $C_1 = I$ and

$$C_{2k} = (A_1^*A_3^*\ldots A_{2k-1}^*)^{-1}(A_2A_4\ldots A_{2k}),$$
$$C_{2k+1} = C_{2k}^*, \qquad k = 1, 2, \ldots.$$

The inequalities

$$||B_{2k+1}|| = ||B_{2k+2}|| \leqq \left(\prod_{j=0}^{k} ||A_{2j+1}||\right)\left(\prod_{j=1}^{k} ||A_{2j}^{-1}||\right),$$

$$||C_{2k}|| = ||C_{2k+1}|| \leqq \left(\prod_{j=1}^{k} ||A_{2j}||\right)\left(\prod_{j=1}^{k} ||A_{2j-1}^{-1}||\right),$$

and

$$||A_k|| \leqq ||A|| \quad \text{for all } k$$

together with either of the inequalities hypothesized in the lemma show that the B_j and C_j are uniformly bounded. Hence $B = \sum_{j=1}^{\infty} \oplus B_j$ and $C = \sum_{j=1}^{\infty} \oplus C_j$ are bounded operators on $\mathscr{H}$ and $A = BC$.

The operator B is a direct sum of operator matrices of the form $X \oplus X^*$. The equation

$$\begin{pmatrix} X & 0 \\ 0 & X^* \end{pmatrix} = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

implies that $X \oplus X^*$ belongs to $\mathscr{M}\mathscr{J}$, and hence so does B.

The operator C can be written as $C_1 \oplus D = I \oplus D$, where D is of the same form as B. Hence $C \in \mathscr{M}\mathscr{J}$, so that $A \in \mathscr{M}^2\mathscr{J}^2$.

The bounds given for $||B||$ and $||C||$ in the lemma are immediate.

THEOREM 1. *Let A be a normal operator on $\mathscr{H}$ with pure point spectrum. Then A is the product of two commuting normal operators in $\mathscr{M}\mathscr{J}$.*

Proof. First assume that A is invertible. By Zorn's Lemma there is a maximal class $\{\mathscr{H}_\alpha\}$ of mutually orthogonal subspaces of $\mathscr{H}$ such that, for each α, (i) $\mathscr{H}_\alpha$ reduces A, and (ii) the restriction A_α of A to $\mathscr{H}_\alpha$ can be represented as $\operatorname{diag}(z_{\alpha 1}, z_{\alpha 2}, \ldots)$, where the sequence $\{|z_{\alpha j}|\}_{j=1}^{\infty}$ is monotone. Then the orthogonal complement, $\mathscr{H}_0$, of $\sum_\alpha \oplus \mathscr{H}_\alpha$ is necessarily finite-dimensional and reduces A; let A_0 denote the restriction of A to $\mathscr{H}_0$.

Let A_β be a fixed member of $\{A_\alpha\}$ and incorporate A_0 into A_β, i.e., redefine A_β as $A_0 \oplus A_\beta$. Then $A = \sum_\alpha \oplus A_\alpha$; all of the operators in this direct sum satisfy the conditions of Lemma 1. Hence $A_\alpha = B_\alpha C_\alpha$ for each α, where the operators B_α and C_α are the members of $\mathscr{M}\mathscr{J}$ constructed as in Lemma 1 for A_α. We observe that the B_α and C_α are all diagonal and hence $B_\alpha C_\alpha = C_\alpha B_\alpha$. Furthermore, it follows from Lemma 1 that

$$||B_\alpha|| \leqq ||A|| \quad \text{and} \quad ||C_\alpha|| \leqq ||A|| \cdot ||A^{-1}||$$

for all $\alpha \neq \beta$. Hence $B = \sum_\alpha \oplus B_\alpha$ and $C = \sum_\alpha \oplus C_\alpha$ are bounded normal operators belonging to $\mathscr{M}\mathscr{J}$, and $A = BC = CB$.

