MOMENT PROBLEMS AND QUASI-HAUSDORFF TRANSFORMATIONS

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1. Introduction. The sequence to sequence quasi-Hausdorff transformations were defined by Hardy [1] 11.19 p. 277 as follows. For a given sequence $\{\mu_n\}$ $(n \geq 0)$ of real or complex numbers, define the operator Δ by $\Delta^o \mu_n = \mu_n, \Delta\mu_n = \mu_n - \mu_{n+1}, \Delta^k = \Delta(\Delta^{k-1})$ for $k > 1$. $\{t_m\}$ $(m \geq 0)$ is called the sequence to sequence quasi-Hausdorff transform by means of $\{\mu_n\}$ (or, in short, the $[QH, \mu_n]$ transform) of $\{s_n\}$ $(n \geq 0)$ if $t_m = \sum_{n=m}^{\infty} \binom{n}{m} \Delta^{n-m} \mu_m s_n$, $m \geq 0$, provided that the sums on the right-hand side converge for all $m \geq 0$. Ramanujan in [11] and [12] has defined the series to series quasi-Hausdorff transformations and has proved necessary and sufficient conditions for the regularity of the two kinds of transformations.

It is our purpose to generalize the quasi-Hausdorff transformations by an idea similar to the one used by Jakimovski [3] p. 17 to define the generalized Hausdorff transformations. In 3 we shall bring necessary and sufficient conditions in order that the generalized quasi-Hausdorff transformation is conservative or regular. In 5 we will deal with some moment problems, the solutions of which are connected with the quasi-Hausdorff transformations. We will obtain necessary and sufficient conditions on a sequence $\{\mu_n\}$ $(n \geq 0)$ in order that it has the representation $\mu_n = \int_o^1 t^{\lambda_n} d\alpha(t)$, $n = 0, 1, 2, \ldots,$ where $\alpha(t)$ is of bounded variation in $[0, 1]$ or $\mu_n = \int_o^1 t^{\lambda_n} f(t)dt$, $n = 0, 1, 2, \ldots,$ where $f(t)$ belongs to a prescribed set of functions. Finally I would like to thank the referee for shortening the proofs of Theorems 3.1 and 5.1.

2. Definitions. Let the sequence $\{\lambda_i\}$ $(i \geq 0)$ satisfy the following properties

(2.1) $$0 \leq \lambda_o < \lambda_1 < \ldots < \lambda_n < \ldots \nearrow \infty, \quad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty .$$

We will obtain now the general form of the transformations of the form $t_m = \sum_{n=m}^{\infty} \lambda^*_{nm} s_n$ which commute with the transformation $t_m = \lambda_m (s_m - s_{m+1})$, this is, what are the λ^*_{nm}'s which satisfy the system of equations

$$\lambda_m \left[\sum_{n=m}^{\infty} \lambda^*_{nm} s_n - \sum_{n=m+1}^{\infty} \lambda^*_{n, m+1} s_n \right] = \sum_{n=m}^{\infty} \lambda^*_{nm} \lambda_n (s_n - s_{n+1}),$$

$m = 0, 1, 2, \ldots$.

A formal solution of these equations yields the following. Let $\{\mu_n\}$ be an arbitrary sequence of real or complex numbers and define $\lambda^*_{nn} = \mu_n$, $n \geq 0$, then

(2.2) $\lambda^*_{nm} = (-1)^{n-m} \lambda_m \ldots \lambda_{n-1} [\mu_m, \ldots, \mu_n]$, $0 \leq m \leq n = 0, 1, 2, \ldots,$

where

(2.3) $[\mu_m, \ldots, \mu_n] = \sum_{i=m}^{n} \mu_i / w'_{nm} (\lambda_i)$, $\qquad 0 \leq m \leq n = 0, 1, 2, \ldots,$

where $w_{nm}(x) = (x - \lambda_m) \ldots (x - \lambda_n)$, $\qquad 0 \leq m \leq n = 0, 1, 2, \ldots$.

