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NON-HYPERUNIFORMITY OF GIBBS POINT PROCESSES WITH SHORT-RANGE INTERACTIONS

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Abstract

We investigate the hyperuniformity of marked Gibbs point processes that have weak dependencies among distant points whilst the interactions of close points are kept arbitrary. Various stability and range assumptions are imposed on the Papangelou intensity in order to prove that the resulting point process is not hyperuniform. The scope of our results covers many frequently used models, including Gibbs point processes with a superstable, lower-regular, integrable pair potential, as well as the Widom–Rowlinson model with random radii and Gibbs point processes with interactions based on Voronoi tessellations and nearest-neighbour graphs.

Keywords: Papangelou intensity; marked point process; density fluctuation; local energy; GNZ equation; Voronoi tessellation; nearest-neighbour graph; Widom–Rowlinson model

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1. Introduction

Point processes form the main building block of stochastic geometry. They serve to construct a broad spectrum of geometric models used for analysing spatial data, for instance in material science, particle physics, telecommunications, and biology. The theory of point processes is now available for very abstract topological settings (see e.g. [5, 26]), but in the current paper, we restrict ourselves to point configurations randomly sprinkled in $\mathbb{R}^d$. This restriction is natural and still allows us to build many random geometric structures by using certain connections among points (e.g. random graphs or tessellations), by adding certain marks to each point (a number, set, etc.), or by a combination of these methods.

A stationary (translation-invariant) point process Γ in $\mathbb{R}^d$ is called *hyperuniform* if it exhibits small density fluctuations. Formally, Γ is hyperuniform if the number of points of the process in a bounded domain $N_\Lambda := N_\Lambda(\Gamma)$ fluctuates at a lower order then the volume of the set, i.e.,

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \frac{\mathbf{Var}(N_\Lambda(\Gamma))}{|\Lambda|} = 0, \qquad (1.1)$$

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where by $\Lambda \nearrow \mathbb{R}^d$ we understand any sequence of increasing sets $\Lambda_1 \subseteq \Lambda_2 \subseteq \ldots$ tending to the whole space $\mathbb{R}^d$. Historically, hyperuniformity is related to the study of the compressibility of matter. In statistical mechanics, compressibility is a measure of the relative volume change of a fluid or solid in response to a pressure, which can be understood as the relative size of fluctuations in particle density. Nonetheless, knowing the order of the variance is of interdisciplinary interest. In material science, the concept of hyperuniformity enables the characterization of naturally organized structures such as crystals and quasicrystals (see [2, 20]). The quantitative characterization of fluctuations in the number of particles has a long history in statistical physics as well. Under certain constraints on the two-point correlation function (in the physics literature referred to as sum rules), the authors of [18] (with an extension in [17]) showed for infinite classical systems of particles with long-range interactions that the variance of N_Λ should increase as the surface area. A rigorous result for one-dimensional Coulomb systems is shown in [15]. In the theory of random matrices, one-dimensional point patterns associated with the eigenvalues have been characterized by their density fluctuations (see e.g. [19]). The measurement of galaxy density fluctuations is a standard approach to studying the structure of the universe (see [21]). The concepts of hyperuniformity and density fluctuations have been identified across many other areas of fundamental science, such as computer science, number theory, and the biological sciences. Yet the theoretical understanding of such systems is still limited. The first attempt to rigorously handle the concept of hyperuniformity in the physics of matter was established in the seminal paper [29] and the subsequent papers of the authors. The current state of the art is summarized in the survey [28].

It seems to be a fundamental question of great practical interest to determine whether a given point process is hyperuniform or not. Unsurprisingly, this question has become a popular topic of study for researchers in stochastic geometry and related fields. In particular, the study of Coulomb and Riesz gases in the context of the theory of point processes is of considerable interest in both mathematics and physics. Some answers have been given for $d = 2, 3$ in [4, 16]. By estimating the structure factor, the authors of [12, 14] provide tests of hyperuniformity based on point samples. It is also of great value in stochastic geometry and spatial statistics to be able not only to prove hyperuniformity, but also to verify when it does not hold. Nondegeneracy of the asymptotic variance (1.1) is one of the key assumptions for geometric central limit theorems (as in [3, 22, 27]).

The question of hyperuniformity is immediate for some special point processes, especially if the exact distribution of N_Λ is tractable. This is the case for a stationary Poisson point process with intensity $\lambda > 0$, which obviously is not hyperuniform. Here, the number of points in any set is independent of the outside configurations. In general, it is unclear what happens if interactions between the points inside the set and outside are introduced into the model. A standard approach to generating point patterns with interactions among points is to consider a Gibbs modification of some underlying measure, typically a Poisson point process through energy. The simplest model takes into account only interactions between pairs of points; this is known as a Gibbs point process with pairwise interactions. Interactions among k-tuples of points and other more complicated types of interactions are also widely studied in stochastic geometry and spatial statistics.

In this paper, we show that short-range Gibbs point processes are not hyperuniform. By a short-range process, we simply mean a point process such that interactions among points weaken with the distance. This phenomenon can be interpreted through the Papangelou intensity. A point process Γ has a Papangelou intensity λ^* if for any non-negative function f we can write

$$\mathbb{E}\left[\sum_{x\in\Gamma} f(x, \Gamma\setminus\{x\})\right] = \int \mathbb{E}f(x, \Gamma)\lambda^*(x, \Gamma)\mathrm{d}\,x.$$

Intuitively, we interpret $\lambda^*(x, \gamma)\mathrm{d}\,x$ as the conditional probability of observing a point in the infinitesimally small neighbourhood of x given that Γ agrees with a configuration γ outside this neighbourhood. For example, if $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ measures the interaction between two points, then a Gibbs point process with pairwise interactions has a Papangelou intensity of the form

$$\lambda^*(x, \gamma) = ze^{-\beta \sum_{y\in\gamma} \varphi(x,y)},$$

where $z > 0$, $\beta \geq 0$ are usually called activity and inverse temperature. Now, a point process is short-range if

$$\frac{\lambda^*(x, \gamma \cup \{y\})}{\lambda^*(x, \gamma)} \to 1 \tag{1.2}$$

fast enough as $\|x - y\| \to \infty$ for almost all configurations γ. The interpretation is straightforward. Any point y in the configuration γ plays only a negligible role when a new point x is introduced far away from y. For a Gibbs point process with pair potential φ, (1.2) simply translates to the condition that $\varphi(x, y) \to 0$ fast enough.

In this paper, we prove a kind of compressibility of any scale in the bulk of the interacting particle systems. Our main theorem is formulated for infinite-volume Gibbs point processes with a general Papangelou intensity λ^*. We claim that if the Papangelou intensity satisfies some moment conditions, it is enough to verify that

$$\int \left(\mathbb{E}\left|1 - \frac{\lambda^*(0, \Gamma \cup \{y\})}{\lambda^*(0, \Gamma)}\right|^{\alpha}\right)^{1/\alpha} \mathrm{d}\,y < \infty \tag{1.3}$$

for some $\alpha \geq 1$ to ensure that Γ is not hyperuniform. Specially for pairwise interactions with potential Φ, (1.3) translates into the integrability condition

$$\int |1 - e^{-\beta\Phi(x)}|\mathrm{d}\,x < \infty,$$

which, combined with superstability and lower-regularity as understood in [25], generates a non-hyperuniform point process. In the context of pair potentials, the authors of [11, 25] also provide some variance estimates, but only for the finite-volume Gibbs measures. A more general question than the initial one is addressed in [30], where the authors study volume-order fluctuations of score functions (including the number of points). The setting is for short-range Gibbs processes that are dominated by a Poisson point process. It corresponds to purely repulsive interactions, which can be translated to the Papangelou intensity by assuming $\lambda^* \geq C > 0$ everywhere. Such a setting is also discussed here in Section 3.2. For our main theorem, we consider more general interactions that can be arbitrary within a short range and vanishing at a long range. The theorem also covers processes that cannot be coupled with a Poisson point process. The techniques in the two papers are completely different. To show our results, we use an approach based on the Georgii–Nguyen–Zessin (GNZ) formalism, rather than the algorithmic construction of the Gibbs point processes used in [30].

The organization of the paper is as follows. In Section 2, we introduce all the necessary notation covering the theory of point processes with marks, and we present two versions of our main result. First, we state a non-hyperuniformity result for Gibbs point processes with pairwise interactions or interactions that become deterministic for distant points. Next, we state a theorem for Gibbs point processes such that the interactions are random everywhere, but also weak at long distances, which is suitable for a large variety of geometrically based interactions. Section 3 provides some further investigation and a couple of hints for verifying Assumption 1.3. It is followed by Section 4, where we provide a comprehensive application of our main results, discussing first pairwise interactions and then more general geometrical interactions. Finally, in Section 5, we give rigorous proofs of our main statements.

2. Main results

2.1. Notation

To describe marked Gibbs point processes in the space $\mathbb{R}^d$, we proceed as follows. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}^d$ and λ the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B})$. Furthermore, we introduce a complete, separable mark space $\mathbb{M}$, equipped with the associated Borel σ-algebra $\mathcal{B}_\mathbb{M}$ and a finite measure $\lambda_\mathbb{M}$.

We consider point configurations on the space $E := \mathbb{R}^d \times \mathbb{M}$ with σ-field $\mathcal{E} := \mathcal{B} \times \mathcal{B}_\mathbb{M}$, where each point in $\mathbb{R}^d$ is associated with a mark belonging to $\mathbb{M}$. The reference measure on E is the product measure $\nu := \lambda \otimes \lambda_\mathbb{M}$. The marks are mutually independent random variables whose distribution $\mathbb{Q}_\mathbb{M}$ does not depend on the location.

Let $\mathbf{N}$ denote the set of all locally finite configurations in E, i.e.,

$$\mathbf{N} := \left\{ \gamma \subset E;\, |\gamma \cap (\Lambda \times \mathbb{M})| < \infty,\ \forall \Lambda \subset \mathbb{R}^d \text{ bounded} \right\}.$$

Moreover, we denote by $\mathbf{N}_f$ the subset of $\mathbf{N}$ consisting of finite configurations. We endow $\mathbf{N}$ with $\mathcal{N}$, which is the smallest σ-field such that all projections $\gamma \mapsto \gamma \cap B$ are measurable for all $B \in \mathcal{E}$. For a point configuration $\gamma \in \mathbf{N}$ and a fixed set $\Lambda \subset \mathbb{R}^d$, we denote by γ_Λ the restriction of γ to the set $\Lambda \times \mathbb{M}$, i.e. $\gamma_\Lambda := \gamma \cap (\Lambda \times \mathbb{M})$.

By a *point process* we mean a probability measure Γ on $(\mathbf{N}, \mathcal{N})$. For $u \in \mathbb{R}^d$, we let $\tau_u : (x, m) \mapsto (x + u, m)$ be the shift in the position coordinate. If $\gamma = \{\mathbf{x}_1, \mathbf{x}_2, \dots\} \in \mathbf{N}$, we write $\tau_u\gamma = \{\tau_u\mathbf{x}_1, \tau_u\mathbf{x}_2, \dots\}$. Moreover, we call a point process *stationary* if its distribution is invariant with respect to τ_u, i.e., $\Gamma \overset{D}{=} \tau_u\Gamma$ for all $u \in \mathbb{R}^d$.

For a bounded set $\Lambda \subset \mathbb{R}^d$ and $\gamma \in \mathbf{N}$, we denote the number of points of γ occurring in Λ by $N_\Lambda := N_\Lambda(\gamma) = \sum_{\mathbf{x}\in\gamma} \mathbf{1}_{\Lambda\times\mathbb{M}}(\mathbf{x})$. If Γ is a marked point process, then $N_\Lambda(\Gamma)$ is a random variable with values in $\mathbb{N} \cup \{0\}$.

2.2. Gibbs point process

Definition 2.1. Let $\lambda^* : E \times \mathbf{N} \to \mathbb{R}$ be some measurable function. We call Γ a *Gibbs point process* associated with the *Papangelou intensity* λ^* if for all positive measurable $f : E \times \mathbf{N} \to \mathbb{R}$ it solves the Georgii–Nguyen–Zessin equations

$$\int_\mathbf{N} \sum_{\mathbf{x}\in\gamma} f(\mathbf{x}, \gamma \setminus \{\mathbf{x}\})\Gamma(\mathrm{d}\,\gamma) = \int_\mathbf{N} \int_E f(\mathbf{x}, \gamma)\lambda^*(\mathbf{x}, \gamma)\nu(\mathrm{d}\,\mathbf{x})\Gamma(\mathrm{d}\,\gamma). \qquad \text{(GNZ)}$$

Remark 2.1. (*The form of the Papangelou intensity*.) Usually, the Papangelou intensity is given in the form

$$\lambda^*(\mathbf{x}, \gamma) = z \exp\{-\beta h(\mathbf{x}, \gamma)\}, \tag{2.1}$$

where $h : E \times \mathbf{N} \to \mathbb{R}$ is called local energy, $z > 0$ is the activity parameter, and $\beta \geq 0$ is the inverse temperature.

