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A PRODUCT FORMULA AND A NON-NEGATIVE POISSON KERNEL FOR RACAH-WILSON POLYNOMIALS

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1. Introduction. Physicists have long been using Racah's [**7**] 6-j symbols as a representation for the addition coefficients of three angular momenta. Racah himself discovered a series representation of the 6-j symbol which can be expressed as a balanced ${}_4F_3$ series of argument 1, that is, a generalized hypergeometric function such that the sum of the 3 denominator parameters exceeds that of the 4 numerator parameters by 1. What Racah does not seem to have realized or, perhaps, cared to investigate, is that his ${}_4F_3$ functions, with variables and parameters suitably identified, form a system of orthogonal polynomials in a discrete variable. The orthogonality of 6-j symbols as an orthogonality of ${}_4F_3$ polynomials was recognized much later by Biedenharn *et al.* [**3**] in some special cases. Recently J. Wilson [**13**, **14**] introduced a very general system of orthogonal polynomials expressible as balanced ${}_4F_3$ functions of argument 1 orthogonal with respect to an absolutely continuous measure and/or a discrete weight function. Wilson's polynomials contain Racah's 6-j symbols as a special case. These polynomials might rightfully be credited to Wilson alone, but justice might be better served if we call them Racah-Wilson polynomials.

In this paper we shall be basically interested in a particular case of these polynomials, namely, the discrete ones which are orthogonal with respect to a positive discrete measure. For a fixed positive integer N let x be an integer-valued variable that can take on values $0, 1, \ldots, N$. In this setting the Racah-Wilson polynomials are defined by

$$(1.1)\qquad W_n(x) \equiv W_n(x; \alpha, \beta, \gamma, N) = {}_4F_3\left[\begin{matrix} -n, & n+\alpha+\beta+1, & -x, & x+\gamma-N \\ & \alpha+1, & -N, & \beta+\gamma+1 \end{matrix}\right],$$

where $n = 0, 1, 2, \ldots, N$. The unit argument of the hypergeometric function is suppressed for convenience. The values of the parameters are such that the ${}_4F_3$ series has finite values for all possible values of x and n. Note that $W_n(x)$ contains the Hahn and dual Hahn [**6**] polynomials in the limits $\gamma \to \infty$ and $|\alpha| \to \infty$, respectively.

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Let us now consider the weight function

$$(1.2)\quad \rho(x;\alpha,\beta,\gamma,N) = \frac{(\alpha+\beta+1)_x\left(\frac{\alpha+\beta+3}{2}\right)_x(\alpha+1)_x(\beta+\gamma+1)_x(-N)_x}{x!\left(\frac{\alpha+\beta+1}{2}\right)_x(\beta+1)_x(\alpha-\gamma+1)_x(N+\alpha+\beta+2)_x}.$$

Using Dougall's well-known formula [**2**, p. 25] for the sum of a very well-poised terminating ${}_5F_4$ series

$$(1.3)\quad {}_5F_4\left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & c, & -N \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a+N & \end{matrix}\right] = \frac{(1+a)_N(1+a-b-c)_N}{(1+a-b)_N(1+a-c)_N},$$

we obtain

$$(1.4)\quad \sum_{x=0}^{N}\rho(x;\alpha,\beta,\gamma,N) = \frac{(\alpha+\beta+2)_N(-\gamma)_N}{(\beta+1)_N(\alpha-\gamma+1)_N}.$$

The non-negativity of the weight $\rho(x)$ is guaranteed in the following cases:

$$(1.5)\quad \begin{aligned} &\text{(i) } N \leqq \gamma,\ -1<\alpha<\gamma-N,\ -1<\beta; \\ &\text{(ii) } \alpha<-N,\ -\gamma<\beta<-N. \end{aligned}$$

In what follows we shall restrict the parameters to either case (i) or (ii). Using the summation formula (1.3) we also have

$$(1.6)\quad \sum_{x=0}^{N}\rho(x;\alpha,\beta,\gamma,N) = \sum_{n=0}^{N}\rho(n;\alpha,\gamma-N-\alpha-1,N+\alpha+\beta+1,N).$$

Wilson [**14**] showed that

$$(1.7)\quad \sum_{x=0}^{N}\rho(x;\alpha,\gamma-N-\alpha-1,N+\alpha+\beta+1,N)W_m(x)W_n(x) = \frac{(\alpha+\beta+2)_N(-\gamma)_N}{(\beta+1)_N(\alpha-\gamma+1)_N}\times\frac{\delta_{mn}}{\rho(n;\alpha,\beta,\gamma,N)},$$

and

$$(1.8)\quad \sum_{n=0}^{N}\rho(n;\alpha,\beta,\gamma,N)W_n(x)W_n(y) = \frac{(\alpha+\beta+2)_N(-\gamma)_N}{(\beta+1)_N(\alpha-\gamma+1)_N} \times\frac{\delta_{xy}}{\rho(x;\alpha,\gamma-N-\alpha-1,N+\alpha+\beta+1,N)}.$$

It is obvious that $W_n(x)$ is self-dual in the special case $\gamma = N + \alpha + \beta + 1$. In § 2 we prove the following:

THEOREM 1. *Let n, x be non-negative integers and $\alpha, \beta, \gamma, \gamma', M, M', y$ be arbitrary complex numbers such that* $\max(n, x) \leqq M$ *when M is a positive integer, and* $\max(n, y) \leqq M'$ *when M' and y are positive integers. We also assume that $\alpha, \beta, \gamma, \gamma'$ do not take such values as to produce singularities in $W_n(x; \alpha, \beta, \gamma, M)$ and $W_n(y; \alpha, \beta, \gamma', M')$. Then*

$$(1.9)\quad W_n(x; \alpha, \beta, \gamma, M) W_n(y; \alpha, \beta, \gamma', M') = \frac{(\beta+1)_n(\alpha-\gamma'+1)_n}{(\alpha+1)_n(\beta+\gamma'+1)_n}$$

$$\times \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(n+\alpha+\beta+1)_{r+s}(-x-y-\gamma')_{r+s}}{r!s!(-M)_{r+s}(-M')_{r+s}(\alpha-\gamma'+1)_{r+s}}$$

$$\times \frac{(-x)_r(-y)_r(x+\gamma-M)_r(M'+\alpha+1-\gamma'-y)_r(x-M)_s(y-M')_s}{(\alpha+1)_r(\beta+\gamma+1)_r(-x-y-\gamma')_r(\beta+1)_s}$$

$$\times {}_4F_3\left[\begin{matrix} -s, & r-x, & r-y+\gamma-\gamma', M+\beta+1-x \\ & r-x-y-\gamma', M+1-x-s, & \beta+\gamma+1+r \end{matrix}\right].$$

This result may be seen as a generalization of Gasper's formula [**4**] for the product of two Hahn polynomials which itself is an extension of Watson's well-known formula [**12**] for the product of two Jacobi polynomials.