For a given sequence $\{\mu_n\}$ $(n \geq 0)$ of real or complex numbers, $\{t_m\}$ $(m \geq 0)$ is called the generalised sequence to sequence quasi-Hausdorff transform (or, in short, the $[QH, \mu_n; \lambda_n]$ transform) of $\{s_n\}$ $(n \geq 0)$ if

(2.4) $$t_m = \sum_{n=m}^{\infty} \lambda^*_{nm} \quad , \; m \geq 0,$$

(where the λ^*_{nm}'s are defined by (2.2)) provided that the sums on the right-hand side of equation (2.4) exist.

For the sequence $\lambda_n = n$, $n \geq 0$, the $[QH, \mu_n; \lambda_n]$ transform is the known $[QH, \mu_n]$ transform.

For a sequence $\{\mu_n\}$ $(n \geq 0)$, the series $\sum_{n=0}^{\infty} b_m$ is called the generalized series to series quasi-Hausdorff transform by means of

$\{\mu_n\}$ of the series $\sum_{n=0}^{\infty} a_n$ if

(2.5) $$b_m = \sum_{n=m}^{\infty} \lambda_{nm} a_n \quad , m \geq 0,$$

where

(2.6) $$\lambda_{nm} = (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n [\mu_m, \ldots, \mu_n] \quad 0 \leq m \leq n = 0, 1, 2, \ldots$$

$$(\lambda_{nn} = \mu_n \quad n = 0, 1, 2, \ldots),$$

provided that the sums on the right hand side of equation (2.5) exist.

This transform with the $\{\mu_n\}$ $(n \geq 0)$ preassumed to have the representation $\mu_n = \int_0^1 t^{\lambda_n} d\alpha(t)$ $n = 0, 1, 2, \ldots$ where $\alpha(t)$ is of bounded variation was discussed by Jakimovski and the author in [4].

For the sequence $\lambda_n = n$, $n \geq 0$, this transform is the series to series quasi-Hausdorff transform defined by Ramanujan [11].

3. Regularity of the transformations.

THEOREM 3.1. *The sequence* $\{\mu_n\}$ $(n \geq 0)$ *posesses the representation*

(3.1) $$\mu_n = \int_0^1 t^{\lambda_n} d\alpha(t) \qquad n = 0, 1, 2, \ldots$$

where $\alpha(t)$ *is of bounded variation in* $[0, 1]$, *if, and only if*

(3.2) $$\sup_{m \geq 0} \sum_{n=m}^{\infty} |\lambda^*_{nm}| \equiv H < \infty$$

Proof. Suppose, first, that $\{\mu_n\}$ $(n \geq 0)$ possesses the representation (3.1). For n, m, $0 \leq m \leq n = 0, 1, 2, \ldots$ and $0 \leq t \leq 1$ we have $(-1)^{n-m}[t^{\lambda_m}, \ldots, t^{\lambda_n}] \geq 0$ (see [9] p. 46(10)), ($[t^{\lambda_m}, \ldots, t^{\lambda_n}]$ is given by (2.3) for $\mu_n = t^{\lambda_n}$, $n \geq 0$) and by [5] Theorem 2.3 we have $\sum_{n=m}^{\infty} (-1)^{n-m} \lambda_m \ldots \lambda_{n-1} [t^{\lambda_m}, \ldots, t^{\lambda_n}] \leq 1$ for $m \geq 0$.

