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AUTOMORPHISMS OF METABELIAN GROUPS

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ABSTRACT. We investigate the problem of determining when $\mathrm{IA}(F_n(\mathbf{A}_m\mathbf{A}))$ is finitely generated for all n and m, with $n \geq 2$ and $m \neq 1$. If m is a nonsquare free integer then $\mathrm{IA}(F_n(\mathbf{A}_m\mathbf{A}))$ is not finitely generated for all n and if m is a square free integer then $\mathrm{IA}(F_n(\mathbf{A}_m\mathbf{A}))$ is finitely generated for all n, with $n \neq 3$, and $\mathrm{IA}(F_3(\mathbf{A}_m\mathbf{A}))$ is not finitely generated. In case m is square free, Bachmuth and Mochizuki claimed in ([7], Problem 4) that $\mathrm{TR}(\mathbf{A}_m\mathbf{A})$ is 1 or 4. We correct their assertion by proving that $\mathrm{TR}(\mathbf{A}_m\mathbf{A}) = \infty$.

1**. Introduction.** For any group G, let $\mathrm{IA}(G)$ be the IA-automorphism group of G, that is, the kernel of the natural mapping from $\mathrm{Aut}(G)$ into $\mathrm{Aut}(G/G')$, where G' denotes the derived group of G. For each positive integer c, we write by $\gamma_c(G)$ the c-th term of the lower central series of G. So, $\gamma_2(G) = G'$. If $a_1, \ldots, a_c$ are elements of a group G then $[a_1, a_2] = a_1a_2a_1^{-1}a_2^{-1}$ and, for $c \geq 3$, $[a_1, \ldots, a_c] = \big[[a_1, \ldots, a_{c-1}], a_c\big]$. For a positive integer n, with $n \geq 2$, we will denote by F_n the (absolutely) free group of rank n freely generated by the set $\{f_1, \ldots, f_n\}$. If $\mathbf{V}$ is a variety of groups, we write $F_n(\mathbf{V})$ for the free group of rank n in $\mathbf{V}$ and $\mathbf{V}(F_n)$ for the verbal subgroup of F_n corresponding to $\mathbf{V}$. Every element in the image of the natural mapping from $\mathrm{Aut}(F_n)$ into $\mathrm{Aut}\big(F_n(\mathbf{V})\big)$ is called *tame*. For a non-negative integer m, with $m \neq 1$, $\mathbf{A}_m$ denotes the variety of all abelian groups of exponent dividing m, interpreted in such a way that $\mathbf{A}_0 = \mathbf{A}$ is the variety of all abelian groups. Further, $\mathbf{W}_m$ is the variety of all extensions of groups in $\mathbf{A}_m$ by groups in $\mathbf{A}$. We write $\mathbf{W}$ for $\mathbf{W}_0$. In the papers ([5] and [6]), Bachmuth and Mochizuki have proved that $\mathrm{IA}\big(F_n(\mathbf{W})\big)$ is finitely generated for $n \neq 3$ and $\mathrm{IA}\big(F_3(\mathbf{W})\big)$ is not finitely generated. Bachmuth *et al.* (see [3], Theorem C) have shown that $\mathrm{IA}\big(F_2(\mathbf{W}_m)\big)$ is not finitely generated if m is a free integer and finitely generated if m is a square free integer. In this paper we extend the latter result for all n, with $n \geq 2$.

We say $\big\{\mathrm{Aut}\big(F_n(\mathbf{V})\big), n \geq 1\big\}$ has tame range infinite, denoted by $\mathrm{TR}(\mathbf{V}) = \infty$, if there does not exist a positive integer d such that all automorphisms of $F_n(\mathbf{V})$ are tame for all $n \geq d$. Otherwise, we say it has a finite one. We deduce from [5] and [6] that $\mathrm{TR}(\mathbf{W}) = 4$. So, we concentrate on m, with $m \geq 2$. If m is prime, say p, then $\mathrm{TR}(\mathbf{W}_p) = 4$. Indeed, by means of techniques of [6] or more easily of [9], every automorphism of $F_n(\mathbf{W}_p)$ is induced by an automorphism of F_n for all $n \geq 4$. As we shall see in Section 4, $\mathrm{IA}\big(F_3(\mathbf{W}_p)\big)$ is not finitely generated and so $\mathrm{TR}(\mathbf{W}_p) = 4$. The method of proving $\mathrm{IA}\big(F_3(\mathbf{W}_p)\big)$ is not finitely generated is based on ideas of Bachmuth and Mochizuki in [5]. Thus, we may assume m is not prime. If m is nonsquare free, it follows from a result of Bachmuth and

