PERMUTATIONAL PRODUCTS OF GROUPS

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1. Introduction

We deal with questions about the possible embeddings of two given groups A and B in a group P such that the intersection of A and B is a given subgroup H. The data, consisting of the "constituents" A and B with the "amalgamated" subgroup H, form an *amalgam*.[1] According to a classical theorem of Otto Schreier [5], every amalgam of two groups can be embedded in a group F, the "free product of A and B with amalgamated subgroup H" or the "generalized free product" of the amalgam. This has the property that every group P in which the amalgam is embedded and which is generated by the amalgam, is a homomorphic image of it. Hence theorems on the existence of certain embedding groups P can be interpreted also as theorems on the existence of certain normal subgroups of F.

One naturally asks whether given properties of the amalgam can be made to persist in some embedding group P. Thus e.g. if A and B are finite, can P be chosen finite? Or again, if A and B are abelian, can P be chosen abelian? The answer to both questions is known [2] to be positive.

By contrast, if A and B are both nilpotent, it may be impossible to choose P nilpotent: Wiegold has given an example [6, Ex. 7.10, p. 152] of an amalgam of an abelian group A and a nilpotent group B, of nilpotent class 2 only, which cannot be embedded in any nilpotent group. Wiegold's amalgam can be embedded in a metabelian group, for reasons which will appear presently. However, a different example, to be given in § 2, will show that with A and B both nilpotent it may yet be impossible to choose P soluble.

On the other hand, if we impose conditions not only on A and B, but also on H, we may hope for more positive results; and indeed, if H is *central* (*i.e.* a subgroup of the centre) in both A and B, then the so-called "direct product of A and B with amalgamated subgroup H" or "generalized direct product"

[1] This term was introduced by Baer [1]; for a survey of the embedding theory of group amalgams, cf. [4] and [2] and the literature there quoted.

[2] The second one is almost trivial and no explicit statement or proof can be referred to; for the first, cf. [2], Corollary 15.2.

of the amalgam [3] provides an embedding in a group D that belongs to the least variety containing both A and B (or, differently put, all laws or identical relations valid in both A and B are also valid in D): hence if A and B are both nilpotent then D is nilpotent, and if A and B are both soluble, then D is soluble; and the nilpotent class or soluble length of D is the greater of the classes or lengths of A and B.

If H is central in only one of the two constituents, say in A, then the generalized direct product is no longer available; and Wiegold's example (*loc. cit.*) shows that in this case we cannot hope for an embedding in a group of the least variety containing A and B. Nevertheless it is possible to show that in this case, if both A and B are soluble, then the embedding group can be chosen soluble, and if $l(P)$ is its soluble length, then

$$l(P) \leqq l(A) + l(B) - 1.$$

This inequality is best possible. If, in particular, A is abelian (and thus H trivially central in A), then P can be chosen soluble whenever B is soluble, and the length of P can be taken equal to that of B. Hence the amalgam of Wiegold's example must be embeddable in a metabelian group.

The proof of these facts consists in the construction (§ 3) of a suitable group P, and the verification of the properties claimed for P (§ 5). The construction is a simplified version of one used already in [2] § 13; it leads to a permutation group P, which will be called a "permutational product" of the amalgam. It depends not only on the amalgam, but also on a choice of left coset representatives of H in A and B. This dependence on the particular coset representatives chosen would appear to merit closer study; a few results, though not subservient to our main purpose, will be derived here (§4).

2. An example

We take as our groups A and B isomorphic groups, namely the non-abelian group of order 5^3 and exponent 5; and H is to be a subgroup of order 5^2. It follows that H is elementary abelian, and normal in both A and B. To complete the specification of the amalgam, we have to say more precisely what subgroup of A and what subgroup of B the amalgamated subgroup H is to be. Employing the usual commutator notation $[x, y] = x^{-1}y^{-1}xy$, we put

$$A = \mathrm{gp}(a, g, h;\ [a, g] = h^{-1},\ [a, h] = [g, h] = a^5 = g^5 = h^5 = 1),$$
$$B = \mathrm{gp}(b, g, h;\ [b, h] = g^{-1},\ [b, g] = [g, h] = b^5 = g^5 = h^5 = 1),$$
$$H = \mathrm{gp}(g, h;\ [g, h] = g^5 = h^5 = 1).$$