With the aid of (1.9) we prove the following theorem in § 3.

THEOREM 2. *Let z be a non-negative integer such that* $z \leqq \min(M, M')$ *if M, M' are positive integers. Let $x, y, n, \alpha, \beta, \gamma, \gamma', M, M'$ satisfy the same conditions as in Theorem 1. Let $\{a_k\}_{k=0}^{\infty}$ be an arbitrary sequence, real or complex. Then the kernel $K_z(x, y)$ defined by*

$$(1.10)\quad K_z(x, y) \equiv K_z(x, y; \alpha, \beta, \gamma, \gamma', M, M')$$

$$= \sum_{n=0}^{z} \frac{(\alpha+1)_n(\alpha+\beta+1)_n(\beta+\gamma'+1)_n(-1)^n}{n!(\beta+1)_n(\alpha+\beta+1)_{2n}(\alpha-\gamma'+1)_n} \times \lambda_n(z) W_n(x; \alpha, \beta, \gamma, M) W_n(y; \alpha, \beta, \gamma', M')$$

has a representation

$$(1.11)\quad K_z(x, y) = \sum_{r=0}^{z} \sum_{s=0}^{z-r} \frac{(-z)_{r+s} a_{r+s}(-x-y-\gamma')_{r+s}(-x)_r(-y)_r \times (x+\gamma-M)_r}{r!s!(-M)_{r+s}(-M')_{r+s}(\alpha-\gamma'+1)_{r+s} \times (\alpha+1)_r(\beta+\gamma+1)_r}$$

$$\cdot \frac{(M'+\alpha+1-\gamma'-y)_r(x-M)_s(y-M')_s}{(-x-y-\gamma')_r(\beta+1)_s}$$

$$\times {}_4F_3\left[\begin{matrix} -\beta, & r-x, & r-y+\gamma-\gamma', & M+\beta+1-x \\ & r-x-y-\gamma', & M+1-x-s, & \beta+\gamma+1+r \end{matrix}\right],$$

where

$$(1.12)\quad \lambda_n(z) = \sum_{k=0}^{z-n} \frac{(-z)_{n+k}a_{n+k}}{k!(\alpha+\beta+2+2n)_k}.$$

Special cases of this theorem includes the Poisson kernel for $W_n(x)$:

$$(1.13)\quad \sum_{n=0}^{z} \frac{(-z)_n}{(-M)_n} \rho(n;\alpha,\beta,\gamma,M) W_n(x;\alpha,\beta,\gamma,M) W_n(y;\alpha,\beta,\gamma,M')$$

$$= \frac{(M-z)\Gamma(M+z)}{M!(M+\alpha+\beta+2)_z}$$

$$\times \sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(-z)_{r+s}\left(\frac{\alpha+\beta+2}{2}\right)_{r+s}\left(\frac{\alpha+\beta+3}{2}\right)_{r+s}(-x-y-\gamma)_{r+s}}{r!s!(-M')_{r+s}\left(\frac{1-M-z}{2}\right)_{r+s}\left(\frac{2-M-z}{2}\right)_{r+s}(\alpha-\gamma+1)_{r+s}}$$

$$\times \frac{(-x)_r(-y)_r(x+\gamma-M)_r(M'+\alpha+1-\gamma-y)_r(x-M)_s(y-M')_s}{(\alpha+1)_r(\beta+\gamma+1)_r(-x-y-\gamma)_r(\beta+1)_s}$$

$$\times {}_4F_3\left[\begin{matrix}-s, & r-x, & r-y, & M+\beta+1-x \\ & r-x-y-\gamma, & M+1-x-s, & \beta+\gamma+1+r\end{matrix}\right].$$

It is easy to see that for real values of the parameters and for non-negative integral values of x, y, M, M' with $0 \leqq x \leqq M$, $0 \leqq y \leqq M'$, the kernel on the right hand side of (1.13) is non-negative if

$$(1.14)\quad \gamma \geqq \max(M, M'),\ -1 < \alpha < \min(\gamma - M, \gamma - M'),\ -1 < \beta.$$

In § 4 we derive a projection formula for the Racah-Wilson polynomials:

$$(1.15)\quad W_n(x;\alpha+\nu,\beta-\nu,\gamma+\nu,M)$$

$$= \sum_{p=0}^{x} B_p(\alpha,\gamma,\nu,x,M) W_n(x-p;\alpha,\beta,\gamma,M),$$

where

$$(1.16)\quad B_p(\alpha,\gamma,\nu,x,M)$$

$$= \frac{(M-2x-\gamma)_p\left(1+\frac{M-\gamma}{2}-x\right)_p(\nu)_p(M+\alpha+1-\gamma-x)_p \times (-x)_p(\alpha+1)_{x-p}(\gamma+\nu-M+x)_{x-p}}{p!\left(\frac{M-\gamma}{2}-x\right)_p(M-\gamma+1-x)_p(\alpha+\nu+1)_x(\nu-M+x)_x}.$$

With the aid of (1.9) and the above projection formula we finally obtain a product formula in § 5 in the form

$$(1.17)\quad W_n(x;\alpha,\beta,\gamma,M)W_n(y;\alpha,\beta,\gamma,M) = \sum_{z=|y-x|}^{y+x} A(x,y,z)W_n(z;\alpha,\beta,\gamma,M)$$

where the coefficients $A(x, y, z)$ are given in (5.11). Unfortunately, the author was not able to establish the non-negativity of these coefficients, but certain conjectures are made at the end of § 5.