Here we consider only stationary Gibbs point processes. In this case, the related Papangelou intensity λ^* is necessarily translation-invariant simultaneously with respect to both coordinates, meaning that

$$\lambda^*(\mathbf{x}, \gamma) = \lambda^*(\tau_u \mathbf{x}, \tau_u \gamma), \quad \forall u \in \mathbb{R}^d.$$

Remark 2.2. The existence of a point process satisfying Definition 2.1 is not discussed in the present paper; it is always implicitly assumed that we are given a well-defined Gibbs point process. However, note that (1.2) is related to the condition for existence as in [9]. We mention some relevant existence results in the course of the text. In addition, the point process Γ may not be uniquely determined by (2.1). If we make a claim about a Gibbs process with Papangelou intensity λ^*, we mean that it holds for all processes corresponding to this Papangelou intensity.

2.3. Main results

Our first result concerns point processes whose interactions are short-range and for which the ratio (1.2) has an upper bound which is uniform in $\mathbf{N}$. An important class of short-range Gibbs point processes is given by those determined by an integrable pair potential (see Section 4.1 for details). For multibody potentials, we replace the integrability condition by a suitable deterministic bound, which is integrable in the sense of [11, 25], except that we do not force the potentials to be integrable around the origin.

Theorem 2.1. *Let Γ be a stationary Gibbs point process on $\mathbb{R}^d$ with Papangelou intensity λ^*, and let the following assumptions be satisfied:*

- *(a1) For all $m \in \mathbb{M}$, $\mathbb{E}\lambda^*((0, m), \Gamma)^2 < \infty$.*
- *(a2) There exist a function $\phi : \mathbb{R}^d \to \mathbb{R}$ and some $\delta \geq 0$ such that $\int_{\mathbb{R}^d \setminus B(0,\delta)} \phi(x) \mathrm{d}x < \infty$ and, for any $\|x - y\| > \delta$,*

$$\left| 1 - \frac{\lambda^*(\mathbf{x}, \gamma \cup \{\mathbf{y}\})}{\lambda^*(\mathbf{x}, \gamma)} \right| \leq \phi(x - y).$$

Then there is a constant $C_{nhyp} > 0$, not depending on Λ, such that

$$\frac{\mathbf{Var}(N_\Lambda(\Gamma))}{|\Lambda|} \geq C_{nhyp} > 0. \tag{2.2}$$

In particular, Γ is not hyperuniform.

Theorem 2.1 applies to point processes where the interaction among distant points is deterministically bounded. This, however, is often too restrictive for processes with a random range of interaction (e.g. processes with energies based on Voronoi tessellations). For such processes, we provide another result.

Theorem 2.2. *Let Γ be a stationary Gibbs point process on $\mathbb{R}^d$ with Papangelou intensity λ^*. Assume that there are some $\alpha_1, \alpha_2 > 1$, $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$, and $\delta \geq 0$ such that the following hold:*

(A1) $\int_{\mathbb{M}} \mathbb{E}|\lambda^*((0, m), \Gamma)|^{2\alpha_1} \lambda_{\mathbb{M}}(\mathrm{d}\, m) < +\infty$, *and*

(A2) $\int_{\mathbb{R}^d \setminus B(0,\delta)} \left(\int_{\mathbb{M}^2} \mathbb{E} \left| 1 - \frac{\lambda^*((0,m_1),\Gamma\cup\{(y,m_2)\})}{\lambda^*((0,m_1),\Gamma)} \right|^{\alpha_2} \mathrm{d}\, \lambda_{\mathbb{M}}(m_1, m_2) \right)^{1/\alpha_2} \mathrm{d}\, y < \infty$.

Then there exists $\mathcal{C}_{nhyp} > 0$, not depending on Λ, such that

$$\frac{\mathbf{Var}(N_\Lambda(\Gamma))}{|\Lambda|} \geq \mathcal{C}_{nhyp}. \tag{2.3}$$

In particular, Γ is not hyperuniform.

The proofs of Theorems 2.2 and 2.2 are postponed to Section 5. The constant $\mathcal{C}_{nhyp}$ can be given in a closed form directly from the proof. We provide some lower bounds for these constants in special cases in Section 4.

Remark 2.3. (*Assumptions (A1)–(A2).*) Note that $\mathbb{E}\lambda^*(\mathbf{0}, \Gamma) := \lambda$ is the intensity of Γ. Loosely speaking, Assumption (A1) of Theorem 2.2 prevents the point process Γ from having too many points in a unit window. In other words, it forces some stability assumptions (e.g. superstability for pair potentials). On the other hand, Assumption (A2) states that the interactions among points of Γ become negligible with the distance. Again, we do not force the interactions among points close to each other to be bounded. A similar interpretation applies for Assumptions (a1) and (a2) of Theorem 2.1.

Remark 2.4. (*Shape of* Λ.) Note that the proof and the lower bounds (2.2)–(2.7) do not depend on the shape of the window Λ.

Remark 2.5. (*Unmarked case.*) In the unmarked case, one can choose the mark space $\mathbb{M}$ with just one atom m and set $\lambda_{\mathbb{M}}(m) = 1$. The Papangelou intensity obviously does not depend on the choice of m, so we may set $\Gamma' = \{x_i;\ (x_i, m_i) \in \Gamma\}$ to be the unmarked point process and write $\lambda^*((x, m), \Gamma) = \lambda^*(x, \Gamma')$ almost surely (a.s.). Then the assumptions of Theorem 2.2 become

(A1) $\mathbb{E}|\lambda^*(0, \Gamma')|^{2\alpha_1} < +\infty$,

(A2) $\int_{\mathbb{R}^d \setminus B(0,\delta)} \left(\mathbb{E} \left| 1 - \frac{\lambda^*(0,\Gamma'\cup\{y\})}{\lambda^*(0,\Gamma')} \right|^{\alpha_2} \right)^{1/\alpha_2} \mathrm{d}\, y < \infty$.

3. Conditions for Assumptions (a1)–(a2) (resp. (A1)–(A2))

In this section, we provide a short cookbook of conditions that can be imposed on the Papangelou intensity λ^* to guarantee the validity of Assumptions (a1)–(a2) of Theorem 2.1 (resp. Assumptions (A1)–(A2) of Theorem 2.2).

3.1. Stability and range of interaction

Definition 3.1. (*Local stability.*) The Papangelou intensity λ^* is called

- *locally stable from above* if there is a constant $C_1 < \infty$ such that $\lambda^*(\mathbf{x}, \gamma) \leq C_1$ for all $\mathbf{x} \in E$ and $\gamma \in \mathbf{N}$,
- *locally stable from below* if there is a constant $C_2 > 0$ such that $\lambda^*(\mathbf{x}, \gamma) \geq C_2$ for all $\mathbf{x} \in E$ and $\gamma \in \mathbf{N}$, and

- *double locally stable* if it is simultaneously locally stable from above and from below.

Note that a point process whose Papangelou intensity is locally stable from above trivially satisfies Assumption (a1) of Theorem 2.1 (resp. (A1) of Theorem 2.2). Yet this is a very frequent assumption in the literature.

Definition 3.2. (*Range of interaction.*) Let Γ be a Gibbs point process with Papangelou intensity λ^*. Then Γ has

- *finite range of interaction* if there exists $R > 0$ such that for all $\gamma \in \mathcal{N}$

$$\lambda^*(\mathbf{x}, \gamma) = \lambda^*(\mathbf{x}, \gamma \cap B(x, R) \times \mathbb{M}), \quad \forall \mathbf{x} := (x, m) \in E;$$

- *random finite range of interaction* if for all $\mathbf{x} := (x, m) \in E$ there is an a.s. finite random variable $R_{\mathbf{x}} := R_{\mathbf{x}}(\gamma)$ such that

$$\lambda^*(\mathbf{x}, \gamma) = \lambda^*(\mathbf{x}, \gamma \cap B(x, R_{\mathbf{x}}) \times \mathbb{M}), \quad \text{for } \Gamma\text{-almost all } \Gamma;$$

- *decreasing range of interaction* if

$$R_{\mathbf{x}}(\gamma) \geq R_{\mathbf{x}}(\gamma \cup \{\mathbf{y}\}) \quad \forall \mathbf{x}, \mathbf{y} \in E, \gamma \in \mathbf{N}. \tag{3.1}$$

Remark 3.1. Note that (3.1) is in fact a natural condition arising from many geometric models, including Voronoi tessellations, Delaunay triangulations, and k-nearest-neighbours graphs.

Proposition 3.1. *Let Γ be a Gibbs point process associated with Papangelou intensity λ^*. Then the following hold:*

(i) A Gibbs point process whose Papangelou intensity is locally stable from above satisfies Assumption (a1) *of Theorem* 2.1 *(resp.* (A1) *of Theorem* 2.2 *with any $\alpha_1 > 0$).*

(ii) Suppose Γ has a random finite range of interaction and there exists a function $f : \mathbb{R}^+ \to \mathbb{R}$ such that $\lambda^*(\mathbf{x}, \gamma) \leq f(R_{\mathbf{x}}(\gamma))$ for all $\mathbf{x} \in E$ and Γ-almost all γ. If $\mathbb{E}f(R_{\mathbf{0}})^{2\alpha_1} < \infty$ then Γ satisfies Assumption (A1) *of Theorem* 2.2*.*

(iii) If Γ has a finite range of interaction, then it satisfies Assumption (a2) *of Theorem* 2.1 *and Assumption* (A2) *of Theorem* 2.2 *for any $\alpha_2 > 1$.*

(iv) If λ^* is double locally stable and Γ has a decreasing random finite range of interaction such that $\mathbb{E}R_{\mathbf{0}}^{\alpha} < \infty$ for some $\alpha > d$, then Assumption (A2) *of Theorem* 2.2 *is satisfied for any $\alpha_2 \leq \alpha/d$.*

Proof.

(i) Since λ^* is uniformly bounded, any moment is finite.

(ii) For any $m \in \mathbb{M}$, we have

$$\mathbb{E}\lambda^*((0, m), \Gamma)^{2\alpha_1} \leq \mathbb{E}f(R_{(0,m)})^{2\alpha_1} < \infty.$$

Since $\lambda_{\mathbb{M}}$ was assumed to be finite, (A1) is satisfied for any mark distribution.

(iii) Consider $\delta = R$ and $\phi \equiv 0$ in Theorem 2.1, resp. $\delta = R$ in Theorem 2.2.

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(iv) Take $\alpha_2 \in (1, \alpha/d)$. By using the Markov inequality, we arrive at

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{M}^2} \mathbb{E} \left| 1 - \frac{\lambda^*((0, m_1), \Gamma \cup \{(y, m_2)\})}{\lambda^*((0, m_1), \Gamma)} \right|^{\alpha_2} \mathrm{d}\, \lambda_{\mathbb{M}}^2(m_1, m_2) \right)^{1/\alpha_2} \mathrm{d}\, y$$
$$\leq \left(1 + \frac{C_2}{C_1}\right) \int_{\mathbb{R}^d} \left[\int_{\mathbb{M}^2} \mathbb{P}\left(|y| \leq R_{(0,m_1)}(\Gamma \cup \{(y, m_2)\}) \right) \mathrm{d}\, \lambda_{\mathbb{M}}^2(m_1, m_2) \right]^{1/\alpha_2} \mathrm{d}\, y$$
$$\leq \left(1 + \frac{C_2}{C_1}\right) \lambda_{\mathbb{M}}(\mathbb{M})^{1/\alpha_2} \int_{\mathbb{R}^d} \left[\int_{\mathbb{M}} \mathbb{P}\left(|y| \leq R_{(0,m_1)}(\Gamma) \right) \mathrm{d}\, \lambda_{\mathbb{M}}(m_1) \right]^{1/\alpha_2} \mathrm{d}\, y$$
$$\leq \left(1 + \frac{C_2}{C_1}\right) \lambda_{\mathbb{M}}(\mathbb{M})^{1/\alpha_2} \int_{\mathbb{R}^d} \frac{1}{|y|^{\alpha/\alpha_2}} \mathrm{d}\, y \left(\int_{\mathbb{M}} \mathbb{E} R^{\alpha}_{(0,m_1)}(\Gamma) \mathrm{d}\, \lambda_{\mathbb{M}}(m_1) \right)^{1/\alpha_2} < \infty. \qquad \square$$

In conclusion, the task of verifying the assumptions of Theorems 2.1 and 2.2 often translates to estimating the moments, resp. the tail probabilities of the radius of interactions. In more than a few situations, one can benefit from the literature on stabilization and stochastic comparison to the Poisson point process or other random structures. The latter is described in the following section.

3.2. Stochastic comparison tools

In this section, we explore tools to estimate the moments $\mathbb{E} R_{\mathbf{0}}(\Gamma)^{\alpha}$, $\alpha > 0$, of the range of interaction needed in the previous section. Usually, for an infinite-volume Gibbs point process, this is not a straightforward task, since we do not possess the local distribution of the number of points. We use stochastic comparison with some random object which is easier to handle.

For this purpose, we consider the usual order on $\mathbf{N}$. For $\gamma_1, \gamma_2 \in \mathbf{N}$, we write $\gamma_1 \leq \gamma_2$ if $\gamma_1(B) \leq \gamma_2(B)$ for all $B \in \mathcal{E}$. In the language of point sets, this means that γ_1 has fewer points than γ_2. A function $f : \mathbf{N} \to \mathbb{R}$ is called *increasing* if $f(\gamma_1) \leq f(\gamma_2)$ whenever $\gamma_1 \leq \gamma_2$. It is *decreasing* if $-f$ is increasing.