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Mochizuki [8] that $\mathrm{TR}(\mathbf{W}_m) = \infty$. In case m is square free, Bachmuth and Mochizuki claimed in ([7], Problem 4) that $\mathrm{TR}(\mathbf{W}_m)$ is 1 or 4. We correct their assertion by proving that $\mathrm{TR}(\mathbf{W}_m) = \infty$.

For positive integers n and m, with $n, m \geq 2$, let $G(n, m)$ be a free group of rank n in the variety $\mathbf{W}_m$. We write $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where p_i are distinct prime integers and $\alpha_i \in N$. For $i = 1, \ldots, r$, we denote by $G(n, m, i)$ the free group of rank n in the variety $\mathbf{W}_{p_i^{\alpha_i}}$. By a result of Bachmuth and Mochizuki [4], $\mathrm{IA}\big(G(n, m)\big)$ is isomorphic to a direct product $\mathrm{IA}\big(G(n, m, 1)\big) \times \cdots \times \mathrm{IA}\big(G(n, m, r)\big)$. Suppose m is a nonsquare free integer. Then there exists a prime p such that p^2 divides m. Let T be the subgroup of $\mathrm{Aut}\big(G(n, p^2)\big)$ consisting of all tame automorphisms. Since T is finitely generated, $\mathrm{Aut}\big(G(n, p^2)\big) = T\,\mathrm{IA}\big(G(n, p^2)\big)$ and $\mathrm{Aut}\big(G(n, p^2)\big)$ is not finitely generated for all n, with $n \geq 2$, (see [8]), we obtain that $\mathrm{IA}\big(G(n, p^2)\big)$ is not finitely generated for all n. Therefore $\mathrm{IA}\big(G(n, m)\big)$ is not finitely generated. Thus, we may assume m is square free. Our main result deals with this case. In Section 4, we prove the following theorem.

THEOREM. *Let $G(n, m)$ be a free group of rank n in the variety $\mathbf{W}_m$, with $n, m \geq 2$.*
(i) If m is a nonsquare free integer then $\mathrm{IA}\big(G(n, m)\big)$ is not finitely generated for all n.
(ii) If m is a square free integer then $\mathrm{IA}\big(G(n, m)\big)$ is finitely generated for all $n \neq 3$ and $\mathrm{IA}\big(G(3, m)\big)$ is not finitely generated. Further, $\mathrm{IA}\big(G(n, m)\big)$ contains nontame elements for all n.

COROLLARY. *The tame range $\mathrm{TR}(\mathbf{W_m})$ is infinite for any positive integer m, with $m \geq 2$, but not prime.*

2. **Preliminaries.** Let $C = A *_U B$ be the free product of groups A and B with amalgamated subgroup U. An element c of C can be written as $c = c_1 \cdots c_r$ where each c_i belongs to A or B, c_i and c_{i+1} cannot both belong to A or both to B and r is uniquely determined. The number r is called the *length* of c and the length of the identity element is defined to be 0. By the length of a subset Γ we will mean the length of the shortest element in the subset.

We shall state the Subgroup Theorem for amalgamated products as in Cohen [10], who used the theory of groups acting on trees. Let H be a subgroup of C. Following Cohen, let $\{D_\alpha\}$ be a set of double coset representatives for (H, A) in C and $\{D_\beta\}$ be a set of double coset representatives for (H, B) in C. Further, for each D_α, let $\{E_\mu\}$ be a set of double coset representatives containing 1 of $(D_\alpha^{-1} H D_\alpha \cap A, U)$ in A, and for each D_β, let $\{E_\nu\}$ be a set of double coset representatives containing 1 of $(D_\beta^{-1} H D_\beta \cap B, U)$ in B. For each α and associated μ, there exists a unique element D_β corresponding E_ν and $u \in U$ such that $D_\alpha E_\mu \in H D_\beta E_\nu u$. Thus

$$t_{\alpha,\mu} = D_\alpha E_\mu (D_\beta E_\nu u)^{-1} \in H,$$

and $t_{\alpha,\mu} \neq 1$ if and only if $D_\alpha E_\mu$ is neither a (H, A) double coset representative nor a (H, B) double coset representative.

