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PRODUCTS OF NORMAL OPERATORS

PEI YUAN WU

1. Main result. Which bounded linear operator on a complex, separable Hilbert space can be expressed as the product of finitely many normal operators? What is the answer if "normal" is replaced by "Hermitian", "nonnegative" or "positive"? Recall that an operator T is *nonnegative* (resp. *positive*) if $(Tx, x) \geqq 0$ (resp. $(Tx, x) > 0$) for any $x \neq 0$ in the underlying space. The purpose of this paper is to provide complete answers to these questions.

If the space is finite-dimensional, then necessary and sufficient conditions for operators expressible as such are already known. For normal operators, this is easy. By the polar decomposition, every operator is the product of two normal operators. An operator is the product of Hermitian operators if and only if its determinant is real; moreover, in this case, 4 Hermitian operators suffice and 4 is the smallest such number (cf. [**10**]). An operator T is the product of positive (resp. nonnegative) operators if and only if $\det T > 0$ (resp. $\det T \geqq 0$); in this case, 5 positive (resp. nonnegative) operators will do and 5 is the smallest (cf. [**1**] and [**13**]). Thus from now on we will only consider the infinite-dimensional space. For this case, the problems have only been slightly touched upon before. For example, in [**8**, Solution 144 (a)] it was shown that the (simple) unilateral shift is not the product of finitely many normal operators; in [**11**] Radjavi showed that every normal operator is the product of 4 Hermitian operators. Other than these, there seems to be very few in the literature. In this paper, we will completely determine which operators can be expressed as such. It turns out that the classes of operators expressible as products of normal, Hermitian or nonnegative operators are identical. More precisely, we have the following

THEOREM 1.1. *Let T be an operator on a separable, infinite-dimensional Hilbert space. Then the following statements are equivalent*:

(1) *T is the product of finitely many normal operators*;

(2) *T is the product of finitely many Hermitian operators*;

(3) *T is the product of finitely many nonnegative operators*;

(4) *$T = SP$ or PS depending on whether* $\dim \ker T \geqq \dim \ker T^*$ *or* $\dim \ker T \leqq \dim \ker T^*$ *for some operator S which is one-to-one with dense range and some orthogonal projection P*;

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(5) dim ker T = dim ker T^* *or* ran T *is not closed*;

(6) *T is the norm limit of a sequence of invertible operators. Moreover, in this case, T can be expressed as the product of* 3 *normal operators,* 6 *Hermitian operators or* 18 *nonnegative operators.*

The proof of this theorem will be given in the next section. Before that, some corollaries.

COROLLARY 1.2. *Unilateral shifts are not the product of finitely many normal operators.*

COROLLARY 1.3. *Every compact operator is the product of* 3 *normal operators.*

Proof. If T is compact and ran T is closed, then T must be of finite rank whence

$$\dim \ker T = \dim \ker T^* = \infty.$$

Our assertion follows from the equivalence of (1) and (5) in Theorem 1.1 immediately.

COROLLARY 1.4. *Every quasinilpotent operator is the product of* 3 *normal operators.*

Proof. That a quasinilpotent operator is not left (right) Fredholm follows by applying, to the Calkin algebra, the general principle that the boundary of the spectrum of an element in a Banach algebra belongs to the left (right) spectrum (cf. [**2**, p. 13, Theorem 14]). Our assertion then follows from Theorem 1.1.

2. Proof. We first consider the case when the range of T is closed and start with the following

LEMMA 2.1. *T has closed range if and only if* $(T^*T)^{1/2}$ *does.*

Proof. Let

$$\gamma(T) = \inf\{ \|Tx\|: \|x\| = 1, x \perp \ker T\}.$$

Since $\ker T = \ker(T^*T)^{1/2}$ and $\|Tx\| = \|(T^*T)^{1/2}x\|$, we have

$$\gamma(T) = \gamma((T^*T)^{1/2}).$$

The assertion follows from the fact that T has closed range if and only if $\gamma(T) > 0$ (cf. [**4**, Proposition XI.3.16]).

LEMMA 2.2. *If* dim ker T = dim ker T^*, *then T is the product of* 2 *normal operators and the product of* 5 *Hermitian operators. If, in addition, the range of T is closed, then the normal and Hermitian operators may be chosen to have closed ranges too.*

Proof. If dim ker T = dim ker T^*, then $T = UP$, where U is unitary and $P = (T^*T)^{1/2}$ is nonnegative (cf. [**8**, Problem 135]). Since every unitary operator is the product of 4 symmetries (cf. [**8**, Problem 143]; a symmetry is an operator which is both unitary and Hermitian), T is the product of 5 Hermitian operators. The last assertion follows from Lemma 2.1.