2. Proof of theorem 1. The principal tools of the proof are the Pfaff-Saalschutz summation theorem for a balanced ${}_3F_2$ [**1**; **2**; **11**] and Whipple's transformation formula for a balanced ${}_4F_3$ [**2**, p. 56]:

$$(2.1)\quad {}_4F_3\left[\begin{matrix} x, & y, & z, & -n \\ u, & v, & w & \end{matrix}\right]$$
$$= \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}\,{}_4F_3\left[\begin{matrix} u-x, & u-y, & z, & -n \\ u, & 1+z-v-n, & 1+z-w-n & \end{matrix}\right],$$

where

$$u+v+w = 1+x+y+z-n.$$

Using this formula on $W_n(x;\alpha,\beta,\gamma,M)$ and the notation $P_n(x, y)$ for the left hand side of (1.9) we obtain

$$(2.2)\quad P_n(x,y) = \frac{(\gamma+1)_x(-\beta-M)_x}{(\beta+\gamma+1)_x(-M)_x}$$
$$\times \sum_{j=0}^{n}\sum_{q=0}^{x} \frac{(\alpha+n+1)_q(-\beta-n)_q(x+\gamma-M)_q(-x)_q}{q!(\alpha+1)_q(-\beta-M)_q(\gamma+1)_q}$$
$$\times \frac{(-n)_j(n+\alpha+\beta+1)_j(-y)_j(y+\gamma'-M')_j}{j!(\alpha+1)_j(-M')_j(\beta+\gamma'+1)_j}.$$

The Pfaff-Saalschutz theorem gives

$$\frac{(y+\gamma'-M')_j}{q!j!(\alpha+1)_j} = \frac{(y+\gamma'-M'+q)_j}{(\alpha+1+q)_j}$$
$$\times \sum_{r=0}^{q} \frac{(M'+\alpha+1-\gamma'-y)_r}{r!(j-r)!(q-r)!(\alpha+1)_r(M'+1-\gamma'-y-q-j)_r}$$

and

$$\frac{(-n)_j(n+\alpha+\beta+1)_j}{(\beta+\gamma'+1)_j(\beta+1)_{n-q}} = \frac{(\alpha-\gamma'+1)_n(\alpha+1+q)_j}{(\beta+\gamma'+1)_n(\alpha+1+q)_n}$$
$$\times \sum_{s=j-r}^{n-r} \frac{(-n)_{r+s}(n+\alpha+\beta+1)_{r+s}(-\gamma'-q)_{r+s-j}}{(r+s-j)!(\beta+1)_{r+s-q}(\alpha-\gamma'+1)_{r+s}}.$$

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Using these relations in (2.2) we get

$$(2.3)\qquad P_n(x, y) = \frac{(\gamma+1)_x(-\beta-M)_x(\beta+1)_n(\alpha-\gamma'+1)_n}{(\beta+\gamma+1)_x(-M)_x(\alpha+1)_n(\beta+\gamma'+1)_n}$$

$$\times \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(n+\alpha+\beta+1)_{r+s}}{r!(\alpha-\gamma'+1)_{r+s}} \times \frac{(M'+\alpha+1-\gamma'-y)_r(-1)^r}{(\alpha+1)_r}$$

$$\times \sum_{q=r}^{x} \frac{(-1)^q(-x)_q(x+\gamma-M)_q}{(q-r)!(\beta+1)_{r+s-q}(-\beta-M)_q(\gamma+1)_q}$$

$$\times \sum_{j=r}^{r+s} \frac{(-y)_j(y+\gamma'-M'+q)_{j-r}(-\gamma'-q)_{r+s-j}}{(j-r)!(r+s-j)!(-M')_j}.$$

However,

$$(2.4)\qquad \sum_{j=r}^{r+s} \frac{(-y)_j(y+\gamma'-M'+q)_{j-r}(-\gamma'-q)_{r+s-j}}{(j-r)!(r+s-j)!(-M')_j}$$

$$= \frac{(-y)_r(-\gamma'-q)_s}{s!(-M')_r}\, {}_3F_2\left[\begin{matrix}-s, & y+\gamma'-M'+q, & r-y \\ & r-M', & \gamma'+1+q-s\end{matrix}\right]$$

$$= \frac{(-y)_r(y-M')_s(r-y-\gamma'-q)_s}{s!(-M')_{r+s}}.$$

Also,

$$(2.5)\qquad (-1)^r \sum_{q=r}^{x} \frac{(-1)^q(-x)_q(x+\gamma-M)_q(r-y-\gamma'-q)_s}{(q-r)!(\beta+1)_{r+s-q}(-\beta-M)_q(\gamma+1)_q}$$

$$= \frac{(-x)_r(x+\gamma-M)_r(-y-\gamma')_s}{(-\beta-M)_r(\gamma+1)_r(\beta+1)_s}$$

$$\times\, {}_4F_3\left[\begin{matrix}r-x, & x+\gamma-M+r, & -\beta-s, & y+\gamma'+1 \\ & \gamma+1+r, & r-\beta-M, & y+\gamma'+1-s\end{matrix}\right].$$

Since the ${}_4F_3$ series on the right is balanced we may apply (2.1) as often as necessary. Applying it once we get

$$(2.6)\qquad P_n(x, y) = \frac{(\beta+1)_n(\alpha-\gamma'+1)_n}{(\alpha+1)_n(\beta+\gamma'+1)_n}$$

$$\times \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(n+\alpha+\beta+1)_{r+s}(-x)_r(-y)_r(x+\gamma-M)_r}{r!s!(-M)_{r+s}(-M')_{r+s}(\alpha-\gamma'+1)_{r+s}(\beta+\gamma+1)_{r+s}}$$

$$\times \frac{(M'+\alpha+1-\gamma'-y)_r(x-M)_s(y-M')_s(x+\beta+\gamma+1)_s(-y-\gamma')_s}{(\alpha+1)_r(\beta+1)_s}$$

$$\times\, {}_4F_3\left[\begin{matrix}-s, & -\beta-s, & M+\gamma'-\gamma+1+y-x-r-s, & r-x \\ & y+\gamma'+1-s & M+1-x-s, & -\beta-\gamma-x-s\end{matrix}\right].$$

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If we let $\gamma, \gamma' \to \infty$ such that $\gamma' - \gamma$ is finite then the ${}_4F_3$ function approaches 1 and the right hand side reduces to Gasper's formula [**4**, (2.3)] for the product of two Hahn polynomials.

Applying the transformation formula (2.1) once again we obtain (1.9).