Definition 3.3. We say that a point process Γ_1 is stochastically dominated by a point process Γ_2 (and we write $\Gamma_1 \ll \Gamma_2$) if $\int h \mathrm{d}\,\Gamma_1 \leq \int h \mathrm{d}\,\Gamma_2$ for all increasing functions h. Vice versa, we say that Γ_1 stochastically minorates Γ_2.

By the famous Strassen theorem, $\Gamma_1 \ll \Gamma_2$ if and only if there is a coupling of Γ_1 and Γ_2 that is supported in the set $\{(\gamma_1, \gamma_2), \gamma_1 \leq \gamma_2\}$.

First, we recall the 'Poisson sandwich inequality' from [10] which stochastically connects a Gibbs point process with a stationary Poisson point process.

Proposition 3.2. *Let Γ be a Gibbs point process associated with Papangelou intensity λ^*, and let Π_ρ denote a stationary Poisson point process with intensity ρ. Then the following hold:*

(i) *If λ^* is locally stable from below, then $\Pi_{C_1} \ll \Gamma$.*

(ii) *If λ^* is locally stable from above, then $\Gamma \ll \Pi_{C_2}$,*

Here, C_1 and C_2 are the constants from Definition 3.1.

Corollary 3.1. *Let Γ be associated with the Papangelou intensity λ^* and assume that the corresponding range of interaction $R_{\mathbf{x}}$ is decreasing. If λ^* is locally stable from below, then*

$$\mathbb{P}(R_{\mathbf{x}}(\Gamma) > r) \leq \mathbb{P}(R_{\mathbf{x}}(\Pi_{C_1}) > r), \quad \forall r \geq 0,$$

where Π_{C_1} is as in Proposition 3.2.

Remark 3.2. (*Increasing range of interaction.*) If the range of interaction is increasing with respect to the order on $\mathbf{N}$, then Corollary 3.1 provides the inequality in the opposite direction. In this situation, the local stability of λ^* from above produces similar upper bounds for the tail probabilities of $R_{\mathbf{x}}(\Gamma)$.

Consequently, for Γ as in Corollary 3.1 and $f : \mathbb{R} \to \mathbb{R}$ an increasing function, we may verify the assumptions of Proposition 3.1(ii) and Proposition 3.1(iv) by computing

$$\mathbb{E}f(R_{\mathbf{0}}(\Gamma)) = \int_0^\infty \mathbb{P}(f(R_{\mathbf{0}}(\Gamma)) > r)\mathrm{d}\, r \le \int_0^\infty \mathbb{P}(f(R_{\mathbf{0}}(\Pi)) > r)\mathrm{d}\, r = \mathbb{E}f(R_{\mathbf{0}}(\Pi)), \tag{3.2}$$

if $f : \mathbb{R} \to \mathbb{R}$ is some increasing function.

Theorems 2.1 and 2.2 provide non-hyperuniformity results for Gibbs processes with interactions that are weak at long distances. As mentioned before, we assume nothing about the interactions among close points. Such points can generate an unboundedly large amount of energy (imagine Coulomb interaction), and therefore the value $\lambda^*(\mathbf{x}, \gamma)$ can be arbitrarily close to zero when $\mathbf{x}$ is at a small distance from γ, i.e. when $d(\mathbf{x}, \gamma) := \inf\{d(\mathbf{x}, \mathbf{y}); \mathbf{y} \in \gamma\}$ is small, where $d(\mathbf{x}, \mathbf{y})$ is some distance between two marked points. This situation does not allow one to use minoration by a Poisson point process. However, in this situation, we are able to construct a coupling with a Bernoulli field.

For this purpose, we introduce the mapping $I^s : \mathbf{N} \to \{0, 1\}^{\mathbb{Z}^d}$ as follows. We split the space $\mathbb{R}^d$ into a collection of disjoint cubes $\mathcal{D} := \{\mathcal{D}_i; i \in \mathbb{Z}^d\}$ of a common side length $s > 0$ such that there exists $k \in \mathbb{Z}^d$ with the origin being one of the vertices of $\mathcal{D}_k$. Then, we define

$$I^s : \gamma \mapsto (\mathbf{1}\{N_{\mathcal{D}_k}(\gamma) \ge 1\})_{k\in\mathbb{Z}^d}, \quad \gamma \in \mathbf{N}.$$

In fact, in many situations, we do not need to know the exact positions of the points of the process to estimate the range of interaction. The only necessary information is often that there is at least one point in a given region. That is exactly the meaning of I^δ.

In order to formulate our next result, we need some partial order on $\{0, 1\}^{Z^d}$. We write that $l_1 \le l_2$ if $l_1, l_2 \in \{0, 1\}^{\mathbb{Z}^d}$ are such that $l_{1,i} = 1$ implies $l_{2,i} = 1$ for all $i \in \mathbb{Z}^d$. As usual, we say that $R : \{0, 1\}^{Z^d} \to \mathbb{R}$ is increasing if $R(l_1) \le R(l_2)$ whenever $l_1 \le l_2$ and decreasing if $-R$ is increasing. We write $X \ll Y$ for two random variables X, Y with values in $\{0, 1\}^{\otimes\mathbb{Z}^d}$ if $\mathbb{P}(X_i = 1; i \in J) \le \mathbb{P}(Y_i = 1, i \in J)$ for any $J \subset \mathbb{Z}^d$.

Proposition 3.3. *Assume there are constants C_1, C_2, $\delta > 0$ such that $\lambda^*(\mathbf{x}, \gamma) \le C_1$ everywhere and $\lambda^*(\mathbf{x}, \gamma) \ge C_2$ whenever $d(\mathbf{x}, \gamma) \ge \delta$. If Γ is the corresponding Gibbs point process, then for any $\varepsilon > 0$ there exists $p > 0$ such that*

$$B << I^{2\delta+\varepsilon}(\Gamma), \tag{3.3}$$

where B is a random variable with values in $\{0, 1\}^{\otimes\mathbb{Z}^d}$ such that B_i, $i \in \mathbb{Z}^d$, are independent with Bernoulli distribution B(p).

See Section 5.2 for the proof.

Remark 3.3. (*On the constant ε.*) The constant ε in Proposition 3.3 is merely an auxiliary for the proof. We need the side length of $\mathcal{D}_k$ to be slightly bigger than 2δ so that we can fit another box $\mathcal{C}_k$ of side length ε inside $\mathcal{D}_k$ in such a way that it has distance exactly δ from the complement of $\mathcal{D}_k$. Depending on ε, though, one can optimize the value of p for further applications.

Corollary 3.2. *Let the range of interaction $R_\mathbf{x}$ be decreasing with respect to the order in* $\mathbf{N}$*, and suppose there exists a decreasing function $R' : \{0, 1\}^{Z^d} \to \mathbb{R}$ such that $R_\mathbf{x}(\gamma) \le R'(I^\delta(\gamma))$ for all $\gamma \in \mathbf{N}$ and some $\delta > 0$. Then, under the assumptions and notation of Proposition 3.3, we have*

$$\mathbb{P}(R_\mathbf{x}(\Gamma) > r) \le \mathbb{P}(R'(B) > r) \quad \text{for all } r > 0.$$

Note that $R_\mathbf{x}$ and R' are defined on different spaces, and so the assumption of being decreasing has a slightly different meaning.

We know exactly the distribution of B. Therefore, it is usually a simple task to compute $\mathbb{P}(R'(B) > r)$. Then, using Corollary 3.2, we can estimate the moments of $R_\mathbf{0}(\Gamma)$ similarly as in (3.2) for any increasing function $f : \mathbb{R} \to \mathbb{R}$ by

$$\mathbb{E}f(R_\mathbf{0}(\Gamma)) \le \mathbb{E}f(R'(B)).$$

Some applications of both Corollary 3.1 and Corollary 3.2 will be presented in Sections 4.3 and 4.4.

4. Examples

4.1. Pair potentials

A classical example of a Gibbs point process in $\mathbb{R}^d$ is the model with pairwise interactions. In this section, we omit the marks.

Definition 4.1. (Pair potential.) A Gibbs point process has a pair potential if there is a measurable, symmetric function $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that the Papangelou intensity has the form

$$\lambda^*(x, \gamma) = ze^{-\beta \sum_{y\in\gamma} \Phi(x-y)}, \quad x \in \mathbb{R}^d, \gamma \in \mathbf{N}.$$

In the following definition, we recall a classical stability assumption from the pairwise model originating in [25]. Note that these conditions jointly guarantee the existence of a Gibbs point process with pair potential Φ.

Definition 4.2. (*Superstable pairwise interactions.*) Let Γ be a Gibbs point process with pair potential Φ. For any finite configuration $\gamma \in \mathbf{N}_f$, we define the energy of this configuration by

$$H(\gamma) := \sum_{\substack{\{x,y\}\subset\gamma \\ x\neq y}} \Phi(x-y).$$

We say that Φ is

- *superstable* if for any bounded $\Lambda \subset \mathbb{R}^d$ there exist constants $A > 0$, $B \ge 0$ such that

$$H(\gamma_\Lambda) \ge AN_\Lambda(\gamma)^2 - BN_\Lambda(\gamma), \quad \forall \gamma \in \mathbf{N},$$

- *lower-regular* if there is a positive, decreasing function $\varphi : [0, \infty) \to \mathbb{R}$ such that

$$\int_0^\infty x^{d-1}\varphi(x)\mathrm{d}\,x < \infty$$

  and $\Phi(x) \ge -\varphi(|x|)$ for all $x \in \mathbb{R}^d$, and

- *integrable* if

$$\int_{\mathbb{R}^d} \left|1 - e^{-\beta\Phi(y)}\right| \mathrm{d}\, y < \infty. \tag{4.1}$$

Remark 4.1. (*Assumptions (a1)–(a2) for superstable interactions.*) A Gibbs point process with an integrable pair potential Φ automatically satisfies Assumption (a2) of Theorem 2.1 with $\phi = \Phi$, since

$$\frac{\lambda^*(0, \gamma \cup \{y\})}{\lambda^*(0, \gamma)} = 1 - e^{-\Phi(y)}, \quad \forall \gamma \in \mathbf{N}.$$

Therefore, Assumption (a2) of Theorem 2.1 reduces to the standard integrability condition for the pair potential Φ. On the other hand, the existence of the second moment stems from the assumption of superstability and lower-regularity (see [25, Corollary 5.3]).

Corollary 4.1. *A Gibbs point process with superstable, lower-regular, and integrable pair potential is not hyperuniform.*

Remark 4.2 (*Bound on the asymptotic variance.*) Following the proof of Theorem 2.1, one can derive the following lower bound for the asymptotic variance of Theorem 2.1 for the Gibbs point process with integrable pair potential Φ:

$$\mathcal{C}_{nhyp} \geq \frac{(\mathbb{E}\lambda^*(0, \Gamma))^2}{\mathbb{E}\lambda^*(0, \Gamma) + \mathbb{E}\lambda^*(0, \Gamma)^2 \int_{\mathbb{R}^d} \left|1 - e^{-\beta\Phi(x)}\right| \mathrm{d}\, x}. \tag{4.2}$$

Example 4.1. The class of Gibbs point processes with superstable, integrable pairwise interactions is large and covers many standard examples, including the following:

1. The *Strauss process*, i.e. a Gibbs point process Γ with pair potential

$$\Phi(x - y) = \begin{cases} 1 & \text{if} |x - y| \leq R, \\ 0 & \text{otherwise,} \end{cases}$$

   for some $R \in [0, \infty)$. Let $\lambda := \mathbb{E}\lambda^*(0, \Gamma)$ be the intensity of the process; then, directly from (4.2), we get the lower bound

$$\mathcal{C}_{nhyp} \geq \frac{\lambda^2}{z + z^2 |B(0, R)|(1 - e^{-\beta})}.$$

2. *Riesz gases* with $s > d$, i.e. processes with a pair potential of the form $\Phi(x) = \|x\|^{-s}$, $x \in \mathbb{R}^d$, $s \in \mathbb{R}$. The case $s \leq d$ (including that of a Coulomb gas, $s = d - 2$) determines a non-integrable pair potential.

3. The *Lennard–Jones* pair potential, given by

$$\Phi(x) = A\|x\|^{-\alpha_1} - B\|x\|^{-\alpha_2}$$

   for some $A, B > 0$ and $\alpha_1 > \alpha_2 > d$.

