ON A RELATIONSHIP BETWEEN MAGNETOHYDRODYNAMIC PLANETARY EIGENMODES AND SECOND-CLASS INERTIAL EIGENMODES

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Abstract. Stewartson [5] considered second class oscillations in a spherical shell in the presence of a toroidal magnetic field. He followed Hide [2] and supposed the toroidal field to be uniform.

The aims of this paper are twofold: First, we wish to subject the 'thin shell' analysis of Stewartson to a more general toroidal field and in particular illustrate the "acceptability" of Malkus' [4] choice of a uniform current parallel to the axis of rotation as an alternative basic state. Second, we show that for a thick shell the full magnetohydrodynamic equations of motion (for oscillations of the second class) may be reduced to the same equations as govern the motion of an incompressible, inviscid fluid in the absence of a magnetic field.

A formula connecting the magnetohydrodynamic planetary eigenvalues with the second class inertial eigenvalues is obtained, and its usefulness discussed.

1. Introduction

Hide [2] has suggested that if the strength of the toroidal magnetic field in the Earth's core is 100 oersted, then many of the properties of the observed secular changes, including the slow westward drift of the non-dipole components of the Earth's magnetic field, can be accounted for in terms of the interaction of magnetic modes in the core with the Earth's poloidal magnetic field. In order to analyse this suggestion, Stewartson [5] considered second class oscillations in a spherical shell in the presence of a toroidal field. He followed Hide and supposed the toroidal field to be uniform. In his study of the hydromagnetic oscillations in a rotating sphere, Malkus [4] chose a uniform current parallel to the axis of rotation as a basic state.

The aims of this paper are twofold: first, we wish to subject the 'thin shell' analysis of Stewartson to a more general toroidal field (see next section) and in particular to illustrate that Malkus' choice of a uniform current parallel to the axis of rotation as a basic state is as "acceptable" as any other reasonably simple basic state in the light of knowledge at present available. Second, in §3 we

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consider the full magnetohydrodynamic equations of motion (retaining the dependence on the radial coordinate R) when the basic state is that of a uniform current parallel to the axis of rotation. We show that, by means of suitable transformations, the governing equations (for oscillations of the second class) may be reduced to the equations of motion of an inviscid, incompressible fluid in the absence of a magnetic field (see, e.g., Stewartson and Rickard [6]).

The second result (above) is both important and useful. If we accept that a constant current parallel to the axis of rotation generates a suitable toroidal field, then the result enables us to "disregard" the magnetic field completely in our analysis. We shall derive a formula connecting the magnetohydrodynamic planetary eigenmodes with the second class inertial eigenmodes calculated in Stewartson and Rickard's [6] paper.

2. Oscillations in a thin shell

Consider a thin shell of perfectly conducting incompressible inviscid fluid bounded by two rigid concentric spheres and rotating with angular velocity $\mathbf{\Omega}$ about an axis Oz where O is the common centre of the spheres. In addition a toroidal magnetic field is imposed. Stewartson [5] considered a field in which the lines of force were circles having Oz as axis, while the magnitude of the field was constant and equal to H_0. In this section we consider the more general case in which

$$H_0 = f(\theta), \tag{2.1}$$

θ being the angle between the position vector $\boldsymbol{R}$ of a representative point S in the fluid and Oz, and $f(\theta)$ an arbitrary function of θ. We shall follow Stewartson and assume the current lines to be completed with current sheets along the inner and outer boundaries; the conditions on the fluid are then self consistent in that they satisfy the governing equations of motion and of the magnetic field. The boundary conditions are also satisfied if the region external to the fluid is non-conducting.