Hence

$$\sum_{n=m}^{\infty} |\lambda^*_{nm}| \leq \sum_{n=m}^{\infty} \int_o^1 (-1)^{n-m} \lambda_m \cdots \lambda_{n-1} [t^{\lambda_m}, \ldots, t^{\lambda_n}] |d\alpha(t)|$$

$$= \int_o^1 [\sum_{n=m}^{\infty} (-1)^{n-m} \lambda_m \cdots \lambda_n [t^{\lambda_m}, \ldots, t^{\lambda_n}]] |d\alpha(t)|$$

$$\leq \int_o^1 |d\alpha(t)| < \infty .$$

Conversely, suppose first that $\lambda_o > 0$. Then (3.2) implies

$$\sum_{n=m}^{\infty} \frac{1}{\lambda_n} |\lambda_{nm}| \leq H/\lambda_m \qquad m \geq 0.$$

Thus for $N \geq 0$ it follows that

$$H \cdot \sum_{m=o}^{N} \frac{1}{\lambda_m} \geq \sum_{m=o}^{N} \sum_{n=m}^{\infty} \frac{1}{\lambda_n} |\lambda_{nm}|$$

$$\geq \sum_{m=o}^{N} \sum_{n=m}^{N} \frac{1}{\lambda_n} |\lambda_{nm}|$$

$$= \sum_{n=o}^{N} \frac{1}{\lambda_n} \sum_{m=o}^{n} |\lambda_{nm}| .$$

Hence

$$\sum_{n=o}^{N} \left(\frac{1/\lambda_n}{\sum_{m=0}^{N} 1/\lambda_m} \right) \sum_{m=o}^{n} |\lambda_{nm}| \leq H \qquad \text{for } N \geq 0 .$$

Since $\sum_{m=1}^{\infty} 1/\lambda_m = \infty$ it follows that there exists an infinite subsequence $\{n_i\}$ $(i \geq 0)$ such that

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(3.3) $$\sup_{i \geq o} \sum_{m=o}^{n_i} |\lambda_{n_i, m}| \equiv K < \infty .$$

It is readily seen by

$$\lambda_n \cdot \lambda_{n-1,m} = (\lambda_n - \lambda_m)\lambda_{nm} + \lambda_{m+1} \cdot \lambda_{n,m+1} \qquad 0 \leq m < n$$

that $|\lambda_{n-1,m}| \leq |\lambda_{nm}| + |\lambda_{n,m+1}| \qquad 0 \leq m < n$

and thus (3.3) implies

(3.4) $$\sup_{n \geq o} \sum_{m=o}^{n} |\lambda_{nm}| \equiv K < \infty .$$

By Theorem 2.1 of [8], $\{\mu_n\}$ $(n \geq 0)$ possesses the representation (3.1). If $\lambda_o = 0$ exactly the same proof yields the result

(3.5) $$\mu_n = \int_o^1 t^{\lambda_n} d\alpha(t) \qquad n \geq 1 .$$

Let $\beta(t) = \begin{cases} \alpha(t) + \mu_o - \alpha(1) & 0 < t \leq 1 \\ 0 & t = 0 \end{cases}$

then by (3.5)

$$\mu_n = \int_o^1 t^{\lambda_n} d\beta(t) \qquad n = 0, 1, 2, \ldots .$$

This completes the proof of Theorem 3.1.

THEOREM 3.2. <u>The sequence to sequence</u> $[QH, \mu_n ; \lambda_n]$ <u>transformation is conservative if and only if</u> $\{\mu_n\}$ $(n \geq 0)$ <u>possesses the representation</u> (3.1). <u>It is regular, if and only if, in addition,</u> $\alpha(1) - \alpha(0+) = 1$.

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Proof. If the $[QH, \mu_n; \lambda_n]$ transformation is conservative, then by the well known Toeplitz Theorem, (3.2) holds and hence by Theorem 3.1, $\{\mu_n\}$ $(n \geq 0)$ possesses the representation (3.1). Conversely, suppose (3.1) holds, then by Toeplitz Theorem in order that the $[QH, \mu_n; \lambda_n]$ transformation is conservative we have to prove that (3.2) holds and that $\lim_{m\to\infty} \sum_{n=m}^{\infty} \lambda^*_{nm}$ exists (the third condition is trivially fulfilled since the $[QH, \mu_n; \lambda_n]$ transformation is defined by an upper triangular matrix). Now, (3.2) holds by Theorem 3.1 and

$$\sum_{n=m}^{\infty} \lambda^*_{nm} = \sum_{n=m}^{\infty} \int_0^1 (-1)^{n-m} \lambda_m \cdot \ldots \cdot \lambda_{n-1} [t^{\lambda_m}, \ldots, t^{\lambda_n}] d\alpha(t)$$