4.2. Widom–Rowlinson models

We start with a simple model in which the size of the balls is the same deterministic constant for all the points. Thus, for $R > 0$, we define

$$L_R(\gamma) := \bigcup_{x\in\gamma} B(x, R), \quad \gamma \in \mathbf{N}.$$

The Widom–Rowlinson point process is a Gibbs point process Γ with an energy function

$$H(\gamma) = |L_R(\gamma)|, \quad \gamma \in \mathbf{N}_f.$$

The Papangelou intensity has the form

$$\lambda^*(x, \gamma) = z \exp\{-\beta(|L_R(\gamma \cup \{x\})| - |L_R(\gamma)|)\}, \quad x \in \mathbb{R}^d, \gamma \in \mathbf{N},$$

for some $z > 0$, $\beta \geq 0$. Clearly, $ze^{-\beta|B(0,R)|} \leq \lambda^*(x, \gamma) \leq z$; hence all moments of λ^* are finite. Moreover, Γ has a finite range of interaction that produces non-hyperuniformity by (i) and (iii) of Proposition 3.1. Alternatively, we can directly apply Theorem 2.1 with $\delta = 0$ and $\phi(x) = I_{B(0,2R)}(x)e^{\beta|B(0,R)|}$. Similarly, a Gibbs point process with quermass interaction among balls with a deterministic size is non-hyperuniform, i.e. a Gibbs point process with energy

$$H(\gamma) = |L_R(\gamma)| + \mathrm{Per}(L_R(\gamma)) + \chi(L_R(\gamma)), \quad \gamma \in \mathbf{N}_f,$$

where Per is the perimeter and χ the Euler–Poincaré characteristic (the number of connected components minus the number of holes).

Moreover, using the above-mentioned estimates for the Papangelou intensity and following the steps in the proof of Theorem 2.1, we obtain the following lower bound for the asymptotic variance:

$$\mathcal{C}_{nhyp} \geq \frac{e^{-\beta|B(0,R)|}}{1 + ze^{\beta|B(0,R)|}|B(0, 2R)|}.$$

Next, assume each point $x \in \Gamma$ is equipped with a non-negative random variable R_x independently and with the same distribution $\mathbb{Q}$. We are now in the setting of a marked Gibbs point process with $\mathbb{M} = \mathbb{R}_+$ and $\lambda_{\mathbb{M}} = \mathbb{Q}$. As in the previous example, we define

$$L(\gamma) := \bigcup_{(x,R_x)\in\gamma} B(x, R_x), \quad \gamma \in \mathbf{N},$$

and

$$\lambda^*(\mathbf{x}, \gamma) = z \exp\{-\beta(|L(\gamma \cup \{\mathbf{x}\})| - |L(\gamma)|)\}. \tag{4.3}$$

Note that a Gibbs point process associated with Papangelou intensity (4.3) is well defined. The existence of the process with interactions also involving other Minkowski functionals is proved in [6].

Corollary 4.2. *Assume that* $\mathbb{E}_{\mathbb{Q}} e^{\alpha\beta|B(0,1)|R^d} < \infty$ *for some* $\alpha > 1$*; then a Gibbs point process defined by the Papangelou intensity* (4.3) *is non-hyperuniform.*

Proof. As in Example 4.2, $\lambda^* \le z$. Hence, Assumption $(A1)$ of Theorem 2.2 is trivially satisfied by Proposition 3.1(ii). It remains to show that Assumption (A2) is true for some $\alpha_2 > 1$. Furthermore, for any $\mathbf{x} := (x, R_x)$, $\mathbf{y} := (y, R_y) \in E := \mathbb{R}^d \times \mathbb{R}_+$, we have

$$\frac{\lambda^*(\mathbf{x}, \gamma \cup \{\mathbf{y}\})}{\lambda^*(\mathbf{x}, \gamma)} \le e^{\beta |B(x,R_x) \cap B(y,R_y)|},$$

which does not depend on the configuration γ. Therefore, for any $\alpha_2 > 1$,

$$\mathbb{E}\left|1 - \frac{\lambda^*(\mathbf{x}, \Gamma \cup \{\mathbf{y}\})}{\lambda^*(\mathbf{x}, \Gamma)}\right|^{\alpha_2} \le \left|1 - e^{\beta |B(x,R_x) \cap B(y,R_y)|}\right|^{\alpha_2} = \left(e^{\beta |B(x,R_x) \cap B(y,R_y)|} - 1\right)^{\alpha_2}. \tag{4.4}$$

Now, take $\zeta_1, \zeta_2 > 1$ such that $\zeta_1 < \alpha$ and $1/\zeta_1 + 1/\zeta_2 = 1$. Furthermore, set $\alpha_2 = \alpha/\zeta_1$ and take $\zeta > \alpha_2 \zeta_2 d$. We successively use (4.4), Hölder's inequality with ζ_1, ζ_2, and Markov's inequality with ζ to get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}_+ \times \mathbb{R}_+} \mathbb{E}\left|1 - \frac{\lambda^*((0, R_0), \Gamma \cup \{(y, R_y)\})}{\lambda^*((0, R_0), \Gamma)}\right|^{\alpha_2} \mathbb{Q}(\mathrm{d}\, R_0)\mathbb{Q}(\mathrm{d}\, R_y) \right)^{1/\alpha_2} \mathrm{d}\, y \\
&\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}_+ \times \mathbb{R}_+} \left(e^{\beta |B(0,R_0) \cap B(y,R_y)|} - 1\right)^{\alpha_2} \mathbf{1}\{R_0 + R_y \ge |y|\} \mathbb{Q}(\mathrm{d}\, R_0)\mathbb{Q}(\mathrm{d}\, R_y) \right)^{1/\alpha_2} \mathrm{d}\, y \\
&\le \int_{\mathbb{R}^d} \left(\mathbb{E} e^{\beta \alpha_2 \zeta_1 |B(0,R_0)|}\right)^{1/\alpha_2 \zeta_1} \left(\mathbb{P}(R_0 + R_y \ge |y|)\right)^{1/\alpha_2 \zeta_2} \mathrm{d}\, y \\
&= \left(\mathbb{E} e^{\beta \alpha |B(0,1)| R_0^d}\right)^{1/\alpha} \int_{\mathbb{R}^d} \left(\mathbb{P}(R_0 + R_y \ge |y|)\right)^{1/\alpha_2 \zeta_2} \mathrm{d}\, y \\
&\le \left(\mathbb{E} e^{\beta \alpha |B(0,1)| R_0^d}\right)^{1/\alpha} \int_{\mathbb{R}^d} \frac{\left(\mathbb{E}(R_0 + R_y)^{\zeta}\right)^{1/\alpha_2 \zeta_2}}{|y|^{\zeta/\alpha_2 \zeta_3}} \mathrm{d}\, y < \infty.
\end{aligned}$$

□

Remark 4.3. (*Gibbs particle process.*) Generally, we may replace balls of random radius by random compact sets to define the Gibbs particle process. Assuming that the diameters of these sets have suitable exponential moments, the resulting process is again non-hyperuniform. Consequently, the crucial assumption of positive asymptotic variance in [1] is satisfied, enabling us to study the limit behaviour of U-statistics of Gibbs particle processes.

4.3. Voronoi interactions

In the previous example, we had direct control over the sizes of the discs through marks and hence over the range of interaction. Here, we may encounter cells that are very large, that interfere with distant neighbourhoods, and whose distribution is generally unknown. To define a Gibbs point process with interactions among Voronoi cells, for non-empty $\gamma \in \mathbf{N}$ and $x \in \gamma$, we denote the cell around x by

$$C(x, \gamma) := \{z \in \mathbb{R}^d : \|z - x\| \le \|z - y\| \text{ for all } y \in \gamma \setminus \{x\}\}.$$

The set $C(x, \gamma)$ represents those points in $\mathbb{R}^d$ such that x is the nearest point to them out of all points in γ. If $C(x, \gamma \cup \{x, y\}) \cap C(y, \gamma \cup \{x, y\}) \ne \emptyset$, we write $x \overset{\gamma}{\sim} y$. Finally, if $\gamma \in \mathbf{N}$ and $x \notin \gamma$, we write $C(x, \gamma)$ for $C(x, \gamma \cup \{x\})$. The set $C(x, \gamma)$ is in fact a closed convex set, since it can be written as an intersection of closed half-spaces. The set of all closed convex sets is denoted by $\mathcal{C}^d$.

We consider a function $\Phi : \mathcal{C}^d \to \mathbb{R} \cup \{\infty\}$ and write $\Phi(x, \gamma)$ for $\Phi(C(x, \gamma))$. For a finite configuration $\gamma \in \mathbf{N}$, we consider energy

$$H(\gamma) = \sum_{x \in \gamma} \Phi(x, \gamma) \mathbf{1}\{|C(x, \gamma)| < \infty\}. \tag{4.5}$$

The Gibbs point process associated to H is a point process satisfying (GNZ) with Papangelou intensity of the form

$$\lambda^*(x, \gamma) := z \exp\left\{-\beta \Phi(x, \gamma) - \beta \sum_{y \in \gamma} \left[\Phi(y, \gamma \cup \{x\}) - \Phi(y, \gamma)\right]\right\}. \tag{4.6}$$

In order to analyse Assumptions (A1) and (A2) of Theorem 2.2, we provide a list of assumptions imposed on the function Φ.

Definition 4.3. We say that the function Φ is

- *sub-additive* if $\Phi(C) \leq \Phi(C_1) + \Phi(C_2)$ whenever $C = C_1 \cup C_2$, where $C, C_1, C_2 \in \mathcal{C}^d$,
- *increasing* if $\Phi(C) \leq \Phi(C')$ whenever $C, C' \in \mathcal{C}^d$ and $C \subseteq C'$, and
- *controlled by the volume* if there exists a constant $K > 0$ such that $|\Phi(C)| \leq \min\{|C|, K\}$ for all $C \in \mathcal{C}^d$.

Note that, by [8], a Gibbs point process associated with λ^* in (4.6) with Φ as above exists. The conditions on Φ stated above allow us to use stochastic minoration by the Poisson point process in order to gain control over the size of the typical cell.

Proposition 4.1. *If Φ is sub-additive, increasing, and controlled by the volume, then*

$$z e^{-\beta K} \leq \lambda^*(x, \gamma) \leq z e^{\beta K} e^{\beta |C(x, \gamma)|}. \tag{4.7}$$

Proof. For any $y \in \gamma$, we have $C(y, \gamma \cup \{x\}) \subseteq C(y, \gamma)$, while both sides are elements of $\mathcal{C}^d$ and hence $\Phi(y, \gamma \cup \{x\}) \leq \Phi(y, \gamma)$ since Φ is increasing. Consequently,

$$\sum_{y \in \gamma} \left[\Phi(y, \gamma \cup \{x\}) - \Phi(y, \gamma)\right] \leq 0.$$

Moreover, Φ is uniformly bounded from above by K, yielding $-\beta \Phi(x, \gamma) \geq -\beta K$. This proves the lower bound.

To obtain the upper bound, for $\gamma \in \mathbf{N}$ and $y \overset{\gamma}{\sim} x$ we define the set $K_y := C(y, \gamma) \setminus C(y, \gamma \cup \{x\}) \in \mathcal{C}^d$. Then

$$C(x, \gamma) = \bigcup_{y \overset{\gamma}{\sim} x} K_y,$$

but also

$$\sum_{y \overset{\gamma}{\sim} x} |K_y| = |C(x, \gamma)|, \tag{4.8}$$

since $|K_y \cap K_{y'}| = 0$ for any $y \neq y'$.

Note that it is enough to consider only the neighbouring points of x in (4.6), that is, to sum over $\{y \in \gamma; x \overset{\gamma}{\sim} y\}$. Otherwise, adding a point x to the configuration γ (or removing it) does not affect the shape of $C(y, \gamma)$ (or, consequently, the value of $\Phi(y, \gamma)$). Using the sub-additivity of Φ, control by the volume, and (4.8), we arrive at

$$\begin{aligned}\sum_{y\in\gamma}\big[\Phi(y, \gamma\cup\{x\}) - \Phi(y, \gamma)\big] &= \sum_{y\in\Gamma;\, x\overset{\gamma}{\sim}y}\big[\Phi(y, \gamma\cup\{x\}) - \Phi(y, \gamma)\big]\\ &= \sum_{y\in\Gamma;\, x\overset{\gamma}{\sim}y}\big[\Phi(y, \gamma\cup\{x\}) - \Phi(C(y, \gamma\cup\{x\})\cup K_y)\big]\\ &\geq -\sum_{y\in\Gamma;\, x\overset{\gamma}{\sim}y}\Phi(K_y)\\ &\geq -\sum_{y\in\Gamma;\, x\overset{\gamma}{\sim}y}|K_y| = -|C(x, \gamma)|.\end{aligned}$$

The latter combined with the fact that $-\Phi \leq K$ produces the desired upper bound. □

Corollary 4.3. *Let Φ be some function on $\mathcal{C}^d$ that is sub-additive, increasing, and controlled by the volume. Then a Gibbs point process Γ associated with the Papangelou intensity (4.6) is non-hyperuniform for any $\beta < \beta_c$, where β_c is the unique solution of the equation*

$$\beta_c = z e^{-\beta_c K} C_d,$$

with

$$C_d := \frac{1}{3}\frac{|B(0, 1)|_{d-1}}{|B(0, 1)|_d}\left(\sin^{d-1}(\pi/12)\cos(\pi/12)\frac{1}{d} + \int_0^{\pi/12}\sin^d(\theta)\mathrm{d}\theta\right).$$

The proof is postponed to Section 5.

Remark 4.4. (Values of C_d and β_c.) The constant C_d can easily be evaluated. It can be seen that it decreases with the dimension d; for instance $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{36}$, $C_3 \approx 0.006$, etc. As a consequence, the exact value of β_c can be also given in terms of z and K. If $d = 2$ and $z = K = 1$, then $\beta_c \approx 0.03$.