Now let this amalgam be embedded in a group G, and denote by P the subgroup of G generated by the amalgam, that is by A and B. If we show that P is insoluble, then G will also be insoluble. Now H is normal in A and in B

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and hence also in P. Let Π be the group of automorphisms of H induced by the inner automorphisms of P. If we think of H as a 2-dimensional vector space over GF(5), with basis g, h, then Π is (isomorphically) represented as a group of matrices, and as such it is generated by the matrices corresponding to $a \in A$ and $b \in B$. These are $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ and generate the group of all unimodular 2×2 matrices over GF(5). Thus Π is isomorphic to this group, which is a well-known insoluble group, namely the binary icosahedral group. But Π is isomorphic to a factor group of P, namely

$$\Pi \cong P/C(H),$$

where $C(H)$ is the centralizer of H in P. Thus P also must be insoluble.

As we shall see, this example is best possible, in that no similar example could be made with one factor abelian, even if the other factor is allowed to be non-nilpotent (but, of course, still soluble). It also serves to disprove other more or less plausible conjectures. Thus one might hope that an amalgam of groups of exponent n can be embedded in a group of exponent a power of n: but our example shows that this is not the case. Again one might conjecture [3] that if the amalgamated subgroup H is normal in both A and B, then every embedding of A/H and B/H (with the trivial amalgamation) in a group Q can be extended to an embedding of A and B in a group P in the sense that $Q \cong P/H$ (both Q and P being assumed generated by the amalgam they embed); and again our example disproves it if we take Q to be the direct product of A/H and B/H.

3. The permutational product

In [2, § 13] a method was given of embedding an amalgam of groups with a single subgroup amalgamated in a permutation group. We now give a simpler construction of essentially the same embedding group, again as a permutation group, but of smaller degree. We deal here only with the case of two groups, because the extension of the procedure to more than two groups is easy, provided always that the amalgamated subgroup is the same throughout. The two given groups are again A and B, and the amalgamated subgroup is H.

In A we choose a system of left coset representatives, or *left transversal*, S of H. Then every element $a \in A$ has a unique product decomposition

$$a = sh, \qquad s \in S, \quad h \in H.$$

We introduce the notation

$$s = a^{\sigma}, \qquad h = a^{-\sigma+1}$$

for the factors in this decomposition. Similarly we choose a left transversal

[3] and Dr J. Wiegold reminds me that I did once formulate this conjecture.

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T of H in B, and denote the factors in the unique decomposition

$$b = th, \qquad t \in T, \quad h \in H$$

of an element $b \in B$ by

$$t = b^{\tau}, \qquad h = b^{-\tau+1}.$$

Note that the mappings σ, $-\sigma + 1$, τ, $-\tau + 1$ are not in general homomorphisms.

Let K stand for the set product of S, T, and H; that is to say, K consists of the ordered triplets

$$k = (s, t, h), \qquad s \in S, \quad t \in T, \quad h \in H.$$

We now define certain mappings of K into K, which will presently turn out to be permutations of K. To every element $a \in A$ we define a mapping $\rho(a)$: for $k = (s, t, h) \in K$ we put $k^{\rho(a)} = (s', t', h')$, where

$$s'h' = sha, \quad t' = t;$$

or, more briefly,

$$(s, t, h)^{\rho(a)} = \big((sha)^{\sigma}, t, (sha)^{-\sigma+1}\big).$$

Similarly we define a mapping $\rho(b)$ of K into K for every $b \in B$, by putting $k^{\rho(b)} = (s'', t'', h'')$, where

$$s'' = s, \qquad t''h'' = thb,$$

or again

$$(s, t, h)^{\rho(b)} = (s, (thb)^{\tau}, (thb)^{-\tau+1}).$$

We note that if $h^* \in H$ then

$$(shh^*)^{\sigma} = s, \qquad (thh^*)^{\tau} = t,$$

and thus

$$(s, t, h)^{\rho(h^*)} = (s, t, hh^*),$$

irrespective of whether we consider h^* as an element of A or of B: thus on the common part of A and B, our definition is in fact unambiguous.

