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EMBEDDING UP TO HOMOTOPY TYPE IN EUCLIDEAN SPACE

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We give a short proof of the classical Stallings theorem that every finite n-dimensional cellular complex embeds up to homotopy in the $2n$-dimensional Euclidean space. As an application we solve a problem of M. Kreck.

INTRODUCTION

This note was inspired by a question of Kreck during his visit to the Steklov Mathematical Institute in Spring of 1989: *Can every finitely presented group be realised as the fundamental group of a 2-dimensional polyhedron embedded in $\mathbb{R}^4$ and can such polyhedron have a minimal Euler characteristic?*

We recall that not every 2-polyhedron is embeddable in 4-dimensional Euclidean space [4]. It turns out that the answer to the first question follows from the classical theorem of Stallings [9]:

STALLINGS THEOREM. *For every finite n-dimensional ($n > 0$) cellular complex K there exists a polyhedron M, homotopy equivalent to K, which is embeddable in $\mathbb{R}^{2n}$.*

After having seen our solution of his problem in 1991, Kreck kindly informed us about the recent work of Huck [5], through which we became familiar with Stallings' mimeographed notes [9] and some other related papers [**1, 2, 3, 8, 11**]. Our solution of Kreck's problem provides an alternative (and we believe also simpler) proof of the Stallings theorem.

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Preliminaries

Our proof is based on the following two lemmas:

Lemma 1. *Suppose that $f\colon X \to Y$ is a homotopy equivalence between cellular complexes and $g\colon S^n \to X$ is an attaching map for an $(n+1)$-dimensional cell B^{n+1}. Then f can be extended to a homotopy equivalence $F\colon X \cup_g B^{n+1} \to Y \cup_{f\circ g} B^{n+1}$.*

Proof: Standard argument – see for example in [10] or [6]. ∎

Lemma 2. *Suppose that a polyhedron P lies in $\mathbb{R}^{2n} \times O \subset \mathbb{R}^{2n} \times \mathbb{R}^2_+$ where O is a point in the boundary of semi-plane $\mathbb{R}^2_+$ and let $f\colon S^n \to P$ be a map of the boundary of the $(n+1)$-dimensional ball. Then for every $\varepsilon > 0$ there exists a map $g\colon B^{n+1} \to \mathbb{R}^{2n+2}_+ = \mathbb{R}^{2n} \times \mathbb{R}^2_+$ such that*

(1) *the restriction $g\,|_{\mathrm{Int}\,B^{n+1}}$ is an embedding in $\mathrm{Int}\,(\mathbb{R}^{2n+2}_+)$; and*
(2) *the restriction $g\,|_{\partial B^{n+1}} = q$ is ε-close to f and $Im(q) \subset \mathbb{R}^{2n} \times O$.*

Proof: Choose $q\colon S^n \to \mathbb{R}^{2n} \times O$ to be ε-close to f and with general position properties. This means that there are only finitely many points of self-intersection of $q(S^n)$. Let $X \subset S^n$ be the singular set of q.

Fix an arbitrary point v in $\mathbb{R}^{2n} \times O$ from the ε-neighbourhood of P and define a map $g_1\colon B^{n+1} \to \mathbb{R}^{2n} \times O$ as the cone of map q with vertex v. Therefore the map g_1 sends linearly the interval $[x, c]$ to the interval $[q(x), v]$ for each $x \in \partial B^{n+1}$, where c is the centre of the ball B^{n+1}.

Let O be the origin of some orthogonal coordinate system in $\mathbb{R}^2_+$ with the x-axis lying in the boundary and let A, B and C be the points with coordinates $(-1, 1/2)$, $(1, 1/2)$ and $(0, 1)$, respectively. Since X is 0-dimensional one can choose a map $\phi\colon \partial B^{n+1} \to [A, B]$ on the interval $[A, B]$ such that $\phi\,|_X$ is an embedding.