A small disturbance is now given to the steady motion and we wish to determine the periods of free oscillation of the fluid. Take the radius of the shell to be a, the permeability of the fluid to be unity and neglect radial motions. Further measure the fluid velocity $\boldsymbol{u}$ relative to a set of axes rotating about Oz with angular velocity $\mathbf{\Omega}$ and denote its components (O, u_θ, u_ϕ) in spherical polar co-ordinates (a, θ, ϕ), where θ is defined above and ϕ is the angle between the planes OSz and a plane through Oz fixed relative to the rotating axes. From the equation of continuity

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2.2}$$

it follows that

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$$\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) + \frac{\partial}{\partial\phi}(u_\phi) = 0. \tag{2.3}$$

A consistent formal solution of the entire problem can be found by separating variables, i.e. by writing a typical dependent variable Q in the form

$$Q = \mathscr{R}\ell(q(\theta)e^{im\phi+i\omega t}), \tag{2.4}$$

where m is an integer, which may be either positive or negative, ω is a constant to be found, and $q(\theta)$ is a function of θ only. From now on we shall omit the exponential factors and it is understood that the real part is taken. It then follows from (2.3) that there exists a function Ψ such that

$$u_\theta = \mathrm{im}\,\Psi, \qquad u_\phi = -\frac{\partial}{\partial\theta}(\sin\theta\Psi). \tag{2.5}$$

Further, on substituting (2.5) into Maxwell's equations, but neglecting the displacement current, i.e. into

$$\frac{\partial \boldsymbol{H}}{\partial t} = \mathrm{curl}(\boldsymbol{u}\times\boldsymbol{H}), \tag{2.6}$$

where $\boldsymbol{H}$ is the total magnetic field, we obtain for the components of the perturbed magnetic field

$$h_R = 0, \qquad h_\theta = \frac{\mathrm{im}^2\Psi f}{a\omega\sin\theta}, \qquad h_\phi = -\frac{m\partial/\partial\theta(\Psi f)}{a\omega}, \tag{2.7}$$

squares and products of small terms being neglected.

The equation

$$\mathrm{div}\,\boldsymbol{H} = 0 \tag{2.8}$$

is automatically satisfied by (2.7). Finally, we substitute into the equations of momentum

$$\frac{\partial\boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega}\times\boldsymbol{u} + (\boldsymbol{u}.\mathrm{grad})\boldsymbol{u} = -\frac{1}{\rho}\mathrm{grad}\,p + \frac{1}{4\pi\rho}(\mathrm{curl}\,\boldsymbol{H})\times\boldsymbol{H} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{R}). \tag{2.9}$$

where $\boldsymbol{\Omega}$ is the vector angular velocity of the axes, p the pressure and ρ the density of the fluid. If we retain linear terms only (2.9) reduces to

$$\begin{aligned} i\omega u_\theta - 2\Omega u_\phi\cos\theta = -\frac{1}{\rho a}\frac{\partial P}{\partial\theta} + \frac{m}{4\pi\rho\omega a^2}\Bigg[&\frac{\partial/\partial\theta(\Psi f)\partial/\partial\theta(\sin\theta f)}{\sin\theta} \\ &+ \frac{f\partial/\partial\theta(\sin\theta\,\partial/\partial\theta(\Psi f))}{\sin\theta} - \frac{m^2 f^2\Psi}{\sin^2\theta}\Bigg], \end{aligned} \tag{2.10}$$

for the θ component and

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$$\text{(2.11)} \qquad i\omega u_\phi + 2\Omega u_\theta \cos\theta = -\frac{\mathrm{im}\,P}{a\rho\sin\theta} + \frac{\mathrm{im}^2\Psi f\partial/\partial\theta(\sin\theta f)}{4\pi\rho\omega a^2\sin^2\theta},$$

for the ϕ component, where

$$\text{(2.12)} \qquad P = p - \tfrac{1}{2}\rho\Omega^2 R^2\sin^2\theta + \frac{1}{4\pi}\int\left(f^2\cot\theta + f\frac{df}{d\theta}\right)d\theta,$$

is small, and of the form (2.4). There is also an equation of motion in the radial direction but this leads to pressure changes which are either functions of R only or of second order.