3. A generalized Poisson kernel for Racah-Wilson polynomials. Let us now prove Theorem 2. First, we use Theorem 1 on the right hand side of (1.10), change the dummy variable $n - r$ to l, and obtain

$$(3.1)\quad K_z(x, y) = \sum_{r=0}^{z} \sum_{l=0}^{z-r} \sum_{s=0}^{l} \sum_{k=0}^{z-r-l} \frac{(-1)^{r+l}(\alpha+\beta+1)_{r+l}(\alpha+\beta+2)_{2r+2l}}{(r+l)!(\alpha+\beta+1)_{2r+2l}}$$
$$\times \frac{(-z)_{r+l+k}(-r-l)_{r+s}a_{r+l+k}(r+l+\alpha+\beta+1)_{r+s}(-x-y-\gamma')_{r+s}}{r!s!k!(\alpha+\beta+2)_{2r+2l+k}(-M)_{r+s}(-M')_{r+s}(\alpha-\gamma'+1)_{r+s}}$$
$$\times \frac{(-x)_r(-y)_r(x+\gamma-M)_r(M'+\alpha+1-\gamma'-y)_r \times (x-M)_s(y-M')_s}{(\beta+\gamma+1)_r(-x-y-\gamma')_r(\alpha+1)_r(\beta+1)_s} b_{r,s},$$

where

$$(3.2)\quad b_{r,s} = {}_4F_3\left[\begin{matrix} -s, & r-x, & r-y+\gamma-\gamma', & M+\beta+1-x \\ & r-x-y-\gamma', & M+1-x-s, & \beta+\gamma+1+r \end{matrix}\right].$$

Changing the summation variables once again by setting $k + l = m$ we have

$$(3.3)\quad K_z(x, y) = \sum_{r=0}^{z} \sum_{m=0}^{z-r} \sum_{s=0}^{m} \frac{(-z)_{r+m}a_{r+m}(-x-y-\gamma')_{r+s}(-x)_r(-y)_r}{r!s!(-M)_{r+s}(-M')_{r+s}(\alpha-\gamma'+1)_{r+s}}$$
$$\times \frac{(x+\gamma-M)_r(M'+\alpha+1-\gamma'-y)_r(x-M)_s(y-M')_s}{(\beta+\gamma+1)_r(\alpha+1)_r(-x-y-\gamma')_r(\beta+1)_s} b_{r,s}T_{r,m,s},$$

where

$$(3.4)\quad T_{r,m,s} = \sum_{l=s}^{m} \frac{(-1)^{l-s}(\alpha+\beta+1)_{r+l}(\alpha+\beta+2)_{2r+2l} \times (r+l+\alpha+\beta+1)_{r+s}}{(m-l)!(l-s)!(\alpha+\beta+1)_{2r+2l}(\alpha+\beta+2)_{2r+m+l}}.$$

In [**8**], where we worked with Hahn polynomials we found exactly the same series as the right hand side of (3.4) which we could readily sum and obtain

$$(3.5)\quad T_{r,m,s} = \delta_{m,s}.$$

Using (3.5) in (3.3) immediately leads to (1.11).

Some special cases. As in [**8**] we now consider some special cases of

(1.11) and obtain the corresponding bilinear sums for Racah-Wilson polynomials.

(i) $\lambda_n(z)$ *proportional to a balanced* ${}_3F_2$. Let

$$a_k = (\alpha_1)_k(\alpha_2)_k/(\alpha_1 + \alpha_2 - \alpha - \beta - z - 1)_k, \tag{3.6}$$

where α_1, α_2 are arbitrary complex parameters. Then, by the Pfaff-Saalschutz theorem, we can sum the series on the right hand side of (1.12) and obtain

$$\lambda_n(z) = \frac{(\alpha+\beta+2-\alpha_1)_z(\alpha+\beta+2-\alpha_2)_z}{(\alpha+\beta+2)_z(\alpha+\beta+2-\alpha_1-\alpha_2)_z} \times \frac{(-1)^n(-z)_n(\alpha_1)_n(\alpha_2)_n(\alpha+\beta+2)_{2n}}{(\alpha+\beta+2-\alpha_1)_n(\alpha+\beta+2-\alpha_2)_n(\alpha+\beta+2+z)_n}. \tag{3.7}$$

This leads to the bilinear sum

$$\sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(-z)_{r+s}(\alpha_1)_{r+s}(\alpha_2)_{r+s}(-x-y-\gamma')_{r+s} \times (-x)_r(-y)_r(x+\gamma-M)_r}{r!s!(\alpha_1+\alpha_2-\alpha-\beta-z-1)_{r+s}} \times (-M)_{r+s}(-M')_{r+s}(\alpha-\gamma'+1)_{r+s}$$
$$\times \frac{(M'+\alpha+1-\gamma'-y)_r(x-M)_s(y-M')_s}{(\alpha+1)_r(\beta+\gamma+1)_r(-x-y-\gamma')_r(\beta+1)_s} b_{r,s}$$
$$= \frac{(\alpha+\beta+2-\alpha_1)_z(\alpha+\beta+2-\alpha_2)_z}{(\alpha+\beta+2)_z(\alpha+\beta+2-\alpha_1-\alpha_2)_z}$$
$$\times \sum_{n=0}^{z} \frac{(-z)_n(\alpha_1)_n(\alpha_2)_n(\alpha+\beta+1)_n(\alpha+\beta+2)_{2n}}{n!(\alpha+\beta+2-\alpha_1)_r(\alpha+\beta+2-\alpha_2)_n(\alpha+\beta+2+z)_n}$$
$$\times \frac{(\alpha+1)_n(\beta+\gamma'+1)_n}{(\alpha+\beta+1)_{2n}(\beta+1)_n(\alpha-\gamma'+1)_n}$$
$$\times W_n(x;\alpha,\beta,\gamma,M)W_n(y;\alpha,\beta,\gamma',M'). \tag{3.8}$$

Let us assume y, M, M', $\gamma'-\gamma+1$ are positive integers such that $x \leqq y \leqq M'$, and $z \leqq M' \leqq M$. Also let $\beta+1>0$, $M \leqq \gamma \leqq \gamma'$, $-1<\alpha<\gamma'-M'$. Then the kernel on the left hand side of (3.8) is non-negative if either

(a) $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1+\alpha_2 < \alpha+\beta+2$,

or

(b) α_1, α_2 are both negative integers,

or

(c) $\alpha_1, \alpha_2 \leqq -z$.