We now dispose of the restriction that A be invertible. If X is the product of two normal, commuting members of $\mathscr{M}\mathscr{J}$, then so is $X \oplus 0$. Furthermore, if A has finite rank, then $\mathscr{H}$ can be written as the direct sum of two equi-dimensional subspaces relative to which $A = A_1 \oplus 0$. Then the equality

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_1^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

proves the assertion of the theorem on A. These remarks permit us to assume that A has trivial null space and the closure of its range is infinite-dimensional. Hence $A = \sum_{j=1}^\infty \oplus A_j$, where $A_j = \phi(A, r_{j+1}, r_j)$, $r_j = ||A||/2^{j-1}$. We also assume, of course, that A is not invertible.

Let K be the set of integers for which A_j has infinite rank. For each $j \in K$ write $A_j = F_j \oplus D_j$, where F_j has finite rank and D_j is a direct sum of diagonal operators satisfying the monotonicity condition given in the first paragraph of this proof. Now $A = F \oplus \sum_{j \in K} \oplus D_j$, where F is necessarily compact. We have, as before, $D_j = B_j C_j$, where B_j and C_j are commuting normal members of $\mathscr{M}\mathscr{J}$. Furthermore, the uniform bounds given above for the B_α and C_α show that

$$||B_j|| \leqq ||A_j|| \leqq ||A|| \quad \text{and} \quad ||C_j|| \leqq ||A_j|| \cdot ||A_j^{-1}|| \leqq a_j/a_{j+1} = 2$$

for all $j \in K$. Hence $\sum_{j \in K} \oplus B_j$ and $\sum_{j \in K} \oplus C_j$ are both bounded operators in $\mathscr{M}\mathscr{J}$ and their product is $\sum_{j \in K} \oplus D_j$.

Finally, we consider the compact operator F. There is no loss of generality in assuming that F has infinite rank. For otherwise K is necessarily an infinite set, and one can transfer to F one eigenvalue from each of the D_j with $j \in K$; this can be done without disturbing the stated properties of the D_j. The proof is now completed by the observation that F can be written as $\operatorname{diag}(z_1, z_2, \ldots)$ such that $\{|z_j|\}_{j=1}^\infty$ is non-increasing; Lemma 1 is then applicable to F.

THEOREM 2. *Every unitary operator on $\mathscr{H}$ with pure point spectrum is the product of two commuting members of $\mathscr{J}^2$.*

Proof. This is a corollary to Theorem 1 and the proof of Lemma 1.

For compact operators we have the following stronger version of Theorem 1.

THEOREM 3. *Every compact normal operator A on $\mathscr{H}$ is the product of two commuting, compact normal operators in $\mathscr{M}\mathscr{J}$.*

Proof. As in the proof of Theorem 1, we can reduce the general case to the one in which A has infinite rank and trivial null space. Thus, represent A as $\operatorname{diag}(z_1, z_2, \ldots)$ with $\lim_n z_n = 0$. By splitting the sequence $\{z_n\}_{n=1}^{\infty}$ suitably, we express A as $\sum_{k=0}^{\infty} \oplus A_k$, where A_0 has finite rank and A_k has the following property for each $k \geqq 1$: $A_k = \operatorname{diag}(z_{k1}, z_{k2}, \ldots)$, where $|z_{k,j+1}| \leqq |z_{k,j}|/2$ for all j. Incorporating A_0 into A_1 and writing $A = \sum_{k=1}^{\infty} \oplus A_k$, we observe that Lemma 1 is applicable to each A_k. Furthermore, by the structure of the operators B and C in the proof of Lemma 1, we obtain $A_k = D_k F_k$, where D_k and F_k are compact diagonal operators with $||D_k|| = |z_{k1}|$ and $||F_k|| = 1$ for $k \geqq 2$. Let $D_k' = |z_{k1}|^{-1/2} D_k$ and $F_k' = |z_{k1}|^{1/2} F_k$ for all k. Since $\lim_k z_{k1} = 0$, it follows that $D = \sum_{k=1}^{\infty} \oplus D_k'$ and $F = \sum_{k=1}^{\infty} \oplus F_k'$ are both compact, diagonal members of $\mathscr{M}\mathscr{J}$. Then $A = DF$, as required.