It is now possible to eliminate P from equations (2.10) and (2.11) to obtain a second order ordinary differential equation for Ψ from which the eigenvalues of ω leading to an acceptable solution for Ψ can be obtained. We are particularly interested in these eigenvalues when

$$\text{(2.13)} \qquad \frac{(H_{0\max})^2}{4\pi\rho} \ll a^2\Omega^2,$$

where $H_{0\max}$ denotes the maximum strength of the toroidal field H_0 (see (2.1)). It can easily be seen from an examination of (2.9) that there are two kinds of eigenvalues satisfying (2.13). First, ω can be of the order of Ω in which case the magnetic terms may be neglected and the usual equations for inertial waves in a rotating system obtained. As shown by Longuet-Higgins (1964) it may then be established that

$$\text{(2.14)} \qquad \omega = \frac{2\Omega m}{n(n+1)},$$

where n is an integer. Suppose $m = 1$. In the case $n = 1$, the angular velocity of the current-system is just Ω, so the system is stationary relative to fixed axes. However, when $n > 1$ the angular velocity in the rotating frame is always less than Ω, so that the system tends to be carried round with the rotating spheres. On the other hand, if viewed by an observer on the rotating globe the current-system tends to 'follow the sun', that is, to drift westward. Similar remarks apply when $m > 1$. For $m \leqq n$ these waves all have a period greater than one day (except $m = n = 1$) and rotate westwards with respect to our axes.

The second type of waves have the property that $\omega \ll \Omega$ and are found by setting $\omega = 0$ in the left hand sides of (2.10) and (2.11). These waves are the 'planetary oscillations' or 'oscillations of the second class'. Eliminating P from (2.10) and (2.11), we find that for these oscillations

$$\text{(2.15)} \qquad f(\theta)^2\frac{d^2\Psi}{d\theta^2} + f(\theta)\left(2\frac{df(\theta)}{d\theta} + f(\theta)\cot\theta\right)\frac{d\Psi}{d\theta} + (f(\theta)^2(1-m^2)\operatorname{cosec}^2\theta + a^*\sin^2\theta)\Psi = 0$$

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where

$$a^* = -\frac{8\Omega\pi\rho\omega a^2}{m}. \tag{2.16}$$

In the case of a constant toroidal field, $f(\theta) = H_0 =$ constant, and (2.15) reduces to

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Psi}{d\mu}\right] + \left[\frac{1-m^2}{1-\mu^2} + a^2\alpha(1-\mu^2)\right]\Psi = 0, \tag{2.17}$$

where

$$\mu = \cos\theta \text{ and } \alpha = -\frac{8\Omega\pi\rho\omega}{mH_0^2}. \tag{2.18}$$

The eigenvalues of (2.17) are discussed in detail in Stewartson's [5] paper.

Suppose now that $f(\theta)$ may be written in the form

$$f(\theta) = K \sin^b\theta \cos^c\theta, \tag{2.19}$$

where K, b, c are constants. On substituting (2.19) into (2.15) the latter reduces to

$$\begin{aligned}\mu(1-\mu^2)\frac{d^2\Psi}{d\mu^2} &- 2[(b+c+1)\mu^2 - c]\frac{d\Psi}{d\mu} \\ &+ \left[\frac{\alpha^*}{K^2}(1-\mu^2)^{1-b}\mu^{1-2c} + \frac{\mu(1-m^2)}{1-\mu^2}\right]\Psi = 0.\end{aligned} \tag{2.20}$$

In considering the slow oscillations of fluid in a rotating sphere, Malkus [4] assumed that the toroidal field was due to a uniform current parallel to the axis of rotation. It is clear that, since

$$\boldsymbol{H}_0 = \boldsymbol{j}_0 \times \boldsymbol{R}, \tag{2.21}$$

where $\boldsymbol{j}_0$ is the uniform current, we must take $K = aj_0$, $b = 1$, $c = 0$ in (2.19) in this case, j_0 being the magnitude of the current. Equation (2.20) now reduces to

$$(1-\mu^2)\frac{d^2\Psi}{d\mu^2} - 4\mu\frac{d\Psi}{d\mu} + \left(\bar{\alpha} + \frac{1-m^2}{1-\mu^2}\right)\Psi = 0, \tag{2.22}$$

where

$$\bar{\alpha} = -\frac{8\Omega\pi\rho\omega}{mj_0^2}. \tag{2.23}$$

In order that the perturbed components of velocity and magnetic field be finite everywhere we require that Ψ is bounded as $\sin\theta \to 0$, or $\mu \to \pm 1$ (see