SUBGROUP THEOREM (*cf.* [10], THEOREM 3). *Let H be a subgroup of $A *_U B$, where $U = A \cap B$. Then,*

(i) those $t_{\alpha,\mu} \neq 1$ freely generate a free subgroup of H;

(ii) the group K generated by all $H \cap (D_\alpha A D_\alpha^{-1})$ and $H \cap (D_\beta B D_\beta^{-1})$ is the tree product of these groups, two such groups being adjacent if $D_\alpha = D_\beta = 1$ or if $D_\alpha = D_\beta b$, $b \in B$, or $D_\beta = D_\alpha a$ for some $a \in A$; the subgroup amalgamated between two adjacent groups is $H \cap (DUD^{-1})$, where D is the longer of D_α and D_β;

(iii) H is the HNN-*group*

$$\langle K, t_{\alpha,\mu}; t_{\alpha,\mu}(H \cap D_\beta E_\nu U E_\nu^{-1} D_\beta^{-1}) t_{\alpha,\mu}^{-1} = H \cap D_\alpha E_\mu U E_\mu^{-1} D_\alpha^{-1} \rangle$$

over all $t_{\alpha,\mu} \neq 1$ and corresponding D_β, E_ν.

Concerning tree products and HNN-groups, we refer the reader to ([11] and [12]). Since $H \cap D_\alpha E_\mu U E_\mu^{-1} D_\alpha^{-1} \subseteq H \cap D_\alpha A D_\alpha^{-1}$ and $H \cap D_\beta E_\nu U E_\nu^{-1} D_\beta^{-1} \subseteq H \cap D_\beta B D_\beta^{-1}$, we apply a result of Karrass and Solitar (**[11]**, Lemma 6) to obtain the following result.

PROPOSITION 2.1. *In the notation of the Subgroup Theorem, if H is a finitely generated subgroup of $A *_U B$, then only finitely many of the $t_{\alpha,\mu} \neq 1$ and K is the tree product of finitely many of the $H \cap (D_\alpha A D_\alpha^{-1})$ and the $H \cap (D_\beta B D_\beta^{-1})$.*

The proof of the following lemma is elementary.

LEMMA 2.2. *Let R be a principal ideal domain (*PID*), which is not a field, and let $a \in R \setminus \{0\}$ be a nonunit of R. Then the localization R_S of R away from S is a* PID*, where $S = \{a^n : n \geq 0\}$.*

3. **A reduction.** For a fixed prime p, we set $G := F_3(\mathbf{W}_p)$ and $x_i := f_i\big(\mathbf{W}_p(F_3)\big)$, $i = 1, 2, 3$. Thus, G is a free group of rank 3 in $\mathbf{W}_p$ freely generated by x_1, x_2, x_3. We denote by M a (left) free $\mathrm{Z}_p(F_3/F_3')$-module with a basis $\{\lambda_1, \lambda_2, \lambda_3\}$ and by Ω the cartesian product of F_3/F_3' by M. The set Ω becomes a group by defining a multiplication

$$(\tilde{u}, m_1)(\tilde{v}, m_2) = (\tilde{u}\tilde{v}, \tilde{u}m_2 + m_1) = (\widetilde{uv}, \tilde{u}m_2 + m_1),$$

for all $\tilde{u}, \tilde{v} \in F_3/F_3'$, where $\tilde{u} = uF_3'$ and $\tilde{v} = vF_3'$, with $u, v \in F_3$, and $m_1, m_2 \in M$. For $i = 1, 2, 3$, let $s_i = f_i F_3'$. The mapping from G into Ω sending x_i to (s_i, λ_i) is an embedding (see [2], [13]). Further, an element $(g, a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3)$ represents an element of G if and only if $a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = 1 - g$, where $\sigma_i = 1 - s_i$, $i = 1, 2, 3$. We identify G with its image in Ω and let ϕ be an IA-automorphism of G. Then ϕ can be described by

$$\phi\colon (s_j, \lambda_j) \longrightarrow (s_j, a_{1j}\lambda_1 + a_{2j}\lambda_2 + a_{3j}\lambda_3) \tag{3.1}$$

where

$$a_{1j}\sigma_1 + a_{2j}\sigma_2 + a_{3j}\sigma_3 = \sigma_j, \tag{3.2}$$

for all $j \in \{1, 2, 3\}$. The mapping of $\mathrm{IA}(G)$ into $\mathrm{GL}_3\big(\mathrm{Z}_p(F_3/F_3')\big)$ given by $\phi \longrightarrow (a_{ij})$, where ϕ is given by (3.1), is an embedding. By a result of Bachmuth (see [2],

