Integrating sine and cosine Maclaurin remainders

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In order to state our primary results, we must first establish some notation. Let $S_{-1}(x) = \sin x$ and $C_{-1}(x) = \cos x$, then for each non-negative integer n, let

$$S_n(x) = \sin x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \text{and} \quad C_n(x) = \cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k};$$

these are the remainders of the Maclaurin series for sine and cosine, respectively. Note that $S_n'(x) = C_n(x)$ for each $n \geqslant -1$ and $C_n'(x) = -S_{n-1}(x)$ for each $n \geqslant 0$. It is known that

$$\alpha_n \equiv \int_0^\infty \frac{S_n(x)}{x^{2n+3}} dx = \frac{(-1)^{n+1}}{(2n+2)!} \cdot \frac{\pi}{2} \qquad \text{for } n \geqslant -1;$$

$$\beta_n \equiv \int_0^\infty \frac{C_n(x)}{x^{2n+2}} dx = \frac{(-1)^{n+1}}{(2n+1)!} \cdot \frac{\pi}{2} \qquad \text{for } n \geqslant 0.$$

See [1] for several different proofs of the well-known fact that

$$\alpha_{-1} \equiv \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2};$$

the values of α_n and β_n for $n \geqslant 0$ then follow rather easily using induction and integration by parts. (Details are provided in the Appendix.)

Our goal is to prove the following results, valid for all non-negative integers n, without requiring any ideas from complex analysis:

$$I_n \equiv \int_0^\infty \frac{(S_n(x))^2}{x^{4n+4}} dx = \frac{\frac{1}{2}\pi}{(4n+3)((2n+1)!)^2} = \int_0^\infty \frac{(C_n(x))^2}{x^{4n+4}} dx \equiv K_n;$$

$$J_n \equiv \int_0^\infty \frac{(S_{n-1}(x))^2}{x^{4n+2}} dx = \frac{\frac{1}{2}\pi}{(4n+1)((2n)!)^2} = \int_0^\infty \frac{(C_n(x))^2}{x^{4n+2}} dx \equiv L_n.$$

See [2] and [3] for two different methods for evaluating these integrals, both of which use results from complex analysis. The computations presented below to prove these facts are a little tedious, but all of the steps are elementary and accessible to students.

We first establish the values of I_n and K_n using mathematical induction. By direct computation, it is not difficult to show that $I_0 = \frac{1}{6}\pi = K_0$. To avoid clutter here, the details are provided later in the paper. (The reader may wish to try evaluating these two integrals as an exercise; the value of α_{-1} plays a role in these computations.) Now suppose that

$$I_{n-1} = \frac{\frac{1}{2}\pi}{(4n-1)((2n-1)!)^2} = K_{n-1}$$

for some positive integer n. Combining the set of derivatives

$$S_n'(x) = C_n(x), \; S_n''(x) = -S_{n-1}(x), \; S'''(x) = -C_{n-1}(x), \; S''''(x) = S_{n-2}(x)$$

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which are valid for each $n \geqslant 1$, along with the fact (sometimes called the general Leibniz rule) that

$$(fg)'''' = fg'''' + 4f'g''' + 6f''g'' + 4f'''g' + f''''g$$

and thus

$$\left(f^2\right)'''' = 2ff'''' = +8f'f''' + 6f''f'',$$

we find that

$$\left((S_n(x))^2\right)'''' = 2S_n(x)S_{n-2}(x) - 8C_n(x)C_{n-1}(x) + 6(S_{n-1}(x))^2.$$

Simplifying two of the terms in this expression yields

$$C_n(x)C_{n-1}(x) = \left(C_{n-1}(x) - \frac{(-1)^n}{(2n)!}x^{2n}\right)C_{n-1}(x) = (C_{n-1}(x))^2 + \frac{(-1)^{n+1}}{(2n)!}x^{2n}C_{n-1}(x);$$

$$\begin{aligned} S_n(x)S_{n-2}(x) &= \left(S_{n-1}(x) - \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)S_{n-2}(x) \\ &= S_{n-1}(x)S_{n-2}(x) + \frac{(-1)^{n+1}}{(2n+1)!}x^{2n+1}S_{n-2}(x) \\ &= S_{n-1}(x)\left(S_{n-1}(x) + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}\right) + \frac{(-1)^{n+1}}{(2n+1)!}x^{2n+1}S_{n-2}(x) \\ &= (S_{n-1}(x))^2 + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}S_{n-1}(x) + \frac{(-1)^{n+1}}{(2n+1)!}x^{2n+1}S_{n-2}(x). \end{aligned}$$