LEMMA 2. *Let A be an invertible normal operator on $\mathscr{H}$ such that $A = rU \oplus B$, where $r > 0$ and U is unitary. Assume that the rank of B does not exceed that of U. Then $A \in \mathscr{M}^2\mathscr{J}^2$.*

Proof. Express U as a direct sum $\sum_{j=1}^{\infty} \oplus U_j$, where each U_j has the same rank as U. Write $A_1 = rU_1 \oplus B$ and $A_j = rU_j$ for $j \geqq 2$. Then $A = \sum_{j=1}^{\infty} \oplus A_j$ belongs to $\mathscr{M}^2\mathscr{J}^2$ by Lemma 1.

We note here that the Halmos-Kakutani result can be obtained by taking $r = 1$, $B = 0$ (on the trivial subspace), and observing that the operators B and C constructed in the proof of Lemma 1 are both unitary.

THEOREM 4. *Every normal operator on $\mathscr{H}$ belongs to $\mathscr{M}^2\mathscr{J}^2$.*

Proof. Every normal operator is the direct sum of operators each acting on an infinite-dimensional separable subspace. Hence it suffices to consider the case in which $\mathscr{H}$ is separable. In view of Theorem 1, we can assume that the point spectrum of A consists of a finite number of points each of which has finite multiplicity. Hence the continuous spectrum of A is not empty. Furthermore, as in the previous proofs, we assume that A has trivial null space.

Now pick a point z in the continuous spectrum of A and let $r = |z|$. If $\phi(A, r)$ has infinite rank, then $r > 0$, $\phi(A, r) = rU$, where U is unitary; hence $A \in \mathscr{M}^2\mathscr{J}^2$ by Lemma 1. Otherwise there exists a monotone sequence $\{r_j\}$ of positive numbers tending to r such that the operator

$$A_j = \begin{cases} \phi(A, r_j, r_{j+1}), & \{r_j\} \text{ increasing}, \\ \phi(A, r_{j+1}, r_j), & \{r_j\} \text{ decreasing}, \end{cases}$$

has infinite rank for all j. If $\sigma(A)$ contains 0, then we can choose $r = 0$ so that $\{r_j\}$ is decreasing; if $\sigma(A)$ does not contain 0, then, by considering A^{-1} instead of A if necessary, we can again assume $\{r_j\}$ to be decreasing. Hence

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we can also choose $r_1 = ||A||$. Furthermore, in the case where A is invertible, we incorporate $\phi(A, 0, r)$ into A_1. Thus $A = \sum_{j=1}^{\infty} \oplus A_j$ and Lemma 1 is applicable, since $||A_{j+1}|| \cdot ||A_j^{-1}|| \leqq r_{j+1}/r_j$ for $j \geqq 2$.

COROLLARY. *Every operator on $\mathscr{H}$ which is similar to a normal operator is in $\mathscr{M}^4$.*

Proof. If $M_i \in \mathscr{M}$, $i = 1, 2$, and if S is invertible, then

$$S^{-1}M_1M_2S = (S^{-1}M_1(S^*)^{-1})(S^*M_2S).$$

Hence $\mathscr{M}^2$ is invariant under similarity transformations (**5**); hence so is $\mathscr{M}^4$.

If A is similar to a normal operator with pure point spectrum, then it follows from Theorem 1 that A is the product of two commuting members of $\mathscr{M}^2$.