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equations (2.5) and (2.7)). From the theory of differential equations Ψ is bounded as $\mu \to \pm 1$ only for a discrete set of values of $\bar{\alpha}$, which are the eigenvalues we wish to determine. To find the eigenvalues, proceed as follows:

Consider the Associated Legendre Equation

$$(2.24) \qquad (1-\mu^2)\frac{d^2w}{d\mu^2} - 2\mu\frac{dw}{d\mu} + \left[v(v+1) - \frac{m^2}{1-\mu^2}\right]w = 0,$$

where v is an integer. If we write

$$(2.25) \qquad w = (1-\mu^2)^{\frac{1}{2}}\Phi,$$

(2.24) becomes

$$(2.26) \qquad (1-\mu^2)\frac{d^2\Phi}{d\mu^2} - 4\mu\frac{d\Phi}{d\mu} + \left[(v+2)(v-1) + \frac{1-m^2}{1-\mu^2}\right]\Phi = 0,$$

and it follows on comparing (2.26) and (2.22) that

$$(2.27) \qquad \bar{\alpha} = n(n+3),$$

where n is an integer. It is of interest to observe that $\bar{\alpha}$ is independent of the azimuthal wavenumber m, resulting in there being only a single infinity of eigenvalues rather than the double infinity of such values one would expect.

From (2.27) we see that the lowest non-zero eigenmode is $\bar{\alpha} = 4$ (when $n = 1$). In comparison, for the constant toroidal field case (see (2.17), (2.18) and Stewartson, [5], p. 182), the lowest eigenmode is $\alpha a^2 = 4.62$, with azimuthal wave number $m = 1$. It follows from (2.18), that, in the constant toroidal field case, the maximum value of the field strength, $H_{0\max}$, is proportional to the reciprocal of the square root of $(m\alpha)_s$, that is,

$$(2.28) \qquad H_{0\max} = \frac{A}{\sqrt{(m\alpha)_s}},$$

$(m\alpha)_s$ being the smallest value of $(m\alpha)$, and

$$(2.29) \qquad A = -8\Omega\pi\rho\omega.$$

As a consequence of the relationship between j_0 and H_0 in (2.21) essentially the same remarks apply to (2.23) as (2.18). A knowledge of A and a would enable us to evaluate the maximum value of the field strength in the two cases under discussion. Ω is approximately 7×10^{-5} radians/second and for the Earth's core ρ is generally taken as 10 gms./cm.^{3}. The mean radius of the core is approximately 3,500 km. (see Hide, [2]). However the precise value of ω, the 'westward drift' rate of the spherical harmonic components, is uncertain (Hide, p. 640) and here we shall be content to compute

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$$
(2.30) \qquad
\begin{aligned}
\hat{H}_0 &= 100\left[1 - 2\frac{H_{01\max} - H_{02\max}}{H_{01\max} + H_{02\max}}\right] \\
&= 100\left[1 - 2\frac{\sqrt{(m\alpha)_{1s}} - \sqrt{(m\alpha)_{2s}}}{\sqrt{(m\alpha)_{1s}} + \sqrt{(m\alpha)_{2s}}}\right],
\end{aligned}
$$

where $H_{01\max}$ and $H_{02\max}$ are the maximum field strengths' predicted by the 'constant field' and 'constant current' cases respectively, and $(m\alpha)_{1s}$, $(m\alpha)_{2s}$ are the associated values of $(m\alpha)_s$. It will be recalled that Stewartson [5] has suggested that Hide's theory of hydromagnetic oscillations in the Earth's core being responsible for the westward drift of the geomagnetic secular variation is untenable, and that instead the theory should be used to put an upper bound on the possible strength of the toroidal field. In this latter respect $\hat{H}_0$ is a measure of how close theories based on 'constant field' and 'constant current' are in agreement; the factor 100 means that it is expressed as a 'percentage'. It is easy to compute $\hat{H}_0$ from (2.30) and we find that it represents an agreement of slightly better than 92%.