It is an easy exercise to show that the function $(S_n(x))^2$ behaves like x^{4n+2} for large values of x and like x^{4n+6} for values of x near 0, while the function $(C_n(x))^2$ behaves like x^{4n} for large values of x and like x^{4n+4} for values of x near 0; these facts justify the limits at ∞ and 0 that are omitted in the following computations involving improper integrals. Combining these results and integrating by parts four times, we find that

$$\begin{aligned} &\frac{(4n+3)!}{(4n-1)!}\int_0^\infty \frac{(S_n(x))^2}{x^{4n+4}}dx \\ &\quad = \int_0^\infty \frac{\left((S_n(x))^2\right)''''}{x^{4n}}dx \\ &\quad = \int_0^\infty \frac{2S_n(x)S_{n-2}(x) - 8C_n(x)C_{n-1}(x) + 6(S_{n-1}(x))^2}{x^{4n}}dx \\ &\quad = 8\int_0^\infty \frac{(S_{n-1}(x))^2}{x^{4n}}dx - 8\int_0^\infty \frac{(C_{n-1}(x))^2}{x^{4n}}dx - \frac{8(-1)^{n+1}}{(2n)!}\int_0^\infty \frac{C_{n-1}(x)}{x^{2n}}dx \\ &\qquad + \frac{2(-1)^{n-1}}{(2n-1)!}\int_0^\infty \frac{S_{n-1}(x)}{x^{2n+1}}dx + \frac{2(-1)^{n+1}}{(2n+1)!}\int_0^\infty \frac{S_{n-2}(x)}{x^{2n-1}}dx. \end{aligned}$$

Using the induction hypothesis $I_{n-1} = K_{n-1}$ and the integrals represented by α_n and β_n, we have

$$\frac{(4n+3)!}{(4n-1)!}\int_0^\infty \frac{(S_n(x))^2}{x^{4n+4}}dx$$

$$= \frac{8(-1)^n}{(2n)!}\beta_{n-1} - \frac{2(-1)^n}{(2n-1)!}\alpha_{n-1} - \frac{2(-1)^n}{(2n+1)!}\alpha_{n-2}$$

$$= \frac{8(-1)^n}{(2n)!}\cdot\frac{(-1)^n}{(2n-1)!}\cdot\frac{\pi}{2} - \frac{2(-1)^n}{(2n-1)!}\cdot\frac{(-1)^n}{(2n)!}\cdot\frac{\pi}{2} - \frac{2(-1)^n}{(2n+1)!}\cdot\frac{(-1)^{n-1}}{(2n-2)!}\cdot\frac{\pi}{2}$$

$$= \left(\frac{8}{(2n)!(2n-1)!} - \frac{2}{(2n-1)!(2n)!} + \frac{2}{(2n+1)!(2n-2)!}\right)\frac{\pi}{2}$$

$$= \frac{\frac{1}{2}\pi}{((2n+1)!)}\big(6(2n+1)(2n)(2n+1) + 2(2n+1)(2n)(2n-1)\big)$$

$$= \frac{\frac{1}{2}\pi}{((2n+1)!)}\cdot 2n(2n+1)(12n+6+4n-2)$$

$$= \frac{\frac{1}{2}\pi}{((2n+1)!)}\cdot 4n(4n+2)(4n+1)$$

and thus

$$I_n = \int_0^\infty \frac{(S_n(x))^2}{x^{4n+4}} = \frac{\frac{1}{2}\pi}{(4n+3)((2n+1)!)^2}.$$

Using the fact that

$$\left((S_n(x))^2 + (C_n(x))^2\right)' = 2\left(S_n(x)C_n(x) - C(x)S_{n-1}(x)\right) = 2C_n(x)\cdot\frac{(-1)^{n+1}}{(2n+1)!}x^{2n+1}$$

and the known value for β_n, we obtain

$$\begin{aligned} I_n + K_n &= \int_0^\infty \frac{(S(x))^2 + (C(x))^2}{x^{4n+4}}dx \\ &= \frac{1}{4n+3}\int_0^\infty \frac{\left(((S_n(x))^2 + (C_n(x))^2\right)}{x^{4n+3}}dx \\ &= \frac{2}{4n+3}\cdot\frac{(-1)^{n+1}}{(2n+1)!}\int_0^\infty \frac{C_n(x)}{x^{2n+2}}dx \\ &= \frac{2}{4n+3}\cdot\frac{(-1)^{n+1}}{(2n+1)!}\cdot\frac{(-1)^{n+1}}{(2n+1)!}\cdot\frac{\pi}{2} \\ &= 2I_n. \end{aligned}$$

This shows that $K_n = I_n$. Hence, the values for I_n and K_n for all $n \geqslant 0$ follow by induction.