If $a, a' \in A$, then, as clearly $(sha)^{\sigma}(sha)^{-\sigma+1} = sha$,

$$(s, t, h)^{\rho(a)\rho(a')} = \big((sha)^{\sigma}, t, (sha)^{-\sigma+1}\big)^{\rho(a')} = \big((shaa')^{\sigma}, t, (shaa')^{-\sigma+1}\big) = (s, t, h)^{\rho(aa')},$$

and as the triplet $(s, t, h) \in K$ is arbitrary, we conclude that

$$\rho(a)\rho(a') = \rho(aa').$$

It follows that the $\rho(a)$, for a ranging over A, form a group $\rho(A)$, say, and that the mapping of $a \in A$ onto $\rho(a) \in \rho(A)$ is an epimorphism; and it follows

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incidentally that the $\rho(a)$ are permutations of K. Moreover, the epimorphism is an isomorphism; for if $\rho(a) = \iota$, the identity mapping of K, then for all s, h, and hence for some s, h,

$$(sha)^\sigma = s, \qquad (sha)^{-\sigma+1} = h,$$

thus $sha = sh$ and $a = 1$. Similarly the $\rho(b)$, with b ranging over B, form a permutation group $\rho(B)$ of K isomorphic to B.

If t is thought of as fixed, and if s and h are thought of as giving an element of A, namely sh, then $\rho(A)$ can be interpreted as the regular permutation representation of A by right multiplications; on the set of all triplets (s, t, h) we obtain this permutation representation repeated for each $t \in T$. Correspondingly $\rho(B)$ acts as the regular permutation representation of B by right multiplications on each set of (s, t, h) with fixed s.

The intersection of $\rho(A)$ and $\rho(B)$ is $\rho(H)$; for if $\rho(a) \in \rho(B)$ then $\rho(a)$ leaves the first component s of each triplet (s, t, h) fixed: thus

$$(sha)^\sigma = s, \qquad (sha)^{-\sigma+1} = ha \in H,$$

whence $a \in H$. Thus the permutation group P of K generated by $\rho(A)$ and $\rho(B)$ contains isomorphic copies of A and B (namely $\rho(A)$ and $\rho(B)$) intersecting in what corresponds to H under these isomorphisms (namely $\rho(H)$), and so embeds the amalgam. We shall call this group P a *permutational product* of A and B with the amalgamated subgroup H, or of the amalgam. It should, however, be borne in mind that P depends not only on the amalgam but also on the choice of the transversals S and T of H in A and B, respectively.

4. Change of the transversals

In this section we digress to glance briefly at the dependence of the permutational product on the choice of transversals.

Let us first assume that H is central in both A and B. Take arbitrary elements $a \in A$ and $b \in B$ and consider the permutation $\rho(a)\rho(b)$ of K. First we note that

$$(sha)^\sigma = (sah)^\sigma = (sa)^\sigma, \qquad (sha)^{-\sigma+1} = (sa)^{-\sigma+1}h,$$

and

$$(thb)^\tau = (tbh)^\tau = (tb)^\tau, \qquad (thb)^{-\tau+1} = (tb)^{-\tau+1}h,$$

as h is central, and as a factor in H does not affect the coset, nor therefore its representative. Hence

$$\begin{aligned}(s, t, h)^{\rho(a)\rho(b)} &= \big((sha)^\sigma, t, (sha)^{-\sigma+1}\big)^{\rho(b)}\\ &= \big((sa)^\sigma, t, (sa)^{-\sigma+1}h\big)^{\rho(b)}\\ &= \big((sa)^\sigma, (t(sa)^{-\sigma+1}hb)^\tau, (t(sa)^{-\sigma+1}hb)^{-\tau+1}\big)\\ &= \big((sa)^\sigma, (tb)^\tau, (tb)^{-\tau+1}(sa)^{-\sigma+1}h\big)\\ &= \big((sa)^\sigma, (tb)^\tau, (sa)^{-\sigma+1}(tb)^{-\tau+1}h\big) = (s, t, h)^{\rho(b)\rho(a)}.\end{aligned}$$

Thus $\rho(a)\rho(b)$ and $\rho(b)\rho(a)$ have the same effect on every triplet (s, t, h), and so we have

$$\rho(a)\,\rho(b) = \rho(b)\,\rho(a).$$

But this means that $\rho(A)$ and $\rho(B)$ centralize each other in P. Now the only group embedding an amalgam and generated by it, in which the constituents centralize each other — which is possible only if their intersection is central in both — is the generalized direct product of the amalgam; for this is the "freest" group with the stated properties, but every non-trivial normal subgroup contains a non-trivial element of one of the constituents or a quotient of two elements, one from each constituent but outside the amalgamated subgroup, simply because every non-trivial element is of one of these forms: thus no proper homomorphic image embeds the amalgam. We have therefore proved the following result.