Remark 4.5. (*General tessellations*.) We can also formulate Corollary 4.3 for interactions based on more general tessellations, such as the Laguerre and Johnson–Mehl types. In this context, the shape of the cells is determined additionally by marks, which justifies our consideration of marks in Theorems 2.1 and 2.2.

4.4. Interactions based on k-nearest-neighbours graph

For $k \in \mathbb{N}$ and $\gamma \in \mathbf{N}$, denote by $v^i(x, \gamma)$ the ith nearest neighbour of x in γ, $i = 1, \ldots, k$, and let $V^k(x, \gamma) := \{v^i(x, \gamma); i = 1, \ldots, \min\{k, N(\gamma) - 1\}\}$ be the set of the first k neighbours of x in γ. Here, $N(\gamma)$ is the cardinality of γ. If there are two or more points at the same distance from a given point, we use the lexicographic ordering as a tie-breaker to determine the k-nearest-neighbours structure. However, such ties have zero probability for the random point sets considered here.

For a finite configuration γ, we consider an energy of the form

$$H(\gamma) = \sum_{x\in\gamma} \sum_{y\in V^k(x,\gamma)} \Phi(x-y),$$

where $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is some measurable function. Adding a new point x to the configuration will change the nearest-neighbour structure, but only locally. If $x \in V^k(y, \gamma \cup \{x\})$ for some $y \in \gamma$, then $V^k(y, \gamma \cup \{x\}) = V^{k-1}(y, \gamma) \cup \{x\}$. Otherwise, it is obvious that $V^k(y, \gamma \cup \{x\}) = V^k(y, \gamma)$. Because of this fact, the corresponding Papangelou intensity takes the form (2.1) with

$$h(x, \gamma) := \sum_{y\in V^k(x,\gamma)} \Phi(x-y) + \sum_{y\in\gamma} \mathbf{1}_{V^k(y,\gamma\cup\{x\})}(x) \left[\Phi(y-x) - \Phi(y - v^k(y, \gamma))\right]. \qquad (4.9)$$

Proposition 4.2. *There exists a constant N_d, depending only on the dimension d, such that*

$$ze^{-\beta(1+2N_d)k\|\Phi\|_\infty} \le \lambda^*(x, \gamma) \le ze^{\beta(1+2N_d)k\|\Phi\|_\infty}.$$

Proof. In (4.9), the indicator $\mathbf{1}_{V^k(y,\gamma\cup\{x\})}(x)$ takes the value 1 only for finitely many points $y \in \gamma$. The number of such points is random, yet bounded by kN_d, where N_d depends only on the dimension (see [23, Lemma 4.3] for the proof). □

Corollary 4.4. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be such that $\|\Phi\|_\infty < \infty$. Then the Gibbs point process defined by the local energy (4.9) is non-hyperuniform.*

The Papangelou intensity in Corollary 4.4 is double locally stable. Therefore, it is enough to verify the moment condition of Proposition 3.1(iv). Alternatively, if we assume that Φ is decreasing and non-negative, we are allowed to have an explosion around the origin.

Corollary 4.5. *If $\Phi(x) := \Phi(\|x\|)$ is a decreasing and non-negative function on $\mathbb{R}_+$, then the Gibbs point process defined by (4.9) is non-hyperuniform.*

Example 4.2. (*Coulomb interaction.*) For $d \ge 3$, let $\Phi(x) = \frac{1}{\|x\|^{d-2}}$. Then by Corollary 4.5, the Gibbs point process defined by (4.9) is non-hyperuniform.

For the proofs, see Section 5.

5. Proofs of the main results

This section aims to present the proof of the main result of this paper, together with the proofs of our examples and auxiliary propositions that involve more technical details.

5.1. Proofs of Theorem 2.1 and Theorem 2.2

We start by proving Theorem 2.2. A slight modification of the proof then yields the statement of Theorem 2.1.

Proof. We aim to show that there is a constant $F > -1$ not depending on Λ such that

$$\frac{\mathbf{Var}(N_\Lambda)}{\mathbb{E}N_\Lambda} \ge \frac{1}{1+F}. \qquad (5.1)$$

By the stationarity of Γ, this already implies (2.3).

In the spirit of the proof of [25, Proposition 4.1], we define $R_\Lambda := R_\Lambda(\gamma) := \int_{\Lambda\times\mathbb{M}} \lambda^*(\mathbf{x}, \gamma)\nu(\mathrm{d}\,\mathbf{x})$ for $\gamma \in \mathcal{N}$. One has directly by (GNZ) that $\mathbb{E}N_\Lambda = \mathbb{E}R_\Lambda$. Moreover, note that

$$\mathbb{E}R_\Lambda^2 = \int\int_{(\Lambda\times\mathbb{M})^2} \lambda^*(\mathbf{x}, \gamma)\lambda^*(\mathbf{y}, \gamma)\nu(\mathrm{d}\,\mathbf{x})\nu(\mathrm{d}\,\mathbf{y})\Gamma(\mathrm{d}\,\gamma), \tag{5.2}$$

$$\begin{aligned}\mathbb{E}N_\Lambda(N_\Lambda - 1) &= \mathbb{E}\sum_{\mathbf{x}\in\Gamma} \mathbf{1}_{\Lambda\times\mathbb{M}}(\mathbf{x}) \sum_{\mathbf{y}\in\Gamma\setminus\{\mathbf{x}\}} \mathbf{1}_{\Lambda\times\mathbb{M}}(\mathbf{y}) \\ &= \int\int_{(\Lambda\times\mathbb{M})^2} \lambda^*(\mathbf{y}, \gamma)\lambda^*(\mathbf{x}, \gamma\cup\{\mathbf{y}\})\nu(\mathrm{d}\,\mathbf{x})\nu(\mathrm{d}\,\mathbf{y})\Gamma(\mathrm{d}\,\gamma),\end{aligned} \tag{5.3}$$

where for the latter equality we apply (GNZ) twice, first with $f(\mathbf{x}, \gamma) := \mathbf{1}_{\Lambda\times\mathbb{M}}(\mathbf{x}) \sum_{\mathbf{y}\in\gamma} \mathbf{1}_{\Lambda\times\mathbb{M}}(\mathbf{y})$ and then with $g(\mathbf{y}, \gamma) := \mathbf{1}_{\Lambda\times\mathbb{M}}(\mathbf{y}) \int_{\Lambda\times\mathbb{M}} \lambda^*(\mathbf{x}, \gamma\cup\{\mathbf{y}\})\nu(\mathrm{d}\,\mathbf{x})$. Similarly, by the definition of R_Λ and (GNZ), we have

$$\mathbb{E}N_\Lambda R_\Lambda = \mathbb{E}N_\Lambda(N_\Lambda - 1). \tag{5.4}$$

Suppose for a moment that

$$\mathcal{M} := \mathbb{E}R_\Lambda^2 - \mathbb{E}N_\Lambda(N_\Lambda - 1) \le F\mathbb{E}N_\Lambda \tag{5.5}$$

for some $F \in (-1, \infty)$. We then have that

$$\begin{aligned}(F+1)^2(\mathbb{E}N_\Lambda)^2 &= [\mathbb{E}(F+1)N_\Lambda]^2 = [\mathbb{E}(FN_\Lambda + R_\Lambda)]^2 \le \mathbb{E}(FN_\Lambda + R_\Lambda)^2 \\ &= F^2\mathbb{E}N_\Lambda^2 + 2F\mathbb{E}N_\Lambda(N_\Lambda - 1) + \mathbb{E}R_\Lambda^2 \\ &= F^2\mathbb{E}N_\Lambda^2 + (2F+1)\mathbb{E}N_\Lambda(N_\Lambda - 1) + \mathcal{M} \\ &\le F^2\mathbb{E}N_\Lambda^2 + (2F+1)\mathbb{E}N_\Lambda(N_\Lambda - 1) + F\mathbb{E}N_\Lambda \\ &= (F+1)^2\mathbb{E}N_\Lambda^2 - (F+1)\mathbb{E}N_\Lambda.\end{aligned}$$

Here, we have used consecutively (GNZ), the Cauchy–Schwartz inequality, (5.4), and the assumption (5.5). From this, the assertion (2.3) immediately follows.

It remains to check the validity of (5.5). First, we write $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$, where

$$\begin{aligned}\mathcal{M}_1 :=& \int\int_{(\Lambda\times\mathbb{M})^2} \lambda^*(\mathbf{x}, \gamma)\lambda^*(\mathbf{y}, \gamma)\mathbf{1}\{|x-y| > \delta\}\nu(\mathrm{d}\,\mathbf{x})\nu(\mathrm{d}\,\mathbf{y})\Gamma(\mathrm{d}\,\gamma) \\ &- \int\int_{(\Lambda\times\mathbb{M})^2} \lambda^*(\mathbf{y}, \gamma)\lambda^*(\mathbf{x}, \gamma\cup\{\mathbf{y}\})\mathbf{1}\{|x-y| > \delta\}\nu(\mathrm{d}\,\mathbf{x})\nu(\mathrm{d}\,\mathbf{y})\Gamma(\mathrm{d}\,\gamma), \\ \mathcal{M}_2 :=& \int\int_{(\Lambda\times\mathbb{M})^2} \lambda^*(\mathbf{x}, \gamma)\lambda^*(\mathbf{y}, \gamma)\mathbf{1}\{|x-y| \le \delta\}\nu(\mathrm{d}\,\mathbf{x})\nu(\mathrm{d}\,\mathbf{y})\Gamma(\mathrm{d}\,\gamma), \\ \mathcal{M}_3 :=& -\int\int_{(\Lambda\times\mathbb{M})^2} \lambda^*(\mathbf{y}, \gamma)\lambda^*(\mathbf{x}, \gamma\cup\{\mathbf{y}\})\mathbf{1}\{|x-y| \le \delta\}\nu(\mathrm{d}\,\mathbf{x})\nu(\mathrm{d}\,\mathbf{y})\Gamma(\mathrm{d}\,\gamma).\end{aligned}$$

Note that

$$\mathcal{M}_1 = \mathbb{E}\int_\Lambda\int_{\Lambda\setminus B(x,\delta)}\int_{\mathbb{M}^2} \lambda^*(\mathbf{y}, \Gamma)\lambda^*(\mathbf{x}, \Gamma)\left(1 - \frac{\lambda^*(\mathbf{x}, \Gamma\cup\{\mathbf{y}\})}{\lambda^*(\mathbf{x}, \Gamma)}\right)\nu(\mathrm{d}\,\mathbf{x})\nu(\mathrm{d}\,\mathbf{y}). \tag{5.6}$$

To proceed further, we define the quantities

$$D_1 := \int_{\mathbb{M}} \mathbb{E}|\lambda^*((0, m), \Gamma)|^{2\alpha_1} \lambda_{\mathbb{M}}(\mathrm{d}\, m),$$

$$D_2 := \int_{\mathbb{R}^d \setminus B(0,\delta)} \left(\int_{\mathbb{M}^2} \mathbb{E} \left| 1 - \frac{\lambda^*((0, m_1), \Gamma \cup \{(y, m_2)\})}{\lambda^*((0, m_1), \Gamma)} \right|^{\alpha_2} \mathrm{d}\, \lambda_{\mathbb{M}}(m_1, m_2) \right)^{1/\alpha_2} \mathrm{d}\, y,$$

which are both assumed to be finite. Using Fubini's theorem, Hölder's inequality with respect to the product measure $P_\Gamma \otimes \lambda^2_{\mathbb{M}}$ (P_Γ is the distribution of Γ), and finally the stationarity of Γ, we arrive at

$$\begin{aligned}
\mathcal{M}_1 &= \int_\Lambda \int_{\Lambda \setminus B(x,\delta)} \int_{\mathbb{M}^2} \mathbb{E}\lambda^*(\mathbf{y}, \Gamma)\lambda^*(\mathbf{x}, \Gamma) \left(1 - \frac{\lambda^*(\mathbf{x}, \Gamma \cup \{\mathbf{y}\})}{\lambda^*(\mathbf{x}, \Gamma)} \right) \nu(\mathrm{d}\, \mathbf{x})\nu(\mathrm{d}\, \mathbf{y}) \\
&\le \int_\Lambda \int_{\Lambda \setminus B(x,\delta)} \left(\int_{\mathbb{M}^2} \mathbb{E}|\lambda^*((y, m_y), \Gamma)|^{2\alpha_1} \mathrm{d}\, \lambda_{\mathbb{M}}(m_y, m_x) \right. \\
&\quad \left. \times \int_{\mathbb{M}^2} \mathbb{E}|\lambda^*((x, m_x), \Gamma)|^{2\alpha_1} \mathrm{d}\, \lambda_{\mathbb{M}}(m_y, m_x) \right)^{\frac{1}{2\alpha_1}} \\
&\quad \cdot \left(\int_{\mathbb{M}^2} \mathbb{E} \left| 1 - \frac{\lambda^*((x, m_x), \Gamma \cup \{(y, m_y)\})}{\lambda^*((x, m_x), \Gamma)} \mathrm{d}\, \lambda_{\mathbb{M}}(m_x, m_y) \right|^{\alpha_2} \right)^{1/\alpha_2} \mathrm{d}\, x \mathrm{d}\, y \\
&= (\lambda_{\mathbb{M}}(\mathbb{M}) D_1)^{1/\alpha_1} \int_\Lambda \int_{\Lambda \setminus B(x,\delta)} \\
&\quad \times \left(\int_{\mathbb{M}^2} \mathbb{E} \left| 1 - \frac{\lambda^*((0, m_x), \Gamma \cup \{(y - x, m_y)\})}{\lambda^*((0, m_x), \Gamma)} \right|^{\alpha_2} \mathrm{d}\, \lambda^2_{\mathbb{M}}(m_x, m_y) \right)^{1/\alpha_2} \mathrm{d}\, x \mathrm{d}\, y \\
&\le (\lambda_{\mathbb{M}}(\mathbb{M}) D_1)^{1/\alpha_1} \int_\Lambda \int_{B(0,\delta)^C} \\
&\quad \times \left(\int_{\mathbb{M}^2} \mathbb{E} \left| 1 - \frac{\lambda^*((0, m_1), \Gamma \cup \{(z, m_2)\})}{\lambda^*((0, m_1), \Gamma)} \right|^{\alpha_2} \mathrm{d}\, \lambda^2_{\mathbb{M}}(m_1, m_2) \right)^{1/\alpha_2} \mathrm{d}\, x \mathrm{d}\, z \\
&= |\Lambda| \, (\lambda_{\mathbb{M}}(\mathbb{M}) D_1)^{1/\alpha_1} \, D_2 =: F_1{}' |\Lambda| =: F_1 \mathbb{E} N_\Lambda .
\end{aligned}$$