It was hoped to be able to compute the eigenvalues of (2.20) for other $f(\theta)$, in particular some simple functions of θ of the form (2.19), (e.g. $f(\theta) = K \cos \theta$, $f(\theta) = K \cos^2\theta$, $f(\theta) = K \sin \theta \cos \theta$). In all of these cases it is a simple matter to substitute the appropriate values of K, b, c and obtain a second order differential equation analogous to (2.17) and (2.22). However, if on physical grounds we require the perturbed components of velocity and magnetic field to be finite everywhere (see (2.5) and (2.7)) then the results of a preliminary analysis suggest that no other $f(\theta)$ of the form (2.19) exist which allow the determination of a set of eigenvalues consistent with this requirement. If we ease the condition on the perturbed magnetic field and stipulate only that we desire the perturbed velocity components to be finite everywhere the above remarks still apply. This is unfortunate because it would be helpful to compute the eigenvalues and maximum field strengths' (or a relative guide to such in the form of a parameter like $\hat{H}_0$) in other cases. It may be that, except for the 'constant field' and 'constant current' cases considered, (2.19) represents an unacceptable physical situation. It may be possible to overcome this difficulty by considering $f(\theta)$ to consist of a Fourier Series in $\sin \theta$ or $\cos \theta$ but this suggestion has not been examined in detail.

The purpose of this section will have been achieved if the reader agrees that the basic state chosen by Malkus [4], that of a uniform current parallel to the axis of rotation, is not unrepresentative. In view of the remarks made by Hide (see Hide, [2], p. 627) it would appear to have some advantages over the choice of a constant toroidal field.

3. The full magnetohydrodynamical equations

In this section we shall show that when the basic state is a constant current parallel to the axis of rotation, the full magnetohydrodynamic equations (for planetary oscillations, i.e. when (2.13) holds and $\omega \ll \Omega$) can be reduced to the usual equations of motion of a rotating fluid in the absence of a magnetic field by suitable transformations.

We shall again choose a set of spherical polar coordinates (R, θ, ϕ) and suppose that they are rotating about the axis Oz with angular velocity Ω. Let a small disturbance be applied to the steady state and let the resulting components (h_R, h_θ, h_ϕ), (u_R, u_θ, u_ϕ) of the perturbed magnetic and velocity fields be sufficiently small that products may be neglected. Let also the boundaries of the shell be surfaces of revolution with Oz as axis and let the media beyond the shell have zero conductivity. Further let all small dependent variables be of the form

$$\mathscr{R}\ell\{q(R,\theta)e^{im\phi+i\omega t}\}, \tag{3.1}$$

where m is an integer.

From (2.21) it follows that $\boldsymbol{H}_0 = (0, 0, Rj_0 \sin\theta)$, and therefore

$$\boldsymbol{H} = (h_R,\ h_\theta,\ h_\phi + Rj_0 \sin\theta). \tag{3.2}$$

Hence from the equation of continuity

$$\frac{\partial}{\partial R}(R^2 \sin\theta u_R) + \frac{\partial}{\partial\theta}(R\sin\theta u_\theta) + \mathrm{im}Ru_\phi = 0, \tag{3.3}$$

and from Maxwell's equation (2.6), together with (3.1) and (3.2) where necessary,

$$\boldsymbol{h} = (h_R,\ h_\theta,\ h_\phi) = \frac{mj_0}{\omega}\boldsymbol{u}, \tag{3.4}$$

so that div $\boldsymbol{H}$ is automatically zero. Further we observe that the perturbed magnetic field $\boldsymbol{h}$ is parallel to the perturbed velocity field $\boldsymbol{u}$ in our set of rotating coordinates. Finally, on substituting into (2.9) and writing

$$\frac{j_0^2 P}{4\pi\omega} = p - \tfrac{1}{2}\rho\Omega^2 R^2 \sin^2\theta + \frac{j_0^2 R^2 \sin^2\theta}{4\pi} + \frac{R\sin\theta j_0 h_\phi}{4\pi}, \tag{3.5}$$