We define a map g_2 onto the boundary of a concentric ball of half the radius $(1/2)B^{n+1}$ as the composition $g_2 = \phi \circ (\times 2)$. Here $(\times 2)$ sends $\partial\big((1/2)B^{n+1}\big)$ to ∂B^{n+1} homeomorphically. Define $g_2\big(\partial B^{n+1}\big) = O$ and $g_2(c) = C$ and extend g_2 onto $\mathrm{Int}\,(1/2)B^{n+1}$ and onto $\mathrm{Int}\,B^{n+1} - (1/2)B^{n+1}$ linearly. Define $g = (g_1, g_2)$.

The properties (2) and (3) hold by the construction of g. Assume that $g(x) = g(y)$ for some $x, y \in \mathrm{Int}\,B^{n+1}$. We identify B^{n+1} with the set $\{x \in \mathbb{R}^n\colon |x| \leqslant 1\}$. It is easy to see that $x \neq 0$ and $y \neq 0$. If $q(x/|x|) = q(y/|y|)$ then $x/|x|, y/|y| \in X$ and hence $g_2(x) \neq g_2(y)$. If $q(x/|x|) \neq q(y/|y|)$ then the equation $g_1(x) = g_1(y)$ implies that the points $q(x/|x|)$, $q(y/|y|)$ and c are collinear and therefore $|x| \neq |y|$. In that case $g_2(x) \neq g_2(y)$. Contradiction. ∎

Proof of Stallings theorem

We shall use induction on n. Since every finite 1-dimensional complex is homotopy equivalent to a finite disjoint union of wedges of circles, the theorem is true for $n = 1$.

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Let us now verify the inductive step. Let K be an $(n+1)$-dimensional cellular complex and let $K^{(n)}$ denote the n-skeleton of K. By induction, there is a homotopy equivalence $f\colon K^{(n)} \to L$, where L is a polyhedron embeddable in $\mathbb{R}^{2n}$. Let $\{e_i\colon \partial B^{n+1} \to K^{(n)}\}_{i\leqslant m}$ be the family of attaching maps in K for $(n+1)$-dimensional cells. Suppose that $\{\alpha_i\}_{i\leqslant m}$ is a family of angles on the plane with common vertex O such that $\alpha_i \cap \alpha_j = O$ for $i \neq j$. Note that each α_i is homeomorphic to the halfplane $\mathbb{R}^2_+$. Let N be a regular neighbourhood of L in $\mathbb{R}^{2n} \times O$. There is $\delta > 0$ such that the δ-neighbourhood of L is contained in N.

Apply Lemma 2 for L, $f \circ e_i$ and $\varepsilon = \delta$ to obtain maps $g_i\colon B^{n+1} \to \mathbb{R}^{2n} \times \alpha_i$ with the properties (1) and (2). The property (2) implies that the map $g_i = g_i\,|_{\partial B^{n+1}}$ is homotopic to $f \circ e_i$ in N. The property (1) yields an embedding of

$$M = N \underset{q_1}{\cup} B^{n+1} \underset{q_2}{\cup} B^{n+1} \cup \ldots \underset{q_m}{\cup} B^{n+1}$$

in $\mathbb{R}^{2n+2}$. Since we may assume that each map q_i is simplicial with respect to some triangulations on N and ∂B^{n+1} we may regard M as a polyhedron. Lemma 1 implies that M is homotopy equivalent to K.

Epilogue

Note that by the construction $\dim M = 2\dim K - 2$ and that we can also achieve that the Euler characteristics of M be minimal. As a result we get the answer to both Kreck's questions:

Corollary. *For every finitely presented group G there exists a 2-dimensional polyhedron $M \subset \mathbb{R}^4$ with the fundamental group $\pi_1(M) \cong G$. In addition, we may assume that M has minimal Euler characteristic.*

Remark. Another solution of Kreck's problem follows from the recent work of Skopenkov, Ščepin and the second author [7].

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