By Assumptions (A1) and (A2), $F_1 < \infty$. By the same arguments,

$$\begin{aligned}
\mathcal{M}_2 &= \int_\Lambda \int_{\Lambda \cap B(x,\delta)} \int_{\mathbb{M}^2} \mathbb{E}\lambda^*((x, m_x), \Gamma)\lambda^*((y, m_y), \Gamma) \mathrm{d}\, \lambda^2_{\mathbb{M}}(m_x, m_y) \mathrm{d}\, y \mathrm{d}\, x \\
&\le \int_\Lambda \int_{B(x,\delta)} \int_{\mathbb{M}^2} \mathbb{E}|\lambda^*((0, m_x), \Gamma)|^2 \mathrm{d}\, \lambda^2_{\mathbb{M}}(m_x, m_y) \mathrm{d}\, y \mathrm{d}\, x \\
&= |\Lambda| |B(0, \delta)| \lambda_{\mathbb{M}}(\mathbb{M}) \int_{\mathbb{M}} \mathbb{E}|\lambda^*((0, m), \Gamma)|^2 \lambda_{\mathbb{M}}(\mathrm{d}\, m) =: F_2 \mathbb{E} N_\Lambda
\end{aligned}$$

is finite by (A1). Finally, $\mathcal{M}_3 \le 0$ and hence (5.5) holds true with $F = F_1 + F_2$. □

We continue with a proof of Theorem 2.1.

Proof. Clearly, for the constants $\mathcal{M}_2$, $\mathcal{M}_3$ from the proof of Theorem 2.2, one can use the same arguments in order to find finite upper bounds. For $\mathcal{M}_1$, we use the Cauchy–Schwartz inequality to see that

$$\begin{aligned}\mathcal{M}_1 &\le \int_\Lambda \int_{\Lambda\setminus B(x,\delta)} \phi(x-y) \int_{\mathbb{M}^2} \mathbb{E}\lambda^*(\mathbf{y}, \Gamma)\lambda^*(\mathbf{x}, \Gamma)\nu(\mathrm{d}\,\mathbf{x})\nu(\mathrm{d}\,\mathbf{y}) \\ &\le \left(\int_{\mathbb{M}^2} \mathbb{E}\lambda^*((0, m_x), \Gamma)^2 \mathrm{d}\,\lambda_{\mathbb{M}}(m_x, m_y)\right) \int_\Lambda \int_{\Lambda\setminus B(x,\delta)} \phi(x-y)\mathrm{d}\,x\mathrm{d}\,y \\ &\le \left(\int_{\mathbb{M}^2} \mathbb{E}\lambda^*((0, m_x), \Gamma)^2 \mathrm{d}\,\lambda_{\mathbb{M}}(m_x, m_y)\right) |\Lambda| \int_{B(0,\delta)^C} \phi(z)\mathrm{d}\,z := F_1\mathbb{E}N_\Lambda.\end{aligned}$$

The rest of the proof follows the same path as the proof of Theorem 2.2. □

5.2. Proof of Proposition 3.3

Proof. For all $k \in \mathbb{Z}^d$ we denote by $\mathcal{C}_k$ a cube of a side length ε lying in the centre of $\mathcal{D}_k$, with the two cubes having parallel edges. Now, let $k \in \mathbb{Z}^d$ and $\gamma \in \mathbf{N}$ be chosen to be arbitrary, but fixed. Let $\Pi^z_{\mathcal{D}_k}$ be the marked Poisson point process in $\mathcal{D}_k \times \mathbb{M}$ with intensity z and $\mathbf{N}_{\mathcal{D}_k} \subset \mathbf{N}$ the set of configurations restricted to $\mathcal{D}_k \times \mathbb{M}$. It can be shown that Γ also satisfies the Dobrushin–Lanford–Ruelle (DLR) equations (see [7, Theorem 1]). Then, the distribution of points in $\mathcal{D}_k$ given the configuration in $\mathcal{D}_k^C$ is precisely given by

$$\mathbb{P}(\mathrm{d}\,\gamma_{\mathcal{D}_k}|\gamma_{\mathcal{D}_k^C}) = \frac{1}{Z^{\beta,z}(\gamma_{\mathcal{D}_k^C})} \prod_{n=1}^{N_{\mathcal{D}_k}(\gamma_{\mathcal{D}_k})} \lambda^*(\mathbf{x}_n, \gamma_{\mathcal{D}_k^C} \cup \{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\})\Pi^z_{\mathcal{D}_k}(\mathrm{d}\,\gamma_{\mathcal{D}_k}),$$

where

$$Z^{\beta,z}(\gamma_{\mathcal{D}_k^C}) = \int_{\mathbf{N}_{\mathcal{D}_k}} \prod_{n=1}^{N_{\mathcal{D}_k}(\gamma_{\mathcal{D}_k})} \lambda^*(\mathbf{x}_n, \gamma_{\mathcal{D}_k^C} \cup \{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\})\Pi^z_{\mathcal{D}_k}(\mathrm{d}\,\gamma_{\mathcal{D}_k}).$$

Since λ^* is locally stable from above,

$$\begin{aligned}Z^{\beta,z}(\gamma_{\mathcal{D}_k^C}) &\le \int_{\mathbf{N}_{\mathcal{D}_k}} {C_1}^{N_{\mathcal{D}_k}(\gamma_{\mathcal{D}_k})}\Pi^z_{\mathcal{D}_k}(\mathrm{d}\,\gamma_{\mathcal{D}_k}) \\ &= \sum_{n=0}^{\infty} C_1^n \frac{z^n(2\delta+\varepsilon)^{dn}}{n!} e^{-z(2\delta+\varepsilon)^d} =: Z^{\beta,z} < \infty.\end{aligned}$$

The constant $Z^{\beta,z}$ depends on δ, ε, C_1, and d, but not on k or on the configuration $\gamma_{\mathcal{D}_k^C}$. Using the fact that $d(\mathbf{x}, \gamma_{\mathcal{D}_k^C}) \ge \delta$ for any $\mathbf{x} \in \mathcal{C}_k \times \mathbb{M}$, we have that

$$\begin{aligned}&\mathbb{P}((I^{2\delta+\varepsilon}(\Gamma))_k = 1|\gamma_{\mathcal{D}_k^C}) = \mathbb{P}(N_{\mathcal{D}_k}(\Gamma) \ge 1|\gamma_{\mathcal{D}_k^C}) \\ &\quad\ge \mathbb{P}(N_{\mathcal{D}_k}(\Gamma) = N_{\mathcal{C}_k}(\Gamma) = 1|\gamma_{\mathcal{D}_k^C}) \\ &\quad= \frac{1}{Z^{\beta,z}(\gamma_{\mathcal{D}_k^C})} \int_{\mathbf{N}_{\mathcal{D}_k}} \mathbf{1}\{N_{\mathcal{D}_k}(\gamma) = N_{\mathcal{C}_k}(\gamma) = 1\}\lambda^*(\gamma_{\mathcal{C}_k}, \gamma_{\mathcal{D}_k^C})\Pi^z_{\mathcal{D}_k}(\gamma_{\mathcal{D}_k}) \\ &\quad\ge \frac{C_2}{Z^{z,\beta}} \int_{\mathbf{N}_{\mathcal{D}_k}} \mathbf{1}\{N_{\mathcal{D}_k}(\gamma) = N_{\mathcal{C}_k}(\gamma) = 1\}\Pi^z_{\mathcal{D}_k}(\gamma_{\mathcal{D}_k})\end{aligned}$$

$$= \frac{C_2}{Z^{z,\beta}} \mathbb{P}(\Pi^z_{\mathcal{D}_k} \cap (\mathcal{C}_k \times \mathbb{M}) = 1)\mathbb{P}(\Pi^z_{\mathcal{D}_k} \cap (\mathcal{D}_k \setminus \mathcal{C}_k \times \mathbb{M}) = 0)$$
$$= \frac{C_2}{Z^{z,\beta}} z\varepsilon^d e^{-z(2\delta+\varepsilon)^d} =: p > 0,$$

where p depends neither on k nor on the boundary condition $\gamma_{\mathcal{D}_k^C}$. Note that the quantity $\lambda^*(\gamma_{\mathcal{C}_k}, \gamma_{\mathcal{D}_k^C})$ is well defined, since $\gamma_{\mathcal{C}_k}$ is assumed to be a.s. a one-point set.

To prove the statement, we construct a disagreement coupling of $I^{2\delta+\varepsilon}(\Gamma)$ and $B \sim B(p)^{\otimes \mathbb{Z}^d}$. For this purpose, let I be a finite index set and $X := (X_i)_{i\in I}$ random variables, not necessarily independent or identically distributed, with values in $\{0, 1\}$. Assume that

$$\mathbb{P}(X_i = 1 | X_j; j \in I \setminus \{i\}) > p, \quad \forall i \in I. \tag{5.7}$$

Then also

$$\mathbb{P}(X_i = 1) = \sum_{x_j \in \{0,1\}; j \in I\setminus\{i\}} \mathbb{P}(X_i = 1 | X_j = x_j; j \in I \setminus \{i\})\mathbb{P}(X_j = x_j; j \in I \setminus \{i\}) > p,$$

and similarly,

$$\mathbb{P}(X_i = 1 | X_j; j \in J) > p$$

for any $J \subseteq I \setminus \{i\}$.

Let $U := (U_i)_{i\in I}$ be a vector of independent and identically distributed (i.i.d.) uniform random variables on [0,1] and define $B_i := \mathbf{1}[U_i < p]$, $i \in I$. Clearly, $B_I := (B_i)_{i\in I}$ is a vector of i.i.d. Bernoulli variables with parameter p.

For the coupling, we define $Z_{i_1} = \mathbf{1}[U_{i_1} < \mathbb{P}(X_{i_1} = 1)]$, $i_1 \in I$. It is easy to check that $\mathbb{P}(Z_{i_1} = z) = \mathbb{P}(X_{i_1} = z)$ for $z \in \{0, 1\}$ and $\mathbb{P}(B_{i_1} \le Z_{i_1}) = 1$. Inductively, for $k \ge 1$, let $((Z_{i_1}, \ldots Z_{i_k}), (B_{i_1}, \ldots B_{i_k}))$ be a coupling of $(X_{i_1}, \ldots X_{i_k})$ and $(B_{i_1}, \ldots B_{i_k})$, such that $\mathbb{P}(Z_{i_j} \ge B_{i_j}) = 1$ for any $j = 1, \ldots, k$. Then we define

$$Z_{i_{k+1}} := \mathbf{1}[U_{i_{k+1}} < \mathbb{P}(X_{i_{k+1}} | X_{i_j} = Z_{i_j}; j = 1, \ldots, k)].$$

Again, $\mathbb{P}(Z_{i_{k+1}} \ge B_{i_{k+1}}) = 1$, and for $z_1, \ldots z_k \in \{0, 1\}$ we compute

$$\begin{aligned}
&\mathbb{P}(Z_{i_1} = z_1, \ldots, Z_{i_k} = z_k, Z_{i_{k+1}} = 1)\\
&\quad = \mathbb{P}(Z_{i_1} = z_1, \ldots, Z_{i_k} = z_k)\mathbb{P}(Z_{i_{k+1}} = 1 | Z_{i_1} = z_1, \ldots, Z_{i_k} = z_k)\\
&\quad = \mathbb{P}(X_{i_1} = z_1, \ldots, X_{i_k} = z_k)\mathbb{P}(U_{i_{k+1}} < \mathbb{P}(X_{i_{k+1}} = 1 | X_{i_1} = z_1, \ldots, X_{i_k} = z_k))\\
&\quad = \mathbb{P}(X_{i_1} = z_1, \ldots, X_{i_k} = z_k)\mathbb{P}(X_{i_{k+1}} = 1 | X_{i_1} = z_1, \ldots, X_{i_k} = z_k)\\
&\quad = \mathbb{P}(X_{i_1} = z_1, \ldots, X_{i_k} = z_k, X_{i_{k+1}} = 1).
\end{aligned}$$

Let $Z := (Z_i)_{i\in I}$. We have shown that (Z, B_I) is a coupling of X and B_I yielding

$$B_I << X.$$

This remains true if we turn to the limit and take $I_n \nearrow \mathbb{Z}^d$. The choice $X = (I^{2\delta+\varepsilon}(\Gamma))_{i\in\mathbb{Z}^d}$ satisfies the assumption (5.7), and hence, by the arguments above,

$$B << I^{2\delta+\varepsilon}(\Gamma).$$

□

5.3. Proof of Corollary 4.3

Proof. As before, one needs to verify Assumptions (A1)–(A2) of Theorem 2.2. Here, it is enough to do so in the form without marks.