(by Lebesgue Theorem on dominated convergence)

$$= \int_0^1 \left[\sum_{n=m}^{\infty} (-1)^{n-m} \lambda_m \cdot \ldots \cdot \lambda_{n-1} [t^{\lambda_m}, \ldots, t^{\lambda_n}] \right] d\alpha(t)$$

$$= \int_{0+}^1 d\alpha(t) = \alpha(1) - \alpha(0+),$$

since by [5] Theorem 2.3

$$\sum_{n=m}^{\infty} (-1)^{n-m} \lambda_m \cdot \ldots \cdot \lambda_{n-1} [t^{\lambda_m}, \ldots, t^{\lambda_n}] = \begin{cases} 1 & \text{for } 0 < t \leq 1. \\ 0 & \text{for } t = 0 \end{cases}$$

In order that the $[QH, \mu_n; \lambda_n]$ transformation is regular it is necessary and sufficient that $\lim_{m\to\infty} \sum_{n=m}^{\infty} \lambda^*_{nm} = 1$ or in other words $\alpha(1) - \alpha(0+) = 1$ in addition to the other properties. This completes our proof.

We have similar results for the series to series quasi-Hausdorff transform, namely

THEOREM 3.3. Suppose that $\lambda_0 = 0$. The series to series quasi-Hausdorff transformation is conservative if and only if $\{\mu_n\}$ $(n \geq 0)$ possesses the representation (3.1). It is regular, if and only if, in addition, $\alpha(1) - \alpha(0) = 1$.

Proof. Necessary and sufficient conditions in order that the series to series generalized quasi-Hausdorff transformation is

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conservative are by Vermes's theorem (see [12] Lemmas 2, 3)

(3.6) $$\sup_{k\geq 0} \sum_{n=0}^{\infty} \left| \sum_{m=0}^{k} (\lambda_{nm}-\lambda_{n+1,m}) \right| \equiv H < \infty$$

(for $n < m$, $\lambda_{nm}=0$)

(3.7) $$\lim_{k\to\infty} \sum_{m=0}^{k} \lambda_{nm} \quad \text{exists for } n = 0, 1, 2, \ldots .$$

It is regular if, and only if, in addition,

(3.8) $$\lim_{k\to\infty} \sum_{m=0}^{k} \lambda_{nm} = 1, \qquad n = 0, 1, 2, \ldots .$$

By Hausdorff [2] (16)

$$\sum_{m=0}^{k} (\lambda_{nm}-\lambda_{n+1,m}) = \lambda_{k+1} \cdot \lambda_{n+1,k+1}/\lambda_{n+1} = \lambda^*_{n+1,\ k+1},$$

hence condition (3.6) is condition (3.2).

Suppose, first, that the transformation is conservative, then by Theorem 3.1 $\{\mu_n\}$ $(n \geq 0)$ possesses the representation (3.1). Conversely, if (3.1) holds, then by Theorem 3.1 we get the conclusion that (3.6) holds. Moreover, by [2] (7) $\lim_{k\to\infty} \sum_{m=0}^{k} \lambda_{nm} = \mu_o$, hence (3.7) is satisfied.

We have regularity if, and only if, in addition,

$$\mu_o = \int_o^1 d\alpha(t) = 1.$$

This completes the proof.