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Proposition 2), $(a_{ij}) \in \mathrm{GL}_3(\mathrm{Z}_p(F_3/F_3'))$ is in the image of this embedding if and only if the columns of (a_{ij}) satisfy the condition (3.2).

We adopt the convention that a 2×2 matrix (a_{ij}) over a ring R is written as $(a_{11}, a_{12}; a_{21}, a_{22})$. Following [5], we identify $\mathrm{Z}_p(F_3/F_3')$ with $\mathrm{Z}_p[s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}]$ and each element ϕ of $\mathrm{IA}(G)$ can be uniquely represented by an element of $\mathrm{GL}_3(\mathrm{Z}_p[s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}])$ of the form

$$(3.3) \qquad \begin{pmatrix} 1 + \sigma_2 a_1 + \sigma_3 a_2 & -\sigma_2 b_3 - \sigma_3 b_1 & -\sigma_3 c_3 + \sigma_2 c_2 \\ -\sigma_1 a_1 + \sigma_3 a_3 & 1 + \sigma_3 b_2 + \sigma_1 b_3 & -\sigma_3 c_1 - \sigma_1 c_2 \\ -\sigma_1 a_2 - \sigma_2 a_3 & -\sigma_2 b_2 + \sigma_1 b_1 & 1 + \sigma_1 c_3 + \sigma_2 c_1 \end{pmatrix}$$

Further, there exists a representation τ of $\mathrm{IA}(G)$ into $\mathrm{GL}_2(\mathrm{Z}_p[s_1^{\pm 1}, s_2^{\pm 1}])$ by sending a matrix of the form (3.3) into a matrix of the form

$$(3.4) \qquad (1 + \sigma_1 \hat{b}_3 + \sigma_2 \hat{a}_1, -\hat{c}_2; -\sigma_1\sigma_2 \hat{b}_2 + \sigma_1^2 \hat{b}_1 + \sigma_1\sigma_2 \hat{a}_2 + \sigma_2^2 \hat{a}_3, 1 + \sigma_1 \hat{c}_3 + \sigma_2 \hat{c}_1)$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are the images of a, b and c in $\mathrm{Z}_p[s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}]$ via the natural mapping from $\mathrm{Z}_p[s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}]$ into $\mathrm{Z}_p[s_1^{\pm 1}, s_2^{\pm 1}]$ by sending s_3 into 1 and $\mathrm{Z}_p[s_1^{\pm 1}, s_2^{\pm 1}]$ is mapped identical onto itself. Let A be the image of the representation τ and $B = A \cap \mathrm{SL}_2(\mathrm{Z}_p[s_1^{\pm 1}, s_2^{\pm 1}])$. Similar arguments as in the proof of Lemma 1 of [5], we have the following result.

LEMMA 3.1. *If A is finitely generated then B is finitely generated.*

4. **Proof of Theorem.** Let R be a PID with a quotient field Q and t an indeterminate over R. Write $\mathrm{SL}_2(Q[t])^S$ for $S^{-1}(\mathrm{SL}_2(Q[t]))S$ where $S = (t, 0; 0, 1)$. By Ihara's Theorem we obtain $\mathrm{SL}_2(Q[t, t^{-1}]) = \mathrm{SL}_2(Q[t]) *_U \mathrm{SL}_2(Q[t])^S$, where $U = \mathrm{SL}_2(Q[t]) \cap \mathrm{SL}_2(Q[t])^S$.