A somewhat simpler form can be derived from (3.8) if we let α_1 or α_2

equal $\alpha - \gamma' + 1$. Suppose $\alpha_2 = \alpha - \gamma' + 1$. Then (3.8) gives

$$(3.9)\quad \sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(-z)_{r+s}(\alpha_1)_{r+s}(-x-y-\gamma')_{r+s}(-x)_r(-y)_r(x+\gamma-M)_r \times (M'+\alpha+1-\gamma'-y)_r}{r!s!(\alpha_1-\gamma'-\beta-z)_{r+s}(-M)_{r+s}(-M')_{r+s}(\beta+\gamma+1)_r \times (-x-y-\gamma')_r}$$
$$\times \frac{(x-M)_s(y-M')_s}{(\alpha+1)_r(\beta+1)_s} b_{r,s} = \frac{(\alpha+\beta+2-\alpha_1)_z(\beta+\gamma'+1)_z}{(\alpha+\beta+2)_z(\beta+\gamma'+1-\alpha_1)_z}$$
$$\times \sum_{n=0}^{z} \rho(n;\alpha,\beta,\alpha_1-\beta-1,z)W_n(x;\alpha,\beta,\gamma,M)W_n(y;\alpha,\beta,\gamma',M').$$

If we multiply both sides by $(\beta+\gamma'+1-\alpha_1)_z$ and let $\alpha_1 \to \beta+\gamma'+1$ we obtain

$$(3.10)\quad \sum_{n=0}^{z} \frac{(-z)_n(M'+\alpha+\beta+2)_n}{(-M')_n(z+\alpha+\beta+2)_n} \rho(n;\alpha,\beta,\gamma',M')W_n(x;\alpha,\beta,\gamma,M)$$
$$\times W_n(y;\alpha,\beta,\gamma',M') = \frac{(x-M)_z(y-M')_z(-x-y-\gamma')_z \times (\alpha+\beta+2)_z}{(-M)_z(-M')_z(\alpha-\gamma'+1)_z(\beta+1)_z}$$
$$\times \sum_{r=0}^{z} \frac{(-x)_r(-y)_r(-z)_r(x+\gamma-M)_r}{r!(\alpha+1)_r(\beta+\gamma+1)_r(-x-y-\gamma')_r}$$
$$\times \frac{(M'+\alpha+1-\gamma'-y)_r(-\beta-z)_r}{(M+1-x-z)_r(M'+1-y-z)_r}$$
$$\times {}_4F_3\left[\begin{matrix} r-z, & r-x, & r-y+\gamma-\gamma', & M+\beta+1-x \\ & r-x-y-\gamma', & M+1-x-z+r, & \beta+\gamma+1+r \end{matrix}\right]$$

provided $x + z \leqq y + z \leqq M' \leqq M$. In this form the non-negativity of the kernel on the right hand side is self-evident for $x \leqq y \leqq M'$, $z \leqq M' \leqq M$, $M \leqq \gamma \leqq \gamma'$, $-1 < \alpha < \gamma' - M'$, and $\beta + 1 > 0$. However, in the case $M \leqq \min(x + z, y + z)$ a sequence of Whipple transformations of the ${}_4F_3$ function on the right is necessary to bring the kernel to the following convenient form

$$(3.11)\quad \frac{(M+\beta+1-x)_x(\gamma'-\gamma+M+y-x-z)_x(\alpha+\beta+2)_z \times (-y-\gamma')_z}{(\beta+\gamma+1)_x(\gamma'+1+y-z)_x(-M')_z(\alpha-\gamma'+1)_z}$$
$$\times \sum_{r=0}^{z} \frac{(-x)_r(-y)_r(-z)_r(x+\gamma-M)_r(M'+\alpha+1-\gamma'-y)_r(y-M')_{z-r}}{r!(-M)_r(-\beta-M)_r(\alpha+1)_r(\gamma-\gamma'+1-M-y+x+z)_r(\beta+1)_{z-r}}$$
$$\times {}_4F_3\left[\begin{matrix} z-M, & r-x, & r-y-M-\beta-1-\gamma', & \gamma-M+x+r \\ & r-M, & r-\beta-M, & \gamma-\gamma'+1-M-y+x+z+r \end{matrix}\right].$$

Under the conditions stated above this kernel is also non-negative. One can see easily that in the special case $z = M = M'$, $\gamma = \gamma'$ this becomes proportional to δ_{xy}, as expected.

(ii) $\lambda_n(z)$ *proportional to a balanced* ${}_4F_3$. Let us now choose

$$a_k = (\alpha_1)_k(\alpha_2)_k(-M)_k/(\beta_1)_k(\alpha_1+\alpha_2-\beta_1-M-z-\alpha-\beta-1)_k. \tag{3.12}$$

The corresponding $\lambda_n(z)$ becomes a multiple of a balanced ${}_4F_3$. Using (2.1) we may have the following alternative forms,

$$\begin{aligned}\lambda_n(z) &= \frac{(-z)_n(-M)_n(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n(\alpha_1+\alpha_2-\beta_1-M-z-\alpha-\beta-1)_n}\\ &\quad\times {}_4F_3\left[\begin{matrix}\alpha_1+n, & \alpha_2+n, & n-M, & n-z\\ \beta_1+n, & \alpha_1+\alpha_2-\beta_1-M-z-\alpha-\beta-1+n, & & \alpha+\beta+2+2n\end{matrix}\right]\\ &= \frac{(\beta_1+M)_z(\alpha_1+\alpha_2-\beta_1-z-\alpha-\beta-1)_z}{(\beta_1)_z(\alpha_1+\alpha_2-\beta_1-M-z-\alpha-\beta-1)_z}\\ &\quad\times \frac{(-z)_n(-M)_n(\alpha_1)_n(\alpha_2)_n}{(1-\beta_1-M-z)_n(\alpha+\beta+2-\alpha_1-\alpha_2+\beta_1)_n}\\ &\quad\times {}_4F_3\left[\begin{matrix}\alpha+\beta+2-\alpha_1+n, & \alpha+\beta+2-\alpha_2+n, & n-M, & n-z\\ 1-\beta_1-M-z+n, & \alpha+\beta+2-\alpha_1-\alpha_2+\beta_1+n, & & \alpha+\beta+2+2n\end{matrix}\right],\end{aligned} \tag{3.13}$$

where $\alpha_1, \alpha_2, \beta_1$ are arbitrary except for the normal restriction that none of the factors in the denominators should vanish. If we now use the special values

$$\alpha_1 = \tfrac{1}{2}(\alpha+\beta+3),\ \alpha_2 = \tfrac{1}{2}(\alpha+\beta+2),\ \beta_1 = \tfrac{1}{2}(1-M-z), \tag{3.14}$$

on the second form on the right hand side of (3.13), we obtain

$$\begin{aligned}\lambda_n(z) &= \frac{(1+M-z)_{2z}}{(1-M-z)_{2z}} \times \frac{(-z)_n(-M)_n(\alpha+\beta+2)_{2n}}{(-M-z)_{2n}}\\ &\quad\times {}_4F_3\left[\begin{matrix}\frac{\alpha+\beta+1}{2}+n, & \frac{\alpha+\beta+2}{2}+n, & n-M, & n-z\\ \alpha+\beta+2+2n, & \frac{1-M-z}{2}+n, & n-\frac{M+z}{2} & \end{matrix}\right].\end{aligned} \tag{3.15}$$