4. Miscellaneous results.

LEMMA 4.1. *For any two sequences* $\{\mu_n\}$ $(n \geq 0)$, $\{\nu_n\}$ $(n \geq 0)$ *we have for* $0 \leq m \leq n = 0, 1, 2, \ldots$

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$$(4.1)\quad [\mu_m\nu_m, \ldots, \mu_n\nu_n] = \sum_{k=m}^{n} [\mu_m, \ldots, \mu_k]\,[\nu_k, \ldots, \nu_n]\ .$$

Proof. We prove (4.1) by induction on $n \geq m$. For $n = m$, (4.1) is trivially satisfied. Suppose (4.1) is true for n and we shall prove it for $n+1$. By (2.3) it is easily proved that

$$[\mu_m\nu_m, \ldots, \mu_{n+1}\nu_{n+1}] = \frac{[\mu_m\nu_m, \ldots, \mu_n\nu_n] - [\mu_m\nu_m, \ldots, \mu_{n-1}\nu_{n-1}, \mu_{n+1}\nu_{n+1}]}{\lambda_n - \lambda_{n+1}}$$

by our assumption

$$\frac{1}{\lambda_n - \lambda_{n+1}} \left[\sum_{k=m}^{n-1} ([\mu_m, \ldots, \mu_k][\nu_k, \ldots, \nu_n] - [\mu_m, \ldots, \mu_k] \right.$$

$$\left. [\nu_k, \ldots, \nu_{n-1}, \nu_{n+1}]) + [\mu_m, \ldots, \mu_n]\,\nu_n - [\mu_m, \ldots, \mu_{n-1}, \mu_{n+1}]\,\nu_{n+1} \right]$$

$$= \sum_{k=m}^{n+1} [\mu_m, \ldots, \mu_k]\,[\nu_k, \ldots, \nu_{n+1}].$$

THEOREM 4.1. Every two conservative sequence to sequence generalized quasi-Hausdorff transformations commute.

Proof. Let $\{s_n\}$ $(n \geq 0)$ be a bounded sequence and let $\{t_m\}$ $(m \geq 0)$ and $\{r_m\}$ $(m \geq 0)$ be the $[QH, \mu_n;\lambda_n]\cdot[QH, \nu_n;\lambda_n]$ and the $[QH, \nu_n;\lambda_n]\cdot[QH, \mu_n;\lambda_n]$ transforms of $\{s_n\}$ $(n \geq 0)$, respectively.

Denote $\lambda^*_{nm}(\mu) = (-1)^{n-m}\lambda_m \cdot \ldots \cdot \lambda_{n-1}\,[\mu_m, \ldots, \mu_n]$ $\quad 0 \leq m \leq n = 0, 1, 2, \ldots$

$\lambda^*_{nm}(\nu) = (-1)^{n-m}\lambda_m \cdot \ldots \cdot \lambda_{n-1}\,[\nu_m, \ldots, \nu_n]$ $\quad 0 \leq m \leq n = 0, 1, 2, \ldots$

We have

$$t_m = \sum_{n=m}^{\infty} \lambda_{nm}(\mu) \sum_{k=n}^{\infty} \lambda_{kn}(\nu)\, s_k$$

and since

$$\sum_{n=m}^{\infty} |\lambda_{nm}(\mu)| \sum_{k=n}^{\infty} |\lambda_{kn}(\nu)|\ |s_k| < \infty$$

we can change order of summation and obtain

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$$t_m = \sum_{k=m}^{\infty} s_k \sum_{n=m}^{k} \lambda_{nm}(\mu)\, \lambda_{kn}(\nu)$$

by Lemma 4.1

$$= \sum_{k=m}^{\infty} s_k (-1)^{k-m} \lambda_m \cdot \ldots \cdot \lambda_{k-1} [\mu_m \nu_m, \ldots, \mu_k \nu_k]$$

and again by Lemma 4.1

$$= \sum_{k=m}^{\infty} s_k \sum_{n=m}^{k} \lambda_{nm}(\nu)\, \lambda_{kn}(\mu)$$

$$= \tau_m .$$

This completes our proof.