THEOREM 1. *If the amalgamated subgroup H is central in the constituents A and B, then the permutational product is the direct product of A and B amalgamating H.*

In particular we see that in this case the structure of the permutational product is independent of the transversals S and T that go into its construction. This fact is also a corollary of the following theorem.

THEOREM 2. *Let the amalgamated subgroup H be central in one of the constituents, say in A. Then the isomorphism type of the permutational product is independent of the transversal T of H in the other constituent, B.*

PROOF. Let T, T' be two left transversals of H in B; if the factorizations of an element $b \in B$ are

$$b = th = t'h', \qquad t \in T, \quad t' \in T', \quad h, h' \in H,$$

we put $t = b^{\tau}$, $h = b^{-\tau+1}$, $t' = b^{\tau'}$, $h' = b^{-\tau'+1}$. As before, K is the set of triplets (s, t, h) with $s \in S$, $t \in T$, $h \in H$; and we denote by K' the set of triplets (s, t', h) with $t' \in T'$. The permutations $\rho(a)$, $\rho(b)$ of K are as before, and they generate the permutational product P; the permutations $\rho'(a)$, $\rho'(b)$ of K' are analogously defined, and they generate the permutational product P'. We define a mapping φ of K onto K' by

$$(s, t, h)^{\varphi} = \big(s, (th)^{\tau'}, (th)^{-\tau'+1}\big).$$

This is clearly one-to-one, and its inverse is given by

$$(s, t', h')^{\varphi^{-1}} = \big(s, (t'h')^{\tau}, (t'h')^{-\tau+1}\big).$$

Differently put, $(s, t, h) \in K$ and $(s', t', h') \in K'$ correspond to each other if, and only if,

$$s = s', \qquad th = t'h'.$$

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With this notation, we compare $\varphi^{-1}\rho(a)\varphi$ with $\rho'(a)$ and $\varphi^{-1}\rho(b)\varphi$ with $\rho'(b)$:

$$\begin{aligned}(s', t', h')^{\varphi^{-1}\rho(a)\varphi} &= (s, t, h)^{\rho(a)\varphi}\\ &= \big((sha)^{\sigma}, t, (sha)^{-\sigma+1}\big)^{\varphi}\\ &= \big((sa)^{\sigma}, t, (sa)^{-\sigma+1}h\big)^{\varphi}\\ &= \big((sa)^{\sigma}, t', (sa)^{-\sigma+1}h'\big),\end{aligned}$$

because

$$t(sa)^{-\sigma+1}h = th(sa)^{-\sigma+1} = t'h'(sa)^{-\sigma+1} = t'(sa)^{-\sigma+1}h'.$$

$$\begin{aligned}(s', t', h')^{\rho'(a)} &= \big((s'h'a)^{\sigma}, t', (s'h'a)^{-\sigma+1}\big)\\ &= \big((s'a)^{\sigma}, t', (s'a)^{-\sigma+1}h'\big) = (s', t', h')^{\varphi^{-1}\rho(a)\varphi},\end{aligned}$$

as $s' = s$. As this is true for all $(s', t', h') \in K'$, we have

$$\varphi^{-1}\rho(a)\varphi = \rho'(a).$$

Next

$$\begin{aligned}(s', t', h')^{\varphi^{-1}\rho(b)\varphi} &= (s, t, h)^{\rho(b)\varphi}\\ &= \big(s, (thb)^{\tau}, (thb)^{-\tau+1}\big)^{\varphi}\\ &= \big(s, (thb)^{\tau'}, (thb)^{-\tau'+1}\big)\\ &= \big(s', (t'h'b)^{\tau'}, (t'h'b)^{-\tau'+1}\big) = (s', t', h')^{\rho(b')},\end{aligned}$$

showing that also

$$\varphi^{-1}\rho(b)\varphi = \rho'(b).$$

Thus φ transforms P into P', and the theorem follows.