By Proposition 4.1, the Papangelou intensity of Γ is locally stable from below, and hence by Proposition 3.2 there exists a Poisson point process Π such that $\Pi \ll \Gamma$ and Π has the intensity $ze^{-K\beta}$, where K is as in (4.7) and z, β are the parameters defining the Papangelou intensity (2.1) of the Gibbs point process Γ. This stochastic minoration can be interpreted via coupling of Π and Γ such that $\gamma' \subseteq \gamma$ whenever $\gamma' \sim \Pi$ and $\gamma \sim \Gamma$. For Voronoi tessellations, it translates to $C(x, \gamma) \subseteq C(x, \gamma')$ for any $x \in \gamma \cap \gamma'$. Combining this with the upper bound from Proposition 4.1, we may write

$$\mathbb{E}|\lambda^*(0, \Gamma)|^{2\alpha_1} \le z^{2\alpha_1} e^{2\alpha_1 \beta K} \mathbb{E} e^{2\alpha_1 \beta |C(0,\Pi)|}.$$

Now we aim to construct a ball $B(0,R)$ such that $R = R(\Pi)$ is a random variable depending on Π and $C(0, \Pi) \subseteq B(0, R)$ a.s. We use exactly the approach in the proof of [24, Lemma 5.1], the only difference being that we are interested in more precise estimates of the tail probabilities $\mathbb{P}(R > t)$, $t > 0$.

Let $K_1, \ldots, K_J$ be set of circular cones with apices in the origin with angular radii $\pi/6$. We do not expect the cones to have zero-volume intersections, yet we choose J to be the minimum value such that $\cup_{j=1}^J K_j = \mathbb{R}^d$. Note that J depends on the dimension and is finite. For each $j = 1, \ldots, J$ choose $x_j \in \gamma' \cup K_j$ to be the closest point to the origin.

Denote by $H_x(y) := \{z \in \mathbb{R}^d; \|z - y\| \le \|z - x\|\}$ the closed half-space induced by points that are closer to y than to x. By the definition of a Voronoi cell,

$$C(0, \Pi) = \bigcap_{x \in \gamma'} H_x(0) \subseteq \bigcap_{j=1}^J H_{x_j}(0) \subseteq \bigcup_{j=1}^J H_{x_j}(0) \cap K_j \quad \text{a.s.}$$

Set $R := \max\{\|x_j\|; j = 1, \ldots, J\}$. We need to verify that $C(0, \Pi) \subseteq B(0, R)$ a.s. To do so, we choose $y \in H_{x_j}(0) \cap K_j$ and show that $\|y\| \le \|x_j\|$ for any $j \in \{1, \ldots, J\}$. It follows from a simple computation that

$$\|y\|^2 \le \|y - x_j\|^2 = \|x_j\|^2 + \|y\|^2 - 2\langle x_j, y\rangle \le \|x_j\|^2 + \|y\|^2 - \|x_j\|\,\|y\|,$$

where the first inequality holds since $y \in H_{x_j}(0)$ and the last one from the fact that $y \in K_j$ (note that $z_1, z_2 \in K_j$ implies $\langle z_1, z_2\rangle \ge 1/2\|z_1\|\,\|z_2\|$). Finally, we have that $C(0, \Gamma) \subseteq B(0, R(\Pi))$ a.s.

Using void probabilities of the Poisson point process Π, we arrive at the estimate

$$\begin{aligned}
\mathbb{P}(R > t) &\le \sum_{j=1}^J \mathbb{P}(\|x_j\| > t) = \sum_{j=1}^J \mathbb{P}(\Pi \cap B(0, t) \cap K_j = \emptyset) \\
&= \sum_{j=1}^J \exp\{-ze^{-\beta K} |K_j \cap B(0, t)|\} \\
&= J \exp\{-ze^{-\beta K} t^d c_d\},
\end{aligned}$$

where $c_d := |K_1 \cap B(0, 1)|_d$. We leave it to the reader to check that

$$c_d = |B(0, 1)|_{d-1} \left(\sin^{d-1}(\pi/12) \cos(\pi/12) \frac{1}{d} + \int_0^{\pi/12} \sin^d(\theta) \mathrm{d}\theta \right).$$

Ultimately, for (A1), we have that

$$\begin{aligned}\mathbb{E}\lambda^*(0, \Gamma)^{2\alpha_1} &\le z^{2\alpha_1} e^{2\alpha_1\beta K}\mathbb{E}_\Pi e^{2\alpha_1\beta|B(0,1)|_d R^d} \\ &= z^{2\alpha_1} e^{2\alpha_1\beta K}\int_0^\infty \mathbb{P}(\exp\{2\alpha_1\beta|B(0, 1)|_d R^d\} > t)\mathrm{d}\, t \\ &= z^{2\alpha_1} e^{2\alpha_1\beta K} d\int_0^\infty u^{d-1}\exp\{2\alpha_1\beta|B(0, 1)|_d u^d\}\mathbb{P}(R > u)\mathrm{d}\, u \\ &\le z^{2\alpha_1} e^{2\alpha_1\beta K} d\int_0^\infty u^{d-1}\exp\{2\alpha_1\beta|B(0, 1)|_d u^d\} J \exp\{-ze^{-\beta K}u^d c_d\}\mathrm{d}\, u.\end{aligned} \tag{5.8}$$

The latter integral converges as long as

$$\beta < \frac{1}{2\alpha_1|B(0, 1)|_d} ze^{-\beta K} c_d. \tag{5.9}$$

Similarly, for (A2), we have from the assumptions on Φ that

$$0 \le \frac{\lambda^*(0, \gamma \cup \{y\})}{\lambda^*(0, \gamma)} \le e^{2\beta K} e^{\beta|C(0,\gamma)|}, \quad \Gamma\text{−a.s.}$$

Therefore, by an additional application of Fubini's theorem and the fact that $1/\alpha_2 \le 1$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(\mathbb{E}\left|1 - \frac{\lambda^*(0, \Gamma \cup \{y\})}{\lambda^*(0, \Gamma)}\right|^{\alpha_2}\right)^{1/\alpha_2}\mathrm{d}\, y \\ &\quad\le \int_{\mathbb{R}^d}\mathbb{E}\left(\frac{\lambda^*(0, \Gamma \cup \{y\})}{\lambda^*(0, \Gamma)}\right)^{\alpha_2}\mathbf{1}\{|y| \le R\}\mathrm{d}\, y \\ &\quad\le \mathbb{E}\int_{B(0,R)} e^{2\alpha_2\beta K} e^{\alpha_2\beta|B(0,R)|_d}\mathrm{d}\, y \\ &\quad= e^{2\alpha_2\beta K}|B(0, 1)|_d \mathbb{E}R^d e^{\alpha_2\beta|B(0,1)|_d R^d}.\end{aligned}$$

A computation in the same spirit as in (5.8) shows that the expectation above is finite as long as

$$\beta < \frac{1}{\alpha_2|B(0, 1)|_d} ze^{-\beta K} c_d. \tag{5.10}$$

The two inequalities (5.9) and (5.10) are optimal for $\alpha_1 = \frac{3}{2}$ and $\alpha_2 = 3$. □

Remark 5.1. We highlight two facts regarding the construction of the estimates of the tail probabilities of R. First, the construction of the estimates is not hindered by the fact that the cones $K_1, \ldots, K_J$ may intersect and we may possibly choose some point $x \in \gamma'$ multiple times. Second, it can be seen from the exponential form of the tail probabilities that the circular cones are the optimal choice for this construction. They give us more precise estimates than any other solids would do (e.g. non-circular and non-intersecting).

5.4. Proofs of Corollary 4.4 and Corollary 4.5

We begin with a proof of Corollary 4.4.

Proof. The Papangelou intensity is double locally stable by Proposition 4.2; ergo Assumption (A1) of Theorem 2.2 is justified by Proposition 3.1(i).

It remains to validate (A2). For that, we construct $R := R(\gamma)$ such that R is the range of interaction, i.e.

$$\lambda^*(0, \gamma) = \lambda^*(0, \gamma \cap B(0, R)) \quad \Gamma\text{−a.s.}$$

The construction is almost the same as in the proof of [22, Lemma 6.1], where the authors prove stabilization for a planar undirected k-nearest-neighbours graph (there is an edge between x, y whenever $x \in V^k(y, \gamma)$ or $y \in V^k(x, \gamma)$). Here, we consider a directed graph, i.e. such that there is an edge pointing from x to y whenever $y \in V^k(x, \gamma)$. The idea, however, remains the same.

Let J be the smallest integer such that $K_1, \ldots, K_J$ are cones with apex at the origin and angular radius at most $\pi/6$, such that $\cup_{j=1}^J K_j = \mathbb{R}^d$. Note that, unlike in the proof of Corollary 4.3, we do not need to optimize the shape of the cones. In fact, by Proposition 3.1(iv), all we need to show is that the radius R has finite αth moment for some $\alpha > d$ and set $\alpha_2 = \alpha/d$. Let $D := D(\gamma)$ be the smallest $t > 0$ such that there are at least $k + 1$ points in each $K_j \cap B(0, t)$, $j = 1, \ldots, J$, and set $R = 2D$.

First, for any $y \in \gamma$ such that $0 \in V^k(y, \gamma)$ there exists $j \in \{1, \ldots, J\}$ such that $y \in K_j \cap B(0, D)$. Otherwise, y would have at least $k + 1$ points that are closer to y than the origin, by the construction of D, and that contradicts $0 \in V^k(y, \gamma)$. In addition, $V^k(y, \gamma) \in B(0, R)$. No point from $B(0, R)^C$ can be among the k nearest neighbours of y, because $y \in K_j \cap B(0, D)$ implies that there are at least k points closer to y than a potential neighbour outside $B(0,R)$. We conclude that R is a decreasing range of interaction for the Papangelou intensity λ^* as in Definition 3.2.

By Proposition 4.2 and Proposition 3.2, there is a Poisson point process Π with intensity $\lambda := ze^{-\beta(1+2N_d)kN_d\|\Phi\|_\infty}$ such that $\Pi \ll \Gamma$. By Corollary 3.1, for any $t > 0$,

$$\begin{aligned}\mathbb{P}(D(\Gamma) > t) \leq \mathbb{P}(D(\Pi) > t) &\leq \sum_{j=1}^{J} \mathbb{P}(\#(\Pi \cap T_j \cap B(0, t)) \leq k) \\ &= \sum_{j=1}^{J} \sum_{i=1}^{k} \frac{(\lambda |T_j \cap B(0, t)|)^i}{i!} e^{-\lambda |T_j \cap B(0,1)| t^d}.\end{aligned}$$

Let $\alpha > d$. Then

$$\begin{aligned}\mathbb{E}R(\Gamma)^\alpha \leq \mathbb{E}R(\Pi)^\alpha &= \int_0^\infty \mathbb{P}(R(\Pi)^\alpha > t) \mathrm{d}\, t \\ &= \int_0^\infty \mathbb{P}((2D(\Pi))^\alpha > t) \mathrm{d}\, t \\ &= \int_0^\infty 2^\alpha u^{\alpha-1} \mathbb{P}(D(\Pi) > u) \mathrm{d}\, u \\ &\leq \sum_{j=1}^{J} \sum_{i=1}^{k} \frac{2^\alpha}{i!} \int_0^\infty u^{\alpha-1} (\lambda |T_j \cap B(0, u)|)^i e^{-\lambda |T_j \cap B(0,1)| u^d} \mathrm{d}\, u.\end{aligned}$$

The latter term is finite, as it is a finite sum of converging integrals. Thus, Assumption (A2) of Theorem 2.2 is satisfied for $\alpha_2 = \alpha/d$ by Proposition 3.1(iv). □

Next we prove Corollary 4.5.