3. The next result gives a necessary condition that an operator (not necessarily normal) be in $\mathscr{M}^3$.

THEOREM 5. *Let A be an operator on $\mathscr{H}$ and let $\tau(A)$ be the closure of its numerical range. Then $A \in \mathscr{M}^3$ implies that $\tau(A)$ contains either a real or a pure imaginary number.*

Proof. Assume that $A = M_1M_2M_3$, $M_i \in \mathscr{M}$. If $0 \in \sigma(A)$, then $0 \in \tau(A)$. Hence assume that A is invertible, so that the M_i are invertible. Now let $B = M_2M_3$ and observe that B and B^* are similar. Hence α can be chosen such that both α and $\bar{\alpha}$ are in the approximate point spectrum of B. (This follows, e.g., from the well-known fact that the boundary points of $\sigma(B)$ are in its approximate point spectrum; the symmetric property of $\sigma(B)$ relative to the real axis should of course be employed also.)

Choose sequences $\{x_n\}$ and $\{y_n\}$ of unit vectors in $\mathscr{H}$ such that $x_n' = (B - \alpha)x_n$ and $y_n' = (B - \bar{\alpha})y_n$ converge to 0. Now $A = M_1B$ implies that

$$(Ax_n, x_n) = \alpha(M_1x_n, x_n) + (M_1x_n', x_n),$$
$$(Ay_n, y_n) = \bar{\alpha}(M_1y_n, y_n) + (M_1y_n', y_n).$$

Choose convergent subsequences of $\{(M_1x_n, x_n)\}$ and $\{(M_1y_n, y_n)\}$ and call their limits r and s, respectively. Thus $r\alpha$ and $s\bar{\alpha}$ both belong to $\tau(A)$. Since r and s are real, the line segment joining $r\alpha$ to $s\bar{\alpha}$ intersects at least one of the coordinate axes. By the Toeplitz-Hausdorff theorem, the numerical range is convex; see (**2**, p. 110, Problem 166). Hence so is $\tau(A)$; this implies the desired result.

COROLLARY 1. *Let A be an arbitrary operator on $\mathscr{H}$ and let α be any complex number whose square is not real. Then no operator of the form $A + r\alpha I$ belongs to $\mathscr{M}^3$ if r is a sufficiently large positive number.*

Proof. A sufficiently large translation of the compact set $\tau(A)$ in any direction α, where α^2 is not real, results in placing it entirely within one of the open quadrants.

COROLLARY 2. *Let A be a normal operator whose spectrum lies entirely within one of the open quadrants. Then A is not in $\mathscr{M}^3$. In particular, no unitary operator whose spectrum is a subset of one of the four open arcs $(n\pi/2, (n+1)\pi/2)$ is in $\mathscr{J}^3$.*

Proof. The convex hull of $\sigma(A)$ coincides with $\tau(A)$; see (**2**, p. 112).

4. The following remarks are pertinent.

(i) An arbitrary operator on $\mathscr{H}$ does not have to be the product of a finite number of self-adjoint operators; an example is the unilateral shift (**2**, p. 270). However, it follows from the polar decomposition theorem together with the Halmos-Kakutani result that if A is invertible, then $A \in \mathscr{M}\mathscr{J}^4$. A sufficient condition for an arbitrary operator A to be in $\mathscr{M}^2\mathscr{J}$ is that A can be expressed as $A_1 \oplus 0$, where A_1 and the zero operator 0 act on equi-dimensional subspaces. (A proof of this was given in the course of proving Theorem 1.)

(ii) A normal (unitary) operator is in $\mathscr{M}\mathscr{J}$ ($\mathscr{J}^2$) if and only if it is unitarily equivalent to its adjoint (**1**, Theorem 6.3; **5**, Theorems 3 and 4).

(iii) Let $\mathscr{H}$ be finite-dimensional. Then an arbitrary operator is in $\mathscr{M}^4$ if and only if its determinant is real (**4**, Theorem 2); this is also a necessary and sufficient condition for the results of the present paper on normal and unitary operators to be true.

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University of Toronto,
Toronto, Ontario