where P is of the form (3.1), we obtain

$$\frac{\partial P}{\partial R} = \mathrm{im}^2 u_R - m(\bar{\alpha}+2)\sin\theta\ u_\phi, \tag{3.6a}$$

$$\frac{1}{R}\frac{\partial P}{\partial\theta} = \mathrm{im}^2 u_\theta - m(\bar{\alpha}+2)\cos\theta\ u_\phi, \tag{3.6b}$$

$$\frac{\mathrm{im}\ P}{R\sin\theta} = \mathrm{im}^2 u_R + m(\bar{\alpha}+2)(\sin\theta u_R + \cos\theta u_\theta). \tag{3.6c}$$

In deriving the above equations it has been assumed that

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$$\frac{j_0^2}{4\pi\rho} \ll \Omega^2 \text{ and } \omega \ll \Omega,$$

that is, we are focusing our attention on the planetary waves.

If we now write

$$(3.7) \qquad \begin{aligned} P &= -iR\sin\theta\bar{P}, & u_\theta &= \frac{2\bar{U}}{m(\bar{\alpha}+2)}, \\ u_\phi &= \frac{2i\bar{V}}{m(\bar{\alpha}+2)}, & u_R &= \frac{2\bar{W}}{m(\bar{\alpha}+2)\sqrt{(1-\mu^2)}} \end{aligned}$$

and

$$(3.8) \qquad \bar{\alpha} = \frac{2(m-\bar{\omega})}{\bar{\omega}}$$

the governing equations (3.6a)–(3.6c) reduce to

$$(3.9a) \qquad \frac{\bar{\omega}\bar{W}}{1-\mu^2} - 2\bar{V} = -\left(R\frac{\partial\bar{P}}{\partial R} + \bar{P}\right),$$

$$(3.9b) \qquad \bar{\omega}\bar{U} - 2\mu\bar{V} = (1-\mu^2)\frac{\partial\bar{P}}{\partial\mu} - \mu\bar{P},$$

$$(3.9c) \qquad \bar{\omega}\bar{V} - 2\mu\bar{U} - 2\bar{W} = -m\bar{P}.$$

These equations are identical with the equations of motion of an incompressible inviscid fluid in the absence of a magnetic field (see, e.g., Stewartson and Rickard, [6], p. 762). The 'magnetohydrodynamic planetary eigenmodes' of this paper are related to the inertial eigenmodes of Stewartson and Rickard's paper by equation (3.8). The relationships (3.4), (3.8) were also obtained independently by Malkus [4] in his study of hydromagnetic oscillations in a rotating fluid sphere.

4. Summary

In this section we summarize the results of this paper, clarifying the underlying assumptions and explaining the full significance of (3.8). The magnetohydrodynamic equations appropriate to our problem are (2.2), (2.6), (2.8) and (2.9). In deriving these equations it was assumed that

(i) the displacement current is negligible, a legitimate procedure when the speeds involved are much less than the speed of light;

(ii) the fluid is incompressible, a valid assumption since the speed of flow $\boldsymbol{u}$ is much less than the speed of sound in the Earth's core;

(iii) the fluid is perfectly conducting;

(iv) the fluid may be taken as inviscid, The reader is referred to Hide [1] for a justification of (iii), (iv) and a more complete discussion of (i), (ii).