The ${}_4F_3$ series on the right can be summed by means of the summation formula [**11**, p. 65]

$$(3.16)\quad {}_4F_3\left[\begin{matrix} d, & 1+f-g, & \frac{1}{2}f, & \frac{1}{2}+\frac{1}{2}f \\ 1+f, & \frac{1}{2}+\frac{1}{2}(f-g+d), & 1+\frac{1}{2}(f-g+d) \end{matrix}\right] = \frac{\Gamma(g-f)\Gamma(g-d)}{\Gamma(g)\Gamma(g-f-d)},$$

provided d or f is a negative integer. Using this in (3.15) and simplifying, we get

$$(3.17)\quad \frac{(\alpha+1)_n(\beta+\gamma'+1)_n(\alpha+\beta+1)_n}{n!(\beta+1)_n(\alpha-\gamma'+1)_n(\alpha+\beta+1)_{2n}}(-1)^n\lambda_n(z) = \frac{(M+\alpha+\beta+2)_z M!}{(M-z)\Gamma(M+z)}\frac{(-z)_n}{(-M)_n}\rho(n;\alpha,\beta,\gamma',M).$$

The Poisson kernel (1.13) for Racah-Wilson polynomials is obtained by substituting (3.17) in (1.10) and then using Theorem 2.

An immediate consequence of (1.13) is the following theorem.

THEOREM 3. *Let* x, y, M, M' *be non-negative integers with* $0 \leqq x \leqq M$, $0 \leqq y \leqq M'$. *Let* α, β, γ *be the real parameters satisfying the inequalities* (1.14). *If* $a_0, a_1, \ldots, a_M$ *are constants such that*

$$\sum_{n=0}^{M} a_n W_n(x;\alpha,\beta,\gamma,M) \geqq 0,$$

then

$$\sum_{n=0}^{z} \frac{(-z)_n}{(-M)_n} a_n W_n(y;\alpha,\beta,\gamma,M') \geqq 0$$

for $z = 0, 1, \ldots, \min(M, M')$.

This constitutes a generalization of Gasper's Theorem 2 of [**4**].

A second kernel of interest may be obtained from (3.13) by choosing

$$(3.18)\quad \alpha_1 = \tfrac{1}{2}(\alpha+\beta+2), \alpha_2 = \tfrac{1}{2}(\alpha+\beta+1), \beta_1 = -\tfrac{1}{2}(M+z)$$

and using the first line on the right of (3.13). Following a similar calculation we get

$$(3.19)\quad \frac{(\alpha+1)_n(\beta+\gamma'+1)_n(\alpha+\beta+1)_n}{n!(\beta+1)_n(\alpha-\gamma'+1)_n(\alpha+\beta+1)_{2n}}(-1)^n\lambda_n(z) = \frac{(M+\alpha+\beta+2)_z}{(M+1)_z}\rho(n;\alpha,\beta,\gamma',M)\left[\frac{\alpha+\beta+1}{2n+\alpha+\beta+1}\right]\frac{(-z)_n}{(-M)_n}.$$

This leads to the following result

$$(3.20)\quad \frac{(M+1)_z}{(M+\alpha+\beta+2)_z} \times \sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(-z)_{r+s}\left(\frac{\alpha+\beta+1}{2}\right)_{r+s}\left(\frac{\alpha+\beta+2}{2}\right)_{r+s}(-x-y-\gamma')_{r+s}}{r!s!\left(-\frac{M+z}{2}\right)_{r+s}\left(\frac{1-M-z}{2}\right)_{r+s}(-M')_{r+s}(\alpha-\gamma'+1)_{r+s}}$$

$$\times \frac{(x+\gamma-M)_r(M'+\alpha+1-\gamma'-y)_r(-x)_r(-y)_r \times (x-M)_s(y-M')_s}{(\alpha+1)_r(-x-y-\gamma')_r(\beta+\gamma+1)_r(\beta+1)_s}$$

$$\times {}_4F_3\left[\begin{matrix}-s, & r-x, & r-y+\gamma-\gamma', & M+\beta+1-x\\ r-x-y-\gamma', & M+1-x-s, & \beta+\gamma+1+r\end{matrix}\right]$$

$$= \sum_{n=0}^{z}\frac{(-z)_n}{(-M)_n}\rho(n;\alpha,\beta,\gamma',M)\left[\frac{\alpha+\beta+1}{2n+\alpha+\beta+1}\right] \times W_n(x;\alpha,\beta,\gamma,M)W_n(y;\alpha,\beta,\gamma',M').$$

The kernel on the left is non-negative for integral values of x, y, z, M, M' with $0 \leqq x \leqq M$, $0 \leqq y \leqq M'$, $0 \leqq z \leqq \min(M, M')$, and α, β, γ, γ' satisfying the inequalities

$$\max(M, M') \leqq \gamma \leqq \gamma', -1 < \alpha < \min(\gamma - M, \gamma' - M'), \quad -1 < \beta, \alpha+\beta+1 \geqq 0.$$

Eq. (3.20) is an extension of Gasper's second kernel in [**4**, (3.7)].

4. Proof of the projection formula (1.15). In (1.3) we stated Dougall's formula for a terminating very well-poised ${}_5F_4$ series. The general formula, which covers both terminating and non-terminating cases, is given in [**2**, **11**]. Using this formula we can derive the identity

$$(4.1)\quad \frac{(e)_m(e+c-a)_m}{(e+c+d-a)_m} = \frac{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(1+a-d-e)\Gamma(1+a-c-e)}{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)} \times \sum_{p=0}^{\infty}\frac{(a)_p(1+\frac{1}{2}a)_p(c)_p(d)_p(e)_p}{p!(\frac{1}{2}a)_p(1+a-c)_p(1+a-d)_p(1+a-e)_p} \times \frac{(e+p)_m(e-a-p)_m}{(e+d-a)_m},$$

for $m = 0, 1, 2, \ldots$, subject to the restriction that $\mathrm{Re}(1 + a - c - d -$

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$e - m) > 0$ in case the series does not terminate. If we set $a = M - \gamma - 2x, c = \nu, d = M + \alpha + 1 - \gamma - x, e = -x$, then (4.1) gives

$$\frac{(-x)_m(x+\gamma+\nu-M)_m}{(\alpha+\nu+1)_m} = \frac{(\alpha+1)_x(\nu+\gamma-M+x)_x}{(\alpha+\nu+1)_x(\gamma-M+x)_x}$$
$$\times \sum_{p=0}^{x} \frac{(M-\gamma-2x)_p\left(1+\frac{M-\gamma-2x}{2}\right)_p}{p!\left(\frac{M-\nu-2x}{2}\right)_p}$$
$$\times \frac{(\nu)_p(M+\alpha+1-\gamma-x)_p(-x)_p}{(M+1-\gamma-\nu-2x)_p(-\alpha-x)_p(M+1-\nu-x)_p}$$
$$\times \frac{(p-x)_m(x+\gamma-M-p)_m}{(\alpha+1)_m},$$

where $m = 0, 1, \ldots, x$.