On the other hand the permutational product may well be affected by a change in the transversal of H in A. Let A be the abelian group of order 8, type (4, 2), given by

$$A = \mathrm{gp}(a, b, c;\, a^2 = c,\, b^2 = c^2 = [a, b] = 1),$$

and B the non-abelian group of order 6, given by

$$B = \mathrm{gp}(c, d;\, c^2 = (cd)^2 = d^3 = 1),$$

and $H = \mathrm{gp}(c)$, cyclic of order 2. Choose $S = \{1, a, b, ab\}$, $T = \{1, d, d^2\}$; then the permutational product P will be a group of order 72. If instead we choose $S' = \{c, a, b, ab\}$ and T as before (we know that the choice of T is immaterial), then we obtain a permutational product P' of order 648. The verification is computational, and we omit it.

If H is not central in either group, the effect of a change of transversals can be even more drastic. Taking A and B both isomorphic to S_3, the symmetric group of degree 3, and H as cyclic of order 2, one can make a permutational product of order 18 by one choice of transversals, and a permutational product of order 9! by a different choice. In the latter case P is not even soluble.

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More precisely, as H has index 3 in both A and B, and as H, and thus every coset of H, has order 2, there are $2^3 = 8$ choices of transversal of H in A and in B, making 64 pairs of such transversals; thus on the face of it there might be 64 different permutational products of this amalgam. However, it turns out that there are only 3 distinct groups that arise. Calling the transversal consisting of the elements of the cyclic group of order 3, and the transversal consisting of the other 3 elements (namely, those of order 2) "pure", and all other transversals "mixed", the permutational product formed with both transversals pure is a group P_{pp} of order 18, namely the splitting extension of the elementary abelian group of order 9 by the involutory automorphism inverting all elements; with one pure and one mixed transversal one obtains a permutational product P_{pm} of order 162, namely the splitting extension of the Sylow 3-subgroup of the symmetric group S_9 by an involutory automorphism which inverts the two generators of order 3 in the well-known representation of the Sylow subgroup as the wreath product of two cyclic groups of order 3; finally, with both transversals mixed, the permutational product P_{mm} is the direct product of the alternating group A_9 and a cyclic group of order 2. Again the verification is computational, and we omit it.

5. The main theorem

From now on we take the transversals S and T as arbitrarily chosen, but fixed. We had seen that the effect of $\rho(b)$ on the triplets (s, t, h) with fixed s is that of a right multiplication (of $th \in B$) by b. We now generalize these permutations by considering mappings that again permute the triplets (s, t, h) with fixed s among themselves according to a right multiplication, but now by an element of B that may vary with s.

DEFINITION. The mapping γ of the set K of triplets (s, t, h) into itself is a *quasi-multiplication* if there is a function f on S to B such that

$$(s, t, h)^\gamma = (s, t', h')$$

with

$$t'h' = thf(s).$$

There is, of course, a dual notion with A and T in place of B and S; hence we should, strictly, give γ a name that indicates the unsymmetrical rôle played in it by B and S, such as "quasi-B-S-multiplication". However, the dual concept will not be required here, and we leave the asymmetry unindicated.

There is clearly a one-to-one correspondence between the quasi-multiplications and the functions on S to B, and we shall denote the quasi-multiplication that belongs to the function f by $\gamma(f)$. The functions f form a group

B^S, called a "cartesian power" of B, if we introduce the obvious "component-wise" multiplication: thus $ff' = f''$ means

$$f(s)f'(s) = f''(s)$$

for all $s \in S$. The "constant" functions f_b, defined by $f_b(s) = b$ for all $s \in S$, form a subgroup, called the "diagonal" of B^S; this is clearly isomorphic to B.