For reference purposes let us recall below the appropriate equations

$$\frac{\partial \boldsymbol{H}}{\partial t} = \operatorname{curl}(\boldsymbol{u} \times \boldsymbol{H}), \tag{4.1}$$

$$\nabla \cdot \boldsymbol{H} = 0, \tag{4.2}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{4.3}$$

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + (\boldsymbol{u} \cdot \operatorname{grad})\boldsymbol{u} = -\frac{1}{\rho}\operatorname{grad} p + \frac{1}{4\pi\rho}(\operatorname{curl} \boldsymbol{H}) \times \boldsymbol{H} \\ - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{R}). \end{aligned} \tag{4.4}$$

The principal result of this paper may be described as follows. Consider the following separate problems:

(a) The basic state is that of a uniform current $\boldsymbol{j}_0$ parallel to the axis of rotation. The dominant toroidal field is given in terms of $\boldsymbol{j}_0$ by (2.21) and the total field $\boldsymbol{H}$ is given in terms of $\boldsymbol{u}$ by (3.2), (3.4). Further, consider the second class oscillations, for which

$$j_0 \ll 4\pi\rho\Omega^2, \tag{4.5}$$

and

$$\omega \ll \Omega.$$

(b) There is no magnetic field ($\boldsymbol{H} = 0$) and the fluid is no longer conducting.

It has been demonstrated in §3 that by means of the transformation (3.7), (3.8) problem (a) can be reduced to the solution of (3.9a)–(3.9c). In Stewartson and Rickard [6] it is shown that (b) reduces to the solution of the same set of equations. §2 was included primarily to give some justification for the choice of a uniform current parallel to the axis of rotation as basic state.

Appendix A

Explicit expression for α^*

We may rewrite (2.15) as follows:

$$\frac{1}{\sin^3\theta}\frac{d}{d\theta}\left(f(\theta)^2 \sin\theta\frac{d\Psi}{d\theta}\right) + (f(\theta)^2(1-m^2)\operatorname{cosec}^4\theta + \alpha^*)\Psi = 0. \tag{A.1}$$

On multiplying (A.1) by Ψ it follows that

$$\Psi\frac{d}{d\theta}\left(f(\theta)^2 \sin\theta\frac{d\Psi}{d\theta}\right) + (f(\theta)^2(1-m^2)\operatorname{cosec}\theta + \alpha^*\sin^3\theta)\Psi^2 = 0,$$

which may be written in the form

$$\text{(A.2)}\qquad \frac{d}{d\theta}\left(\Psi\frac{d\Psi}{d\theta}f(\theta)^2\sin\theta\right)+(f(\theta)^2(1-m^2)\operatorname{cosec}\theta+\alpha^*\sin^3\theta)\Psi^2 - \left(\frac{d\Psi}{d\theta}\right)^2 f(\theta)^2\sin\theta = 0.$$

We may obtain an explicit expression for α^* by integrating (A.2) w.r.t. θ between the limits 0 and $\pi/2$; thus

$$\text{(A.3)}\qquad \alpha^* = -\frac{\displaystyle\int_0^{\pi/2}\left(\left(\frac{d\Psi}{d\theta}\right)^2 f(\theta)^2\sin\theta-\frac{f(\theta)^2(1-m^2)\Psi^2}{\sin\theta}\right)d\theta}{\displaystyle\int_0^{\pi/2}\sin^3\theta\Psi^2\,d\theta},$$

since

$$\text{(A.4)}\qquad \left[\Psi\frac{d\Psi}{d\theta}f(\theta)^2\sin\theta\right]_0^{\pi/2} = 0.$$

References

[1] R. Hide, *Physics and Chemistry of the Earth* (ed. Ahrens *et al.*) (1, Chapter 5, 91–137 (1956), London, (Pergamon Press)).

[2] R. Hide, 'Free hydromagentic oscillations of the Earth's core and the Geomagnetic Secular Variation', *Phil. Trans. Roy. Soc. A* 259 (1966), 615–650.

[3] M. S. Longuet-Higgins, 'Planetary Waves on a rotating sphere', *Proc. Roy. Soc. A* 279 (1964), 546–473.

[4] W. V. R. Malkus, 'Hydromagnetic Planetary Waves', *J. Fluid Mech.* 28 (1967), 793–802.

[5] K. Stewartson, 'Slow oscillations of fluid in a rotating cavity in the presence of a toroidal magnetic field', *Proc. Roy. Soc. A* 299 (1967), 173–187.

[6] K. Stewartson and J. A. Rickard, 'Pathological oscillations of a rotating fluid', *J. Fluid Mech.* 35 (1969), 759–773.

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