This immediately leads to the projection formula (1.15). It is obvious that the coefficients $B_p(\alpha, \gamma, \nu, x, M)$ are non-negative if

$$\nu \geqq 0, \gamma \geqq M, -1 < \alpha < \gamma - M. \tag{4.3}$$

5. A product formula for $W_n(x)$. Using a Whipple transformation we first derive the following expression for $P_n(x, y)$ from (1.9):

$$P_n(x,y) = \frac{(\beta+1)_n(\alpha-\gamma'+1)_n}{(\alpha+1)_n(\beta+\gamma'+1)_n}\frac{(-\beta)_x(x+\gamma-M)_x}{(M-x+1)_x(\beta+\gamma+1)_x} \tag{5.1}$$
$$\times \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(n+\alpha+\beta+1)_{r+s}}{r!s!(-M')_{r+s}(\beta+1-x)_{r+s}}$$
$$\times \frac{(-x-y-\gamma')_{r+s}(-x)_r(-y)_r(M'+\alpha+1-\gamma'-y)_r(y-M')_s}{(\alpha-\gamma'+1)_{r+s}(\alpha+1)_r(-x-y-\gamma')_r}$$
$$\times {}_4F_3\left[\begin{matrix} r+s-x-y-\gamma', & -x-\gamma, & M+\beta+1-x, & r-x \\ r-x-y-\gamma', & M+1-\gamma-2x, & \beta+1-x+r+s \end{matrix}\right].$$

Let us now consider the double sum

$$U_{n,k} \equiv \frac{(\beta+1)_n(\alpha-\gamma'+1)_n}{(\alpha+1)_n(\beta+\gamma'+1)_n} \tag{5.2}$$
$$\times \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(n+\alpha+\beta+1)_{r+s}(k-x-y-\gamma')_{r+s}}{r!s!(-M')_{r+s}(\beta+1-x+k)_{r+s}(\alpha-\gamma'+1)_{r+s}}$$
$$\times \frac{(k-x)_r(-y)_r(M'+\alpha+1-\gamma'-y)_r(y-M')_s}{(\alpha+1)_r(k-x-y-\gamma')_r},$$

for $k = 0, 1, \ldots, x$.

It is easy to see that

$$(5.3)\quad \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(n+\alpha+\beta+1)_{r+s}(k-x-y-\gamma')_{r+s}(y-M')_s}{s!(-M')_{r+s}(\beta+1-x+k)_{r+s}(\alpha-\gamma'+1)_{r+s}}$$

$$= \sum_{s=0}^{n} \frac{(-n)_s(n+\alpha+\beta+1)_s(k-x-y-\gamma')_s}{s!(-M')_s(\alpha-\gamma'+1)_s} \times \frac{(y-M')_{s-r}(-s)_r(-1)^r}{(\beta+1-x+k)_s}.$$

Also, by virtue of Pfaff-Saalschutz theorem, it can be proved that

$$(5.4)\quad (M'+\beta+1-x-y+k)_r(y-M')_{s-r}(-s)_r(-1)^r/ (\beta+1-x+k)_s = \sum_{l=0}^{r} \binom{r}{l}(x-\beta-k)_l(\beta-x+k)_{r-l} \times (y-M'-l)_s/(\beta+1-x+k-l)_s.$$

Applying (5.4) in (5.3) we obtain

$$(5.5)\quad U_{n,k} = \sum_{r=0}^{x-k} \frac{(k-x)_r(-y)_r(M'+\alpha+1-\gamma'-y)_r}{r!(\alpha+1)_r(k-x-y-\gamma')_r(M'+\beta+1-x-y+k)_r}$$

$$\times \sum_{l=0}^{r}\binom{r}{l}(x-\beta-k)_l(\beta-x+k)_{r-l}$$

$$\times {}_4F_3\left[\begin{matrix}-n, & n+\alpha+\beta+1, & k-x-y-\gamma', & y-M'-l \\ & \beta+1-x+k-l, & -M', & \alpha-\gamma'+1\end{matrix}\right]$$

$$\times \frac{(\beta+1)_n(\alpha-\gamma'+1)_n}{(\alpha+1)_n(\beta+\gamma'+1)_n}$$

$$= \sum_{l=0}^{x-k} \frac{(k-x)_l(-y)_l(M'+\alpha+1-\gamma'-y)_l(x-\beta-k)_l}{l!(\alpha+1)_l(k-x-y-\gamma')_l(M'+\beta+1-x-y+k)_l}$$

$$\times {}_4F_3\left[\begin{matrix}k+l-x, & l-y, & M'+\alpha+1-\gamma'-y+l, & \beta-x+k \\ & k-x-y-\gamma'+l, & M'+\beta+1-x-y+k+l, & \alpha+1+l\end{matrix}\right] V_{n,k,l},$$

where

$$(5.6)\quad V_{n,k,l} = \frac{(\beta+1)_n(\alpha-\gamma'+1)_n}{(\alpha+1)_n(\beta+\gamma'+1)_n}\, {}_4F_3\left[\begin{matrix}-n, & n+\alpha+\beta+1, & y-M'-l, & k-x-y-\gamma' \\ & \alpha-\gamma'+1, & -M', & \beta+1-x+k-l\end{matrix}\right].$$

It should be observed that the ${}_4F_3$ functions in (5.5) and (5.6) are both

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balanced. In particular, the ${}_4F_3$ series on the right of (5.6) is a Racah-Wilson polynomial of degree $2n$ in y.