LEMMA 4.1. *Let π be an irreducible element of R. Then the matrices $A(i) = (1, 0; \frac{1}{\pi^i}, 1)$, with $i \geq 1$, can be chosen as part of a set of double coset representatives of $(\mathrm{SL}_2(R[t]), U)$ in $\mathrm{SL}_2(Q[t])$.*

PROOF. Suppose there exist $1 \leq i < j$ such that $A(i)$ and $A(j)$ are in the same double coset. By setting $t = 0$, we obtain

$$A(i) = (f, g; h, k) A(j) (\delta^{-1}, \xi; 0, \delta)$$

where $(f, g; h, k) \in \mathrm{SL}_2(R)$ and $(\delta^{-1}, \xi; 0, \delta) \in \mathrm{SL}_2(Q)$. Therefore,

$$(4.1) \qquad f + \frac{g}{\pi^j} = \delta$$

$$(4.2) \qquad h + \frac{k}{\pi^j} = \frac{\delta}{\pi^i}$$

$$(4.3) \qquad \xi\left(f + \frac{g}{\pi^j}\right) + g\delta = 0$$

$$(4.4) \qquad \xi\left(h + \frac{k}{\pi^j}\right) + k\delta = 1$$

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From (4.1) and (4.3), we obtain that $\xi = -g$. So (4.4) becomes

$$\delta^{-1} = -\frac{g}{\pi^i} + k \tag{4.5}$$

We write $\delta = \delta_1/\delta_2$, where $(\delta_1, \delta_2) = 1$. We claim π^j divides g. To get a contradiction, we assume that π^j does not divide g. We first show that π divides g. Indeed, if not and since π is prime in R, we obtain from (4.1) that π divides δ_2. Further, we obtain from (4.5) that π divides δ_1 which is a contradiction. Therefore, $g = \pi^\mu g'$, where $g' \in R$ and $(\pi, g') = 1$. Since g is not divided by π^j and π divides g, we have $1 \leq \mu \leq j-1$. Now, either $\mu < i$ or $i \leq \mu \leq j-1$. Suppose $\mu < i$. Then, we obtain from (4.1) that π divides δ_2. Similarly, from (4.5), we have π divides δ_1, which is a contradiction. Thus, $i \leq \mu \leq j-1$. By (4.5), $\delta^{-1} \in R$ and so π^j must divide g. Therefore, we obtain from (4.1) and (4.2) that $\delta \in R$ and π^{j-i} must divide k. Thus, π divides both g and k. Since $(f, g; h, k)$ is invertible over R and the ideal $\langle \pi \rangle$ is properly contained in R, we get the required contradiction.