5. Moment problems. Let $M(u)$ be an even, convex, continuous function satisfying 1. $M(u)/u \to 0$ as $u \to 0$, 2. $M(u)/u \to \infty$ as $u \to \infty$. Denote by $L_M[0, 1]$ the class of all functions integrable over $[0, 1]$ such that $\int_0^1 M[f(x)]dx < \infty$. $L_M[0, 1]$ is the Orlicz class related to $M(u)$. (For details see [6]). Take $M(u) = |u|^p$, $1 < p < \infty$, then $L_M[0, 1]$ is the space $L^p[0, 1]$. $L_M[0, 1]$ is not necessarily a linear space (see [7] Theorem 8.2).

THEOREM 5.1. The sequence $\{\mu_n\}$ $(n \geq 0)$ possesses the representation

(5.1) $$\mu_n = \int_0^1 t^{\lambda_n} f(t)dt \qquad n = 1, 2, \ldots$$

where $f \in L_M[0, 1]$, if, and only if,

(5.2) $$\sup_{m \geq 1} \sum_{n=m}^{\infty} \left[\int_0^1 (-1)^{n-m} \lambda_m \cdot \ldots \cdot \lambda_{n-1} [t^{\lambda_m}, \ldots, t^{\lambda_n}]dt \right] M\left(\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]dt} \right) \equiv H < \infty .$$

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COROLLARY 5.1. The sequence $\{\mu_n\}$ $(n \geq 0)$ possesses the representation (5.1) where $f \in L^p[0,1]$, if, and only if,

$$(5.3) \qquad \sup_{m \geq 1} \sum_{n=m}^{\infty} \frac{|\lambda^*_{nm}|^p}{\left|\int_0^1 \lambda^*_{nm}(t)dt\right|^{p-1}} \equiv H < \infty,$$

where $\lambda^*_{nm}(t) = (-1)^{n-m}\lambda_m \cdot \ldots \cdot \lambda_{n-1}[t^{\lambda_m}, \ldots, t^{\lambda_n}]$, $0 \leq m \leq n = 0, 1, 2, \ldots$

Corollary 5.1 for $\lambda_n = n$, $n \geq 0$, reduces to Ramanujan's Theorem [13]. Corollary 5.1 for $\lambda_n = n + \alpha$, $\alpha \geq 0$, $n \geq 0$, reduces to Jakimovski and Ramanujan Theorem 7 [6].

Proof of Theorem 5.1. Suppose, first, that $\{\mu_n\}$ $(n \geq 0)$ possesses the representation (5.1). Then

$$[\mu_m, \ldots, \mu_n] = \int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}] f(t)\, dt;$$

hence

$$M\left(\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]dt}\right) = M\left(\frac{\int_0^1 \lambda^*_{nm}(t) f(t)dt}{\int_0^1 \lambda^*_{nm}(t)dt}\right)$$

and as $\lambda^*_{nm}(t) \geq 0$ for $0 \leq m \leq n = 0, 1, 2, \ldots$ and $0 \leq t \leq 1$ (see [10] p. 46(10)) we have by Jensen's inequality (see [14] p.23-24)

$$M\left(\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]dt}\right) \leq \frac{\int_0^1 \lambda^*_{nm}(t)\, M[f(t)]dt}{\int_0^1 \lambda^*_{nm}(t)\, dt}.$$

Hence

$$\sum_{n=m}^{\infty} \left[\int_0^1 \lambda^*_{nm}(t)dt\right] M\left(\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]dt}\right) \leq \sum_{n=m}^{\infty} \int_0^1 \lambda^*_{nm}(t) M[f(t)]dt$$

by Levi's Theorem and since $\sum_{n=m}^{\infty} \lambda^*_{nm}(t) \leq 1$ for $0 \leq t \leq 1$

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(see [5] Theorem 2.3)

$$= \int_0^1 [\sum_{n=m}^{\infty} \lambda^*_{nm}(t)] M[f(t)]dt \leq \int_0^1 M[f(t)]dt < \infty .$$

The proof of the sufficiency runs along the lines of the proof of Theorem 3.1 using the proof of Theorem 1 of [9] after having proved that $\{\mu_n\}$ possesses the representation (3.1). This completes the proof.

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