LEMMA 2.3. *Let* $T = N_1 \dots N_n$, *where*

$$\dim \ker N_i = \dim \ker N_i^* \quad \textit{for each } i.$$

If T is one-sided invertible (resp. one-sided Fredholm), then T is invertible (resp. Fredholm).

Proof. We only prove for Fredholmness. Assume that T is left Fredholm. From $T = N_1 \dots N_n$, we infer that N_n is left Fredholm. Since dim ker N_n = dim ker N_n^*, N_n must be Fredholm. Let M be an operator such that $N_nM - 1$ and $MN_n - 1$ are compact. Then TM is left Fredholm and

$$TM - N_1 \dots N_{n-1} = N_1 \dots N_{n-1}(N_nM - 1)$$

is compact. Then we repeat the above arguments to obtain that N_{n-1} is Fredholm. By induction, every N_i is Fredholm, so is T. If T is right Fredholm, consider T^* instead.

The next result characterizes operators expressible as the product of a finite number of normal operators among those with closed range.

PROPOSITION 2.4. *The following statements are equivalent for an operator T with* ran T *closed*:

(1) *T is the product of finitely many normal operators*;
(2) *T is the product of finitely many Hermitian operators*;
(3) *$T = SP$, where S is invertible and P is an orthogonal projection*;
(4) dim ker T = dim ker T^*.

Proof. (3) $\Rightarrow$ (2) by Lemma 2.2 and (2) $\Rightarrow$ (1) is trivial. To prove (1) $\Rightarrow$ (4), assume that dim ker $T < \infty$ and $T = N_1 \dots N_n$, where the N_i's are normal. Then T is left Fredholm and hence, together with $N_1, \dots, N_n$, is Fredholm by Lemma 2.3. Thus

$$\text{ind } T = \text{ind } N_1 + \dots + \text{ind } N_n = 0.$$

It follows that dim ker T = dim ker T^*. If dim ker $T^* < \infty$, consider T^* instead.

Now we prove (4) $\Rightarrow$ (3). Let P be the orthogonal projection onto $(\ker T)^\perp$, let R be an invertible operator from ker T onto ker T^*, and define S by

$$S(x + y) = Rx + Ty \quad \text{for } x \in \ker T \text{ and } y \in (\ker T)^\perp.$$

Then S is invertible and $T = SP$.

Before passing on, a few remarks are in order. First, the equivalence of (3) and (4) in the preceding proposition, together with other equivalent conditions, has been obtained before (cf. [**9**, Theorem 3.2]). Second, (1) or (2) may not imply (4) without the assumption on the closedness of ran T. Examples have been given by Radjavi and Williams [**12**, p. 180] and Gray [7]. It is also evident from our main theorem which we are going to take care of next.

LEMMA 2.5. *If* ran T *is not closed, then there exists a closed, infinite-dimensional subspace* K *of* H *such that* $K \cap \text{ran } T = \{0\}$.

Proof. The assertion follows from Dixmier's proof of a result of von Neumann (cf. [**6**, Theorem 3.6]). Indeed, since TH is not closed, there exist unitary operators V and W on H and a dense operator range L which contains a closed, infinite-dimensional subspace, say, M such that $L \cap VL = \{0\}$ and $WTH \subseteq L$. It follows that $L \cap VWTH = \{0\}$ and therefore $(VW)^{-1}L \cap TH = \{0\}$. Hence $K \equiv (VW)^{-1}M$ is a closed, infinite-dimensional subspace satisfying $K \cap TH = \{0\}$.

PROPOSITION 2.6. *If* ran T *is not closed, then* T *is the product of* 3 *normal operators and the product of* 6 *Hermitian operators.*

Proof. In view of Lemma 2.2, we need only consider the case $\dim \ker T \neq \dim \ker T^*$. Since ran T is closed if and only if ran T^* is, we may further assume that $\dim \ker T > \dim \ker T^*$. Let P be the orthogonal projection onto $(\ker T)^{\perp}$. We are going to construct an operator S with $\dim \ker S = \dim \ker S^*$ such that $T = SP$.

Let $\ker T = H_1 \oplus H_2$, where $\dim H_1 = \dim \ker T^*$, and let K be a closed, infinite-dimensional subspace of $\overline{\text{ran } T}$ such that $K \cap \text{ran } T = \{0\}$. Note that ran T is not closed implies that $\overline{\text{ran } T}$ is infinite-dimensional and hence such K exists by Lemma 2.5. Let S be an operator on H which maps H_1 to $\{0\}$, maps H_2 isometrically into K and equals T on $(\ker T)^{\perp}$. We first show that $\ker S = H_1$. Indeed, if

$$S(x_1 + x_2 + x_3) = 0,$$

where $x_1 \in H_1$, $x_2 \in H_2$ and $x_3 \in (\ker T)^{\perp}$, then

$$Sx_2 = -Sx_3 \in K \cap \text{ran } T = \{0\}.$$

Hence $Sx_2 = Sx_3 = 0$ and therefore

$$x_2 = 0 \quad \text{and} \quad x_3 \in (\ker T)^{\perp} \cap \ker T = \{0\},$$

that is, $x_3 = 0$. This shows that $\ker S = H_1$. On the other hand, we certainly have $\overline{\text{ran } S} = \overline{\text{ran } T}$ whence

$$\ker S^* = \overline{\text{ran } S}^{\perp} = \overline{\text{ran } T}^{\perp} = \ker T^*.$$

It follows that

$$\dim \ker S = \dim H_1 = \dim \ker T^* = \dim \ker S^*$$

as asserted.