LEMMA 1. *The quasi-multiplications form a group Γ isomorphic to B^S; this contains the isomorphic copy $\rho(B)$ of B in P, namely as the subgroup that corresponds to the diagonal of B^S.*

We have already seen that to each $f \in B^S$ there is a unique quasi-multiplication $\gamma = \gamma(f) \in \Gamma$, and conversely. It remains to show that if $ff' = f''$ then $\gamma(f)\gamma(f') = \gamma(f'')$. Now

$$(s, t, h)^{\gamma(f)\gamma(f')} = (s, t', h')^{\gamma(f')} = (s, t'', h''),$$

where $t'h' = thf(s)$ and $t''h'' = t'h'f'(s)$; thus

$$t''h'' = thf(s)f'(s) = thf''(s),$$

whence

$$(s, t, h)^{\gamma(f)\gamma(f')} = (s, t, h)^{\gamma(f'')}.$$

As this is true for all $(s, t, h) \in K$, we have

$$\gamma(f)\gamma(f') = \gamma(f'')$$

as required. The correspondence of $\rho(B)$ and the diagonal is obvious.

It should be noted that in general not the whole of Γ will belong to P; indeed if A and B are countable groups, then P is countably generated, hence countable, whereas Γ may easily be uncountable (namely if S is infinite and B non-trivial).

Now we assume that H is central in A; and we compute

$$\gamma' = \rho(a)^{-1}\gamma\rho(a) = \rho(a^{-1})\gamma\rho(a),$$

where $\gamma = \gamma(f) \in \Gamma$ and $a \in A$. Put

$$(s, t, h)^{\rho(a^{-1})} = (s_1, t_1, h_1),$$
$$(s_1, t_1, h_1)^{\gamma} = (s_2, t_2, h_2),$$
$$(s_2, t_2, h_2)^{\rho(a)} = (s', t', h'),$$

so that

$$(s, t, h)^{\gamma'} = (s', t', h').$$

Then

$$s_1h_1 = sha^{-1}, \qquad t_1 = t,$$
$$s_2 = s_1, \qquad t_2h_2 = t_1h_1f(s_1),$$
$$s'h' = s_2h_2a, \qquad t' = t_2.$$

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Then, substituting step by step,

$$s' = s_2h_2ah'^{-1} = s_1h_2ah'^{-1} = sha^{-1}h_1^{-1}h_2ah'^{-1}$$
$$= shh_1^{-1}h_2h'^{-1},$$

as a permutes with $h_1^{-1}h_2$. Now

$$s' = shh_1^{-1}h_2h'^{-1}$$

implies

$$s' = s, \qquad hh_1^{-1}h_2h'^{-1} = 1.$$

We use this last equation and again substitute step by step, using in between that H is abelian:

$$t'h' = t'hh_1^{-1}h_2 = t_2hh_1^{-1}h_2 = t_2h_2hh_1^{-1}$$
$$= t_1h_1f(s_1)hh_1^{-1} = th_1f(s_1)hh_1^{-1}$$
$$= th(hh_1^{-1})^{-1}f(s_1)hh_1^{-1} = thc,$$

say, where c is an element of B that on the face of it depends on s_1, h, and h_1. However,

$$s_1 = (sha^{-1})^\sigma = (sa^{-1}h)^\sigma = (sa^{-1})^\sigma$$

does not depend on h, but only on s (and, of course, on a — but this has nothing to do with the particular triplet (s, t, h) under consideration); and

$$hh_1^{-1} = s^{-1}s_1a = s^{-1}(sa^{-1})^\sigma a$$

also depends on s (and a) only, not on h. Thus, if we define the elements f_1, g, f' of B^S by

(1) $$f_1(s) = f(s_1) = f((sa^{-1})^\sigma),$$

(2) $$g(s) = hh_1^{-1} = s^{-1}(sa^{-1})^\sigma a,$$

$$f'(s) = (hh_1^{-1})^{-1}f(s_1)hh_1^{-1} = g(s)^{-1}f_1(s)g(s)$$

for all $s \in S$, we see that $f' = g^{-1}f_1g$, and

$$\gamma' = \gamma(f') = \gamma(g^{-1}f_1g);$$

and we note incidentally that g lies in the subgroup H^S of B^S. To summarize, we have shown:

LEMMA 2. *If H is central in A, then $\rho(A)$ normalizes Γ; more precisely, to every $a \in A$ and $\gamma = \gamma(f) \in \Gamma$ there is an element $\gamma' = \gamma(f') \in \Gamma$ such that*

$$\rho(a)^{-1}\gamma(f)\rho(a) = \gamma(f').$$

Here $f' \in B^S$ is given by

$$f' = g^{-1}f_1g,$$

where $f_1 \in B^S$ and $g \in H^S$ are defined by equations (1), (2).