Proof. Since $\Phi \geq 0$ everywhere, the Papangelou intensity is locally stable from above; hence Assumption (A1) of Theorem 2.2 is trivially satisfied by Proposition 3.1(i). For (A2), an easy computation leads to

$$\frac{\lambda^*(0,\gamma\cup\{y\})}{\lambda^*(0,\gamma)} = \exp\Bigg\{-\beta\Big[\Phi(y)-\Phi(v^{k+1}(0,\gamma\cup\{0,y\}))\Big]\mathbf{1}\{y\in V^k(y,\gamma\cup\{0,y\})\}$$
$$-\beta\Big[\Phi(y)-\Phi(y-v^{k+1}(y,\gamma\cup\{0,y\}))\Big]\mathbf{1}\{0\in V^k(y,\gamma\cup\{0,y\})\}$$
$$+\beta\sum_{\substack{x\in\gamma\\ y\in V^k(x,\gamma\cup\{0,y\})\\ 0=v^{k+1}(x,\gamma\cup\{0,y\})}}\Big[\Phi(x)-\Phi(x-v^{k+2}(x,\gamma\cup\{0,y\}))\Big]\Bigg\}$$
$$\le \exp\left\{+\beta\sum_{\substack{x\in\gamma\\ y\in V^k(x,\gamma\cup\{0,y\})\\ 0=v^{k+1}(x,\gamma\cup\{0,y\})}}\Phi(x)\right\},$$

where we used the fact that $\Phi(x)$ is non-negative and decreasing. Let $\delta>0$ and $\zeta<\infty$ be such that $\Phi(x)<\zeta$ whenever $\|x\|\ge\delta/2$. Then, for $\alpha_2>1$,

$$D := \int_{\mathbb{R}^d\setminus B(0,\delta)}\left(\mathbb{E}\left|1-\frac{\lambda^*(0,\Gamma'\cup\{y\})}{\lambda^*(0,\Gamma')}\right|^{\alpha_2}\right)^{1/\alpha_2}\mathrm{d}y$$
$$\le \int_{\mathbb{R}^d\setminus B(0,\delta)}\left[\mathbb{E}\left(\exp\left\{\beta\sum_{\substack{x\in\gamma\\ y\in V^k(x,\Gamma\cup\{0,y\})\\ 0=v^{k+1}(x,\Gamma\cup\{0,y\})}}\Phi(x)\right\}-1\right)^{\alpha_2}\right]^{1/\alpha_2}\mathrm{d}y$$
$$\le \int_{\mathbb{R}^d\setminus B(0,\delta)}\Big[\mathbb{E}\exp\{\beta\alpha_2N_d\zeta\}$$
$$\cdot\,\mathbf{1}\{\exists x\in\Gamma;\, y\in V^k(x,\Gamma\cup\{0,y\}),\, 0=v^{k+1}(x,\Gamma\cup\{0,y\})\}\Big]^{1/\alpha_2}\mathrm{d}y$$
$$= A\int_{\mathbb{R}^d\setminus B(0,\delta)}\Big(\mathbb{P}(\exists x\in\Gamma;\, y\in V^k(x,\Gamma\cup\{0,y\}),\, 0=v^{k+1}(x,\Gamma\cup\{0,y\}))\Big)^{1/\alpha_2}\mathrm{d}y,$$

where $A:=\exp\{\beta N_d\zeta\}<\infty$. In the third line of the latter expression, we used that $\|x\|\ge\|y\|/2\ge\delta/2$. Otherwise, $\|x-0\|\le\|x-y\|$ implies that if $0=v^{k+1}(x,\Gamma\cup\{0,y\})$, then y cannot be among the k nearest neighbours of x.

Again, we let J be the smallest integer such that $K_1,\ldots,K_J$ are cones with apex at the origin and angular radius at most $\pi/6$ such that $\cup_{j=1}^J K_j=\mathbb{R}^d$. For $\gamma\in\mathbf{N}$, we define $D:=D(\gamma)$ as the smallest $t>0$ such that there are at least $k+1$ points of γ in each $K_j\cap B(0,t)$, $j=1,\ldots,J$, and set $R=2D$. Then

$$\mathbb{P}(\exists x\in\Gamma;\, y\in V^k(x,\Gamma\cup\{0,y\}),\, 0=v^{k+1}(x,\Gamma\cup\{0,y\}))\le\mathbb{P}(\|y\|\le R).$$

If $\|y\|>R$, then any point $x\in\gamma$ with $y\in V^k(x,\gamma\cup\{0,y\})$ and $0=v^{k+1}(x,\gamma\cup\{0,y\})$ satisfies $x\in B(0,D)^C$. But then there is $j\in\{1,\ldots,J\}$ with $x\in K_j$ such thatat least $k+1$ points in the cone K_j are closer to x than the origin, contradicting the fact that $0=v^{k+1}(x,\gamma\cup\{0,y\})$.

We are now in a position to apply Corollary 3.2. It can be seen from (4.9) that $\lambda^*(x, \gamma) \le z$. This is due to the fact that Φ is non-negative and decreasing. Furthermore, if $d(x, \gamma) \ge \delta/2$, then $\lambda^*(x, \gamma) \ge z \exp\{-(1 + N_d)k\zeta\}$. Here, we also used the fact that the number of summands is bounded by a deterministic constant (see Proposition 4.2). By Proposition 3.3, for every $\varepsilon > 0$ there is some $p > 0$ such that $I^{2\delta+\varepsilon}(\Gamma)$ is minorated by a Bernoulli field B with parameter p.

For $l \in \mathbb{L}^d := \{0, 1\}^{\mathbb{Z}^d}$, we define $R'(l)$ by taking the smallest $t > 0$ such that, for all $j = 1, \ldots, J$, the cone K_j fully contains at least $k + 1$ cubes $\mathcal{D}_{i_1}, \ldots, \mathcal{D}_{i_{k+1}} \in \mathcal{D}$ such that $l_{i_1}, \ldots, l_{i_{k+1}} = 1$. Then R' is decreasing. By the construction of $I^{2\delta+\varepsilon}(\Gamma)$ and Proposition 3.3 we have that, a.s.,

$$R(\Gamma) \le R'(I^{2\delta+\varepsilon}(\Gamma)) \le R'(B),$$

and hence, by Corollary 3.2,

$$\mathbb{P}(R(\Gamma) > r) \le \mathbb{P}(R'(I^{2\delta+\varepsilon}(\Gamma)) > r) \le \mathbb{P}(R'(B) > r).$$

For $j \in \{1, \ldots, J\}$, take the cube $\mathcal{D}_{i_1} \subset K_j$ that is closest to the origin, and denote by $c_j := \inf_{x \in \mathcal{D}_{i_1}} d(x, 0)$ its distance to the origin. Then there exists another cube $\mathcal{D}_{i_2} \subset K_j$ sharing exactly one vertex with $\mathcal{D}_{i_1}$ at a distance $c_1 + c_2$ from the origin, where c_2 is the body diagonal length of $\mathcal{D}_{i_1}$. Note that c_1^j and c_2 depend on d, δ, and ε, and c_1^j moreover depends on j. Inductively, we construct a chain of cubes $\mathcal{D}_{i_j}$, $j \in \mathbb{N}$, all fully included in the cone K_j. Let $B = (B_i)_{i \in \mathbb{Z}^d}$ be distributed according to $B(p)^{\otimes \mathbb{Z}^d}$. Define $R^j_{diag}(B)$ as the smallest $t = c_1^j + c_2 q$ such that $\sum_{j=1}^q B_{i_j} = k + 1$. To simplify the notation, for the rest of the proof let C be a universal finite constant depending on d, k, δ, ε, α_2, c_2, and c_1^j, $j = 1, \ldots, J$. It can be seen that

$$\begin{aligned}
\mathbb{P}(R'(B) > r) &\le \sum_{j=1}^{J} \mathbb{P}(R^j_{diag}(B) > r) \\
&= \sum_{j=1}^{J} \mathbb{P}\Big(\sum_{j=1}^{\lceil \frac{r - c_1^j}{c_2} \rceil} B_{i_j} \le k \Big) \\
&= \sum_{j=1}^{J} \sum_{n=1}^{k} \binom{\lceil \frac{r - c_1^j}{c_2} \rceil}{n} p^n (1-p)^{\lceil \frac{r - c_1^j}{c_2} \rceil - n} \\
&\le C r^k (1-p)^r.
\end{aligned}$$

Finally, since $p \in (0, 1)$,

$$\begin{aligned}
D &\le C \int_{\mathbb{R}^d \setminus B(0,\delta)} (\mathbb{P}(R'(B) \ge \|y\|))^{1/\alpha_2} \mathrm{d}\, y \\
&\le C \int_{\mathbb{R}^d} \|y\|^{(k/\alpha_2)} ((1-p)^{1/\alpha_2})^{\|y\|} \mathrm{d}\, y < \infty.
\end{aligned}$$

□

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References

[1] BENEŠ, V., HOFER-TEMMEL, C., LAST, G. AND VEČEŘA, J. (2020). Decorrelation of a class of Gibbs particle processes and asymptotic properties of U-statistics. *J. Appl. Prob.* **57**, 928–955.

[2] BJÖRKLUND, M. AND HARTNICK, T. (2023). Hyperuniformity and non-hyperuniformity of quasicrystals. *Math. Ann.* Available at https://doi.org/10.1007/s00208-023-02647-1.

[3] BŁASZCZYSZYN, B., YOGESHWARAN, D. AND YUKICH, J. E. (2019). Limit theory for geometric statistics of point processes having fast decay of correlations. *Ann. Prob.* **47**, 835–895.

[4] CHATTERJEE, S. (2019). Rigidity of the three-dimensional hierarchical Coulomb gas. *Prob. Theory Relat. Fields* **175**, 1123–1176.

[5] DALEY, D. J. AND VERE-JONES, D. (2008). *An Introduction to the Theory of Point Processes*, Vol. **II**, *General Theory and Structure*, 2nd edn. Springer, New York.

[6] DEREUDRE, D. (2009). The existence of quermass-interaction process for nonlocally stable interaction and nonbounded convex grains. *Adv. Appl. Prob.* **41**, 664–681.

[7] DEREUDRE, D. (2019). Introduction to the theory of Gibbs point processes. In *Stochastic Geometry*, ed. D. Coupier, Springer, Cham, pp. 181–229.

[8] DEREUDRE, D., DROUILHET, R. AND GEORGII, H.-O. (2012). Existence of Gibbsian point processes with geometry-dependent interactions. *Prob. Theory Relat. Fields* **153**, 643–670.

[9] DEREUDRE, D. AND VASSEUR, T. (2020). Existence of Gibbs point processes with stable infinite range interaction. *J. Appl. Prob.* **57**, 775–791.

[10] GEORGII, H.-O. AND KÜNETH, T. (1997). Stochastic comparison of point random fields. *J. Appl. Prob.* **34**, 868–881.

[11] GINIBRE, J. (1967). Rigorous lower bound on the compressibility of a classical system. *Phys. Lett. A* **24**, 223–224.

[12] HAWAT, D., GAUTIER, G., BARDENET, R. AND LACHIÈZE-REY, R. (2023). On estimating the structure factor of a point process, with applications to hyperuniformity. *Statist. Comput.* 33, article no. 61.

[13] HILHORST, H.-J. (2005). Asymptotic statistics of the n-sided planar Poisson–Voronoi cell: I. Exact results. *J. Statist. Mech.* **45**, 127–159.

[14] KLATT, M. A., LAST, G. AND HENZE, N. (2024). *A genuine test for hyperuniformity*. Preprint. Available at https://arxiv.org/abs/2210.12790.

[15] KUNZ, H. (1974). The one-dimensional classical electron gas. *Ann. Phys.* **84**, 303–335.

[16] LEBLÉ, T. (2024). *The two-dimensional one-component plasma is hyperuniform*. Preprint. Available at https://arxiv.org/abs/2104.05109.

[17] LEBOWITZ, J. L. (1983). Charge fluctuations in Coulomb systems. *Phys. Rev. A* **27**, 1491–1494.

[18] MARTIN, P. A. AND YALCIN, T. (1980). The charge fluctuations in classical Coulomb systems. *J. Statist. Phys.* **22**, 435–463.

[19] METHA, M. L. (1991). *Random Matrices*. Academic Press, New York.

[20] Ouz, E. C., Socolar, J. E. S., Steinhardt, P. J. and Torquato, S. (2017). Hyperuniformity of quasicrystals. *Phys. Rev. B* 95, article no. 054119.

[21] PEEBLES, P. J. E. (1993). *Principles of Physical Cosmology*. Princeton University Press.

[22] PENROSE, M. D. AND YUKICH, J. E. (2001). Central limit theorems for some graphs in computational geometry. *Ann. Appl. Prob.* **11**, 1005–1041.

[23] PENROSE, M. (2003). *Random Geometric Graphs*. Oxford University Press.

[24] PENROSE, M. (2007). Laws of large numbers in stochastic geometry with statistical applications. *Bernoulli* **14**, 1124–1150.

[25] RUELLE, D. (1970). Superstable interactions in classical statistical mechanics. *Commun. Math. Phys.* **18**, 127–159.

[26] SCHNEIDER, R. AND WEIL, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin.

[27] SCHREIBER, T. AND YUKICH, J. E. (2013). Limit theorems for geometric functionals of Gibbs point processes. *Ann. Inst. H. Poincaré Prob. Statist.* **49**, 1158–1182.
[28] TORQUATO, S. (2018). Hyperuniform states of matter. *Phys. Reports* **745**, 1–95.
[29] TORQUATO, S. AND STILLINGER, F. H. (2003). Local density fluctuations, hyperuniform systems, and order metrics. *Phys. Rev. E* 68, article no. 041113.
[30] XIA, A. AND YUKICH, J. E. (2015). Normal approximation for statistics of Gibbsian input in geometric probability. *Adv. Appl. Prob.* **47**, 934–972.