Let m be a square free integer. If $n \neq 3$ then $\mathrm{IA}\big(G(n, m)\big)$ is finitely generated (see, Introduction). So, we may assume that $n = 3$. To prove that $\mathrm{IA}\big(G(3, m)\big)$, with $m \geq 2$, is not finitely generated, it is enough to show that, for any prime p, $\mathrm{IA}\big(G(3, p)\big)$ is not finitely generated. For a fixed prime p, we recall that $G := F_3(\mathbf{W}_p)$ and $x_i := f_i\big(\mathbf{W}_p(F_3)\big)$, $i = 1, 2, 3$. Thus, G is a free group of rank 3 in the variety $\mathbf{W}_p$ freely generated by x_1, x_2, x_3. We set $R := \mathrm{Z}_p[s_1, s_2^{-1}]$, $\pi := s_1 + 1$ and $A(i) := (1, 0; \frac{1}{(s_1+1)^i}, 1)$, with $i \geq 1$. By Lemma 2.2, we obtain that R is a PID. Further, it is easily checked that π is an irreducible element of R. By Lemma 4.1, the set $\{A(i), i \geq 1\}$ may be included as part of a set of double coset representatives of $\mathrm{SL}_2(R[s_2], U)$ in $\mathrm{SL}_2(Q[s_2])$, where $U = \mathrm{SL}_2(Q[s_2]) \cap \mathrm{SL}_2(Q[s_2])^S$ and $S = (s_2, 0; 0, 1)$. Hence also part of representatives of $\big(B \cap \mathrm{SL}_2(R[s_2]), U\big)$ in $\mathrm{SL}_2(Q[s_2])$. We apply the Subgroup Theorem to B as a subgroup of $\mathrm{SL}_2(Q[s_2, s_2^{-1}])$. Let $\{D_\alpha\}$ be a set of double coset representatives for $(B, \mathrm{SL}_2\big(Q[s_2]\big))$ in $\mathrm{SL}_2(Q[s_2, s_2^{-1}])$ and $\{D_\beta\}$ a set of double coset representatives for $(B, \mathrm{SL}_2(Q[s_2])^S)$ in $\mathrm{SL}_2(Q[s_2, s_2^{-1}])$. Recall that the group K generated by all $B \cap \big(D_\alpha \mathrm{SL}_2(Q[s_2])D_\alpha^{-1}\big)$ and $B \cap \big(D_\beta \mathrm{SL}_2(Q[s_2])^S D_\beta^{-1}\big)$ is the tree product of these groups. To show that B is not finitely generated, it is enough, by Proposition 2.1, to show that infinitely many of the $A(i)$ are not double coset representatives of $\big(B, \mathrm{SL}_2(Q[s_2])^S\big)$ in $\mathrm{SL}_2(Q[s_2, s_2^{-1}])$. To get a contradiction, we suppose that infinitely many of the $A(i)$ are double coset representatives. We assert that K is *not* the tree product of *only finitely many* of the $B \cap D_\alpha \mathrm{SL}_2(Q[s_2])D_\alpha^{-1}$ and the $B \cap D_\beta \mathrm{SL}_2(Q[s_2])^S D_\beta^{-1}$. In particular, we claim that $B \cap A(i)UA(i)^{-1}$ is a proper subgroup of $B \cap A(i)\,\mathrm{SL}_2(Q[s_2])^S A(i)^{-1}$ for all i. To see this, let $\Lambda = (1, \pi^{2i}\sigma_1^2\sigma_2 s_2^{-1}; 0, 1)$. Then $A(i)\Lambda A(i)^{-1} = (1 - \pi^i s_2^{-1}\sigma_1^2\sigma_2, \pi^{2i}\sigma_1^2\sigma_2 s_2^{-1}; -s_2^{-1}\sigma_1^2\sigma_2, 1 + \pi^i s_2^{-1}\sigma_1^2\sigma_2)$. It is easily seen that both Λ and $A(i)\Lambda A(i)^{-1}$ belong to B, also $A(i)\Lambda A(i)^{-1} \in B \cap A(i)\,\mathrm{SL}_2(R[s_2])^S A(i)^{-1}$, but $A(i)\Lambda A(i)^{-1}$ does not belong to $B \cap A(i)UA(i)^{-1}$, which is the required contradiction.

To complete the proof of Theorem (ii), we need some further notation. Recall that $G(n, m)$ is a free group of rank n in the variety $\mathbf{W}_m$, with $n, m \geq 2$. For the next few

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lines, we set $H := G(n, m)$ and $y_i := f_i\big(\mathbf{W}_m(F_n)\big)$, $i = 1, \ldots, m$. Thus, the set $\{y_1, \ldots, y_m\}$ freely generates H. Let $m = p_1 \cdots p_r$, with $r \geq 2$, where p_i are distinct prime integers. Since Z_m is equal to the direct product $I_1 \oplus \cdots \oplus I_r$, where I_i is an ideal of Z_m which is isomorphic to Z_{p_i} as rings, there exists a (necessarily unique) complete set $\{e_1, \ldots, e_r\}$ of pairwise orthogonal idempotents in Z_m such that $I_i = e_i\mathrm{Z}_m$, $i = 1, \ldots, r$. Note that H' may be regarded as a right $\mathrm{Z}_m(H/H')$- module, where the action of H/H' comes from conjugation in H. For $u \in H'$ and $d \in \mathrm{Z}_m(H/H)'$ we write u^d for the image of u under the module action by d.

PROPOSITION 4.1. *Let $n \geq 2$ and, in the notation described above, let λ be the endomorphism of H satisfying $\lambda(y_1) = y_1[y_1, y_2]^{-e_1s_1}$, $\lambda(y_2) = y_2[y_1, y_2]^{(1-e_1)s_2}$, and $\lambda(y_i) = y_i$, $i \geq 3$. Then λ is a non-tame automorphism of H.*