By Lemma 2.2, S is the product of 2 normal operators and the product of 5 Hermitian operators. Since $T = SP$, T is the product of 3 normal operators and the product of 6 Hermitian operators.

Next we consider the product of nonnegative operators. Since any unitary operator is the product of 4 symmetries (cf. [**8**, Problem 143]), we start with the nonnegative factorization of the latter. The proof of the next lemma depends on Ballantine's results on the product of positive finite matrices [**1**].

LEMMA 2.7. *Every symmetry is the product of* 6 *positive invertible operators.*

Proof. Since any symmetry is unitarily equivalent to $T \equiv (-1) \oplus 1$ on $H \equiv H_1 \oplus H_2$, we distinguish three cases depending on the dimension of H_1:

(1) Dim H_1 is infinite. In this case, consider the operator -1 on H_1 as

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \dots .$$

Since each $\left[\begin{smallmatrix} -1 & 0 \\ -0 & -1 \end{smallmatrix}\right]$ is the product of 5 positive matrices [**1**, Theorem 5], T is the product of 5 positive invertible operators.

(2) Dim H_1 is finite and even. Consider T as

$$\begin{bmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \oplus 1.$$

Since the first summand has a positive determinant and is not a negative scalar matrix, it is the product of 4 positive matrices [**1**, Theorem 4]. The same is true for T.

(3) Dim H_1 is finite and odd. In this case, H_2 must be infinite-dimensional. Let $n = \dim H_1$. We have $T = T_1T_2$, where

$$T_1 = \begin{bmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & -1 & 0 \\ & & & 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\oplus \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \oplus \dots$$

and

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$$T_2 = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & 0 & \\ & & & 1 & 1 & \\ & & & & & -1 \end{bmatrix} \oplus \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\oplus \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \oplus \dots,$$

the first summands of T_1 and T_2 being of sizes $(n + 2) \times (n + 2)$ and $(n + 3) \times (n + 3)$, respectively. By [**1**, Theorem 3], each summand of T_1 and T_2 is the product of 3 positive matrices. Hence T is the product of 6 positive invertible operators.

LEMMA 2.8. *Every unitary operator is the product of* 16 *positive invertible operators.*

Proof. The proof in [**8**, Solution 143] of the Halmos-Kakutani result establishes that every unitary operator is the product of 4 symmetries each of which has 1 and -1 as eigenvalues with infinite multiplicity. From the proof of Lemma 2.7, we deduce that each such symmetry is the product of 4 positive invertible operators. Our assertion follows.

PROPOSITION 2.9. *If* dim ker T = dim ker T^*, *then T is the product of* 17 *nonnegative operators*; *if* ran T *is not closed, then T is the product of* 18 *nonnegative operators.*

Proof. The assertions follow from Lemma 2.8 and the proofs of Lemma 2.2 and Proposition 2.6.

COROLLARY 2.10. *An operator is the product of finitely many positive operators if and only if it is one-to-one with dense range. In this case,* 17 *positive operators suffice.*

Now we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. In view of what we have done so far and the fact that the equivalence of (5) and (6) follows easily from [**5**, Theorem 2] or [**3**, Theorem 3], we need only show that (5) implies (4).

If ran T is not closed, then, as proved in Proposition 2.6, in case of dim ker $T >$ dim ker T^*, $T = T_1P$, where T_1 is such that dim ker T_1 = dim ker T_1^* and ker $T_1 \subseteq$ Ker T and P is the orthogonal projection onto $(\ker T)^{\perp}$. From the proof of Proposition 2.4, we have $T_1 = S_1P_1$, where S_1 is one-to-one with dense range and P_1 is the orthogonal projection onto $(\ker T_1)^{\perp}$. Hence $T = S_1(P_1P) = S_1P$ as required.

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3. Miscellanies. In the previous two sections, we have characterized operators which are expressible as products of normal, Hermitian, nonnegative or positive operators. However, except in some trivial cases it is in general difficult to determine the minimum number of such operators which are required to form the product. For special classes of operators, this probably will be easier to handle. One such result is due to Radjavi [**11**]: Every normal operator is the product of 4 Hermitian operators and 4 is the smallest such number. In this section, we present some results which reduce, for certain classes of operators, the number of normal or whatever operators needed. We start with the finite-rank operators.