Using the projection formula (1.15) we now get

$$(5.7)\quad {}_4F_3\left[\begin{matrix}-n, & n+\alpha+\beta+1, & y-M'-l, & k-x-y-\gamma' \\ & \alpha-\gamma'+1, & -M', & \beta+1-x+k-l\end{matrix}\right]$$
$$= \sum_{p=0}^{x+l-k} C_p(x, y, k, l, \beta, \gamma', M')$$
$$\times\ {}_4F_3\left[\begin{matrix}-n, & n+\alpha+\beta+1, & y-M'-l+p, & l-y-\gamma'-p \\ & -M', & \alpha-\gamma'+1, & \beta+1\end{matrix}\right]$$

where

$$(5.8)\quad C_p(x, y, k, l, \beta, \gamma', M')$$
$$= \frac{(y-M'-\beta-l)_{x+l-k}(\gamma'+1+y-l)_{x+l-k}}{(\gamma'+1+2y-M'-2l)_{x+l-k}(-\beta)_{x+l-k}}$$
$$\times \frac{(\gamma'-M'+2y-2l)_p\left(1+\frac{\gamma'-M'}{2}+y-l\right)_p}{p!\left(\frac{\gamma'-M'}{2}+y-l\right)_p(y-M'-\beta-l)_p}$$
$$\times \frac{(\beta+\gamma'+1+y-l)_p(k-x-l)_p(y-M'-l)_p}{(\gamma'+1-M'+x+2y-k-l)_p(\gamma'+1+y-l)_p}.$$

Finally, use of (2.1) gives

$$(5.9)\quad {}_4F_3\left[\begin{matrix}-n, & n+\alpha+\beta+1, & y-M'-l+p, & l-y-\gamma'-p \\ & -M', & \alpha-\gamma'+1, & \beta+1\end{matrix}\right]$$
$$= \frac{(\alpha+1)_n(\beta+\gamma'+1)_n}{(\beta+1)_n(\alpha-\gamma'+1)_n}$$
$$\times\ {}_4F_3\left[\begin{matrix}-n, n+\alpha+\beta+1, l-p-y, y+p-l+\gamma'-M' \\ \alpha+1, \quad -M', \quad \beta+\gamma'+1\end{matrix}\right].$$

Thus we have the product formula

$$(5.10)\quad W_n(x;\alpha,\beta,\gamma,M)W_n(y;\alpha,\beta,\gamma',M') = \frac{(-\beta)_x(x+\gamma-M)_x}{(M-x+1)_x(\beta+\gamma+1)_x}$$
$$\times \sum_{k=0}^{x} \frac{(-x)_k(-x-\gamma)_k(M+\beta+1-x)_k}{k!(\beta+1-x)_k(M+1-\gamma-2x)_k} \sum_{l=0}^{x-k} \frac{(k-x)_l(-y)_l}{l!(\alpha+1)_l}$$
$$\times \frac{(M'+\alpha+1-\gamma'-y)_l(x-\beta-k)_l}{(k-x-y-\gamma')_l(M'+\beta+1-x-y+k)_l}$$
$$\times\ {}_4F_3\left[\begin{matrix}k+l-x, & l-y, & M'+\alpha+1-\gamma'-y+l, & \beta-x+k \\ k-x-y-\gamma'+l, & M'+\beta+1-x-y+k+l, & \alpha+1+l\end{matrix}\right] \sum_{m=0}^{x+l-k} C_m(x, y, k, l, \beta, \gamma', M')$$
$$\times W_n(y-l+m;\alpha,\beta,\gamma',M').$$

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If we now set $y - l + m = z$, then for $y \geqq x$, $\min z = y - x$ and $\max z = y + x$. Hence the product formula (1.17) holds with

$$(5.11)\quad A(z) = \frac{(y-\beta-M)_x(y+\gamma+1)_x(x+\gamma-M)_x}{(M-x+1)_x(\beta+\gamma+1)_x(\gamma+1+2y-M)_x}$$

$$\times \sum_{k=0}^{x} \frac{(-x)_k(-x-\gamma)_k(M+\beta+1-x)_k}{k!(M+1-\gamma-2x)_k(-x-y-\gamma)_k}$$

$$\times \frac{(M-\gamma-x-2y)_k}{(M+\beta+1-x-y)_k}$$

$$\times \sum_{l=0}^{x-k} \frac{(k-x)_l(-y)_l(M+\alpha+1-\gamma-y)_l(M+\beta+1-y)_l \times (-y-\gamma)_l}{l!(\alpha+1)_l(k-x-y-\gamma)_l(M+\beta+1-x-y+k)_l}$$

$$\times \frac{(M-\gamma-x-2y+k)_l(-1)^l}{(M-\gamma-2y)_{2l}}$$

$$\times {}_4F_3\left[\begin{matrix} k+l-x,\ l-y,\ M+\alpha+1-\gamma-y+l,\ \beta-x+k \\ k+l-x-y-\gamma,\ M+\beta+1-x-y+k+l,\ \alpha+1+l \end{matrix}\right]$$

$$\times \frac{(\gamma-M+2y-2l)_{z+l-y}\left(1+\frac{\gamma-M}{2}+y-l\right)_{z+l-y}}{(z+l-y)!\left(\frac{\gamma-M}{2}+y-l\right)_{z+l-y}(y-\beta-M-l)_{z+l-y}}$$

$$\times \frac{(\beta+\gamma+1+y-l)_{z+l-y}(k-x-l)_{z+l-y}(y-M-l)_{z+l-y}}{(\gamma+1-M+x+2y-k-l)_{z+l-y}(y+\gamma+1-l)_{z+l-y}}$$

Although we have not been able to establish the conditions for non-negativity of these coefficients we know that in the limit $\gamma \to \infty$ eq. (1.17) reduces to the product formula for Hahn polynomials for which the non-negativity of the coefficients has been proved in [**9**] and [**10**] under the conditions:

(5.12) (i) $\alpha \leqq \beta < -M$, if $\beta - \alpha$ is a non-negative integer;
(ii) $\alpha < \beta < -M$, $M - 1 < \beta - \alpha$ if $\beta - \alpha$ is not an integer.

We also know that, if we set

$$(5.13)\quad \beta + \gamma = \alpha',\ M + \beta + 1 = -\beta', \text{ so that } \gamma = M + \alpha' + \beta' + 1,$$

and then proceed to the limit $\alpha \to -\infty$, $M, n \to \infty$ such that $n/M = \xi$, then the product formula (1.17) reduces to the linearization formula for the product of two Jacobi polynomials for which the non-negativity of the coefficients has been established by Gasper [**5**] under the conditions

$$-1 < \beta' \leqq \alpha',\ \alpha' + \beta' + 1 \geqq 0.$$

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Our conjecture is, then, that the coefficients $A(z)$ are non-negative under (5.12) as well as the additional conditions

(5.14) (iii) $M \leqq \gamma,\ M + \gamma + 2\beta + 1 \geqq 0.$

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