PROOF. Let μ be the endomorphism of H satisfying $\mu(y_1) = y_1[y_1, y_2]^{e_1s_1s_2}$, $\mu(y_2) = y_2[y_1, y_2]^{-(1-e_1)s_1s_2}$, and $\mu(y_i) = y_i$, $i \geq 3$. It is easily checked that $\mu\lambda = \lambda\mu = 1$, where 1 denotes the identity mapping on H. Hence, λ is an automorphism of H. By Corollary 3.2 of [4], λ may be represented by an $n \times n$ matrix over $\mathrm{Z}_m(F_n/F_n')$, say λ^*. It is easy to see that the determinant of λ^* is equal to $(1 - e_1)s_1 + e_1s_2$. Now, suppose λ is tame automorphism. Thus, there exists an automorphism ψ of F_n such that ψ induces λ on H. Since $H/H' \cong F_n/F_n'$, we may take $\psi \in \mathrm{IA}(F_n)$. Let ϕ be the induced IA-automorphism of ψ on $F_n(\mathbf{W})$. By Theorem 1 of [1], ϕ corresponds to an $n \times n$ matrix $C(\phi)$ over $\mathrm{Z}(F_n/F_n')$ such that $C(\phi)$ induces λ^*. Namely, $C(\phi) = \mathrm{I}_n + (\alpha_{ij})$, where I_n is the identity matrix, $\alpha_{11} = -e_1(1 - s_2) + u_{11}$, $\alpha_{12} = (1 - e_1)(1 - s_2) + u_{12}$, $\alpha_{21} = e_1(1 - s_1) + u_{21}$, $\alpha_{22} = (1-e_1)(1-s_1)+u_{22}$ and $\alpha_{ij} = u_{ij}$ otherwise. Each u_{ij} is a polynomial in the variables $s_1^{\pm 1}, \ldots, s_n^{\pm 1}$ over Z with coefficients multiples of m. But the determinant of $C(\phi)$ equals $\prod_{i=1}^n s_i^{\alpha_i}$, where $\alpha_i \in \mathrm{Z}$. Therefore, by working over Z_m, we have $\prod_{i=1}^n s_i^{\alpha_i} = (1-e_1)s_1+e_1s_2$, which is a contradiction. Thus, λ is a non-tame automorphism of H for all $n \geq 2$.

REFERENCES

1. S. Bachmuth, *Automorphisms of free metabelian groups.* Trans. Amer. Math. Soc. **118**(1965), 93–104.
2. ———, *Automorphisms of a class of metabelian groups*. Trans. Amer. Math. Soc. **127**(1967), 284–293.
3. S. Bachmuth, G. Baumslag, J. Dyer and H. Y. Mochizuki, *Automorphisms of* 2*-generator metabelian groups*. J. London Math. Soc. **36**(1987), 393–406.
4. S. Bachmuth and H. Y. Mochizuki, *Automorphisms of a class of metabelian groups II*. Trans. Amer. Math. Soc. **127**(1967), 294–301.
5. ———, IA-*automorphisms of free metabelian group of rank* 3. J. Algebra **55**(1979), 106–115.
6. ———, $\mathrm{Aut}(F) \longrightarrow \mathrm{Aut}(F/F'')$ *is surjective for free groups F of rank* $n \geq 4$. Trans. Amer. Math. Soc. **292**(1985), 81–101.
7. ———, *The tame range of automorphism groups and* GL_n. Group Theory Proc. Singapore Group Theory Conf., 1987, 241–251, de Gruyter, New York, 1989.
8. ———, *Infinite generation of automorphism groups*. Proceedings of Groups Korea 1988, Pusan, 1988, 25–28, Lecture Notes in Math. **1398**, Springer, Berlin, New York, 1989.
9. R. M. Bryant and J. R. J. Groves, *Automorphisms of free metabelian groups of infinite rank*. Comm. Algebra (11) **18**(1990), 3619–3631.
10. D. E. Cohen, *Subgroups of* HNN*-groups*. J. Austral. Math. Soc. **17**(1974), 394–405.
11. A. Karrass and D. Solitar, *The subgroups of a free product of two groups with an amalgamated subgroup*. Trans. Amer. Math. Soc. **150**(1970), 227–255.

https://doi.org/10.4153/CMB-1998-015-7 Published online by Cambridge University Press

12. ______, *Subgroups of* HNN*-groups and groups with one defining relation*. Canad. J. Math. **23**(1971), 627–543.
13. A. F. Krasnikov, *Nilpotent subgroups of relatively free groups*. Algebra i Logika **17**(1978), 389–401.

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