PROPOSITION 3.1. *Every finite-rank operator is the product of* 2 *normal operators*, 3 *Hermitian operators or* 4 *nonnegative operators. Moreover, the numbers* 2 *and* 3 *are the smallest possible.*

Proof. Let T be a finite-rank operator. Since rank $T =$ rank $T^* < \infty$ implies that

$$\dim \ker T = \dim \ker T^* = \infty,$$

T is the product of 2 normal operators by Lemma 2.2. Here 2 is obviously the smallest.

For the Hermitian product, we first write T as $T = S \oplus 0$ on $H = K \oplus K$. Then

$$T = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

expresses T as the product of 3 Hermitian operators. (This is due to Radjavi [**11**, p. 1423].) To prove the minimality of 3, let $T = i \oplus 0$ on $H = H_1 \oplus H_2$, where $\dim H_1 = 1$ and $\dim H_2 = \infty$. Since $\sigma(T) = \{i, 0\}$ is not symmetric with respect to the real line, T cannot be the product of 2 Hermitian operators (cf. [**12**, p. 179]).

To prove the assertion for the nonnegative product, write $T = T_1 \oplus 0$ on $H = H_1 \oplus H_2$, where $\dim H_1 < \infty$. Let $T_1 = UP$ be the polar decomposition of T_1, where U is unitary and $P = (T_1^*T_1)^{1/2} \geqq 0$. Let z be a complex number such that $z \cdot \det U > 0$, and let

$$S = U \oplus \begin{bmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \quad \text{and} \quad Q = P \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easily seen that $\det S > 0$ and 0 belongs to the interior of the numerical range of S. By [**1**, Theorem 3], S is the product of 3 positive matrices. Hence $T = (S \oplus 0)(Q \oplus 0)$ is the product of 4 nonnegative operators.

The next proposition deals with nonnegative factorization of Hermitian operators.

PROPOSITION 3.2. *Every Hermitian operator is the product of* 7 *nonnegative operators.*

Proof. Let T be a Hermitian operator. Then T can be written as $T = T_1 \oplus T_2$, where T_1 and $-T_2$ are nonnegative. Hence

$$T = (T_1 \oplus (-T_2))(1 \oplus (-1))$$

is the product of 7 nonnegative operators by Lemma 2.7.

Recall that an operator T is *idempotent* if $T^2 = T$, an *involution* if $T^2 = 1$, and *nilpotent of index* 2 if $T^2 = 0$.

PROPOSITION 3.3. (1) *Every idempotent operator is the product of* 2 *nonnegative operators.*

(2) *Every involution is the product of* 2 *Hermitian invertible operators and the product of* 6 *positive invertible operators.*

(3) *Every nilpotent operator of index* 2 *is the product of* 2 *Hermitian operators and the product of* 3 *nonnegative operators.*

Proof. (1) Let T be an idempotent operator. Then $T = X^{-1}PX$, where X is invertible and P is an orthogonal projection. We have

$$T = (X^{-1}X^{-1*})(X^*PX)$$

as the product of 2 nonnegative operators.

(2) Let T be an involution. Then $T = X^{-1}SX$, where X is invertible and S is an operator of the form $1 \oplus (-1)$. We have

$$T = (X^{-1}X^{-1*})(X^*SX)$$

as the product of 2 Hermitian invertible operators. By Lemma 2.7, S is the product of 6 positive invertible operators, say,

$$S = R_1R_2R_3R_4R_5R_6.$$

Hence

$$T = (X^{-1}R_1X^{-1*})(X^*R_2X)(X^{-1}R_3X^{-1*})(X^*R_4X) \times (X^{-1}R_5X^{-1*})(X^*R_6X)$$

is the product of 6 positive invertible operators.

(3) Let T be nilpotent of index 2. Then T can be expressed as $\left[\begin{smallmatrix} 0 & S \\ 0 & 0 \end{smallmatrix}\right]$ on $\ker T \oplus \overline{\operatorname{ran} T^*}$, and

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix}$$

is a product of 2 Hermitian operators. On the other hand, let $r > \|S\|$, and let

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} r & S \\ S^* & r \end{bmatrix} \quad \text{and} \quad T_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easily verified that $T = T_1T_2T_3$ and these T_j's are nonnegative operators.

We conclude this paper by remarking that, in a forthcoming joint paper with M.-D. Choi, we obtained some necessary or sufficient conditions for an operator expressible as the product of 2 normal operators. In particular, under the conditions of Theorem 1.1 the number of 3 (normal operators) cannot be lowered to 2 in general, thanks to an example pointed out to us by K. Davidson.

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National Chiao Tung University,
Hsinchu, Taiwan