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COROLLARY 1. *If H is central in A then*

$$[\rho(A), \Gamma] \leqq \Gamma.$$

As usual, $[X, Y]$ denotes the "cross-commutator" group generated by all $[x, y] = x^{-1}y^{-1}xy$, where $x \in X$, $y \in Y$. The corollary is immediate from the fact that $\rho(A)$ normalizes Γ.

COROLLARY 2. *If H is central in A then*

$$[\rho(A), \Gamma'] \leqq \Gamma'.$$

As usual, $\Gamma' = [\Gamma, \Gamma]$ is the derived group of Γ. As it is characteristic in Γ, it is also normalized by $\rho(A)$, and the corollary follows as before. Corresponding inclusions are valid for all characteristic subgroups of Γ, but will not be required.

COROLLARY 3. *If H is central in A then*

$$[\rho(A), \rho(B)] \leqq \Gamma'.$$

We apply the lemma to $\gamma = \rho(b) \in \rho(B)$; this means that $f = f_b$ is a constant function, and thus $f_1 = f$. Then

$$\rho(a)^{-1}\rho(b)\rho(a) = \gamma(g^{-1}fg) = \gamma(g)^{-1}\rho(b)\gamma(g),$$

and

$$[\rho(a), \rho(b)] = [\gamma(g), \gamma(f)] \in \Gamma'.$$

THEOREM 3. *Let the amalgamated subgroup H be a proper subgroup of both constituents (the amalgam is then called "proper"), and let H be central in one of the constituents. Then the generalized free product F of the amalgam is not simple.*

PROOF. As the permutational product P is generated by $\rho(A)$ and $\rho(B)$ and as $\rho(B)$ is a subgroup of Γ, which by Lemma 2 is normalized by $\rho(A)$, the subgroup $P \cap \Gamma$ of P is normal in P. One easily sees that $\Gamma \cap \rho(A) = \rho(H)$; it follows that

$$P/(P \cap \Gamma) \cong A/H,$$

which is a non-trivial group. On the other hand $P \cap \Gamma$ is not trivial, as it contains $\rho(B)$. Hence P is not simple; and as P is a homomorphic image of F, the theorem follows.

We now turn to the main theorem announced in the introduction.

THEOREM 4. *Let the amalgamated subgroup H be central in one of the constituents, A; let both constituents A and B be soluble, of lengths $l(A)$, $l(B)$, respectively. Then the permutational product P is soluble, and its length $l(P)$ satisfies the inequality*

$$l(P) \leqq l(A) + l(B) - 1.$$

PROOF. As P is generated by $\rho(A)$ and $\rho(B)$, its derived group is

$$P' = \rho(A')\rho(B')[\rho(A), \rho(B)] \leqq \rho(A')\Gamma',$$

using the trivial inclusion $\rho(B') \leqq \Gamma'$ and Corollary 3. Next

$$P'' \leqq \rho(A'')\Gamma''[\rho(A'), \Gamma'] \leqq \rho(A'')\Gamma',$$

by Corollary 2; and continuing thus,

$$P^{(m)} \leqq \rho(A^{(m)})\Gamma'.$$

When in particular $m = l(A)$, the first factor on the right becomes trivial, and

$$P^{(l(A))} \leqq \Gamma'.$$

Now Γ is isomorphic to a cartesian power of B, hence, like B, it is soluble of length $l(B)$. Thus

$$P^{(l(A)+l(B)-1)} \leqq \Gamma^{(l(B))}.$$

This is the trivial group, and the theorem follows.

It is not difficult to make examples to show that the inequality for $l(P)$ cannot in general be improved.

COROLLARY 4. *Every amalgam of an abelian group A and a soluble group B can be embedded in a soluble group P of length $l(P) = l(B)$.*

COROLLARY 5. *If A and B are non-trivial and soluble and H central in A, then the derived series of the generalized free product F of the amalgam descends properly for at least $l(A) + l(B) - 1$ steps.*

It would be interesting to know whether a generalized free product of two soluble groups can be simple when the amalgamated subgroup is not central in either.

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