We now turn to the J_n and L_n integrals. The integrals J_0 and L_0 are easy to evaluate (see the Appendix for details), while the value of J_1 appears as

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part (a) of Problem 105.H in the July 2021 issue of the *Gazette*. However, these values all follow from the calculations below, which do not rely on induction. Suppose that n is a nonnegative integer. Using the fact that

$$\begin{aligned}\tfrac{1}{2}\left((S_n(x))^2\right)'' &= \left(S_n(x)C_n(x)\right)' = (C_n(x))^2 - S_n(x)S_{n-1}(x)\\ &= (C_n(x))^2 - \left(S_{n-1}(x) - \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)S_{n-1}(x)\\ &= (C_n(x))^2 - (S_{n-1}(x))^2 + \frac{(-1)^n}{(2n+1)!}x^{2n+1}S_{n-1}(x)\end{aligned}$$

and the known values for α_{n-1} and I_n, we obtain (using integration by parts)

$$\begin{aligned}I_n &= \frac{1}{(4n+3)(4n+2)}\int_0^\infty \frac{\left((S_n(x))^2\right)''}{x^{4n+2}}dx\\ &= \frac{2}{(4n+3)(4n+2)}\left(\int_0^\infty \frac{(C_n(x))^2}{x^{4n+2}}dx - \int_0^\infty \frac{(S_{n-1}(x))^2}{x^{4n+2}}dx + \frac{(-1)^n}{(2n+1)!}\int_0^\infty \frac{S_{n-1}(x)}{x^{2n+1}}dx\right)\\ &= \frac{1}{(4n+3)(2n+1)}\left(L_n - J_n + \frac{(-1)^n}{(2n+1)!}\cdot\frac{(-1)}{(2n)!}\cdot\frac{\pi}{2}\right)\\ &= \frac{1}{(4n+3)(2n+1)}\left(L_n - J_n\right) + \frac{\frac{1}{2}\pi}{(4n+3)((2n+1)!)^2}\\ &= \frac{1}{(4n+3)(2n+1)}\left(L_n - J_n\right) + I_n.\end{aligned}$$

This shows that $L_n = J_n$ for all $n \geqslant 0$. Paralleling our previous work, using the fact that

$$\left((S_{n-1}(x))^2 + (C_n(x))^2\right)' = 2\left(S_{n-1}(x)C_{n-1}(x) - C_n(x)S_{n-1}(x)\right) = 2S_{n-1}(x)\cdot\frac{(-1)^n}{(2n)!}x^{2n}$$

and the known value for α_{n-1}, we obtain (using integration by parts yet again)

$$\begin{aligned}J_n + L_n &= \int_0^\infty \frac{(S_{n-1}(x))^2 + (C_n(x))^2}{x^{4n+2}}dx\\ &= \frac{1}{4n+1}\int_0^\infty \frac{\left((S_{n-1}(x))^2 + (C_n(x))^2\right)'}{x^{4n+1}}dx\\ &= \frac{2}{4n+1}\cdot\frac{(-1)^n}{(2n)!}\int_0^\infty \frac{S_{n-1}(x)}{x^{2n+1}}dx\\ &= \frac{2}{4n+1}\cdot\frac{(-1)^n}{(2n)!}\cdot\frac{(-1)^n}{(2n)!}\cdot\frac{\pi}{2}.\end{aligned}$$

It follows that

$$J_n = \frac{\frac{1}{2}\pi}{(4n+1)((2n)!)^2} = L_n$$

for all non-negative integers n.

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To find I_0 and K_0, we must evaluate the integrals

$$\int_0^\infty \frac{(\sin x - x)^2}{x^4}dx \qquad \text{and} \qquad \int_0^\infty \frac{(\cos x - 1)^2}{x^4}dx,$$

respectively. With $f(x) = (\sin x - x)^2$ and $g(x) = (\cos x - 1)^2$, we find that

$$f(x) = \sin^2 x - 2x \sin x + x^2; \qquad g(x) = \cos^2 x - 2\cos x + 1;$$
$$f'(x) = \sin(2x) - 2\sin x - 2x\cos x + 2x; \qquad g'(x) = -\sin(2x) + 2\sin x;$$
$$f''(x) = 2\cos(2x) - 4\cos x + 2x\sin x + 2; \qquad g''(x) = -2\cos(2x) + 2\cos x.$$

Note that the Maclaurin series for f has the form $f(x) = \frac{1}{36}x^6 - \frac{1}{360}x^8 + \dots$ while the Maclaurin series for g has the form $g(x) = \frac{1}{4}x^4 - \frac{1}{24}x^6 + \dots$. Using integration by parts and the value of α_{-1}, while omitting the evaluation of some simple limits at ∞ and 0 (using both the explicit series forms for each of the functions), it follows that

$$\begin{aligned}
I_0 = \int_0^\infty \frac{f(x)}{x^4}dx &= \int_0^\infty \frac{f'(x)}{3x^3}dx = \int_0^\infty \frac{f''(x)}{6x^2}dx \\
&= \frac{1}{3}\int_0^\infty \frac{\cos(2x) - 2\cos x + 1}{x^2}dx + \frac{1}{3}\int_0^\infty \frac{\sin x}{x}dx \\
&= \frac{1}{3}\int_0^\infty \frac{-2\sin(2x) + 2\sin x}{x}dx + \frac{1}{3}\int_0^\infty \frac{\sin x}{x}dx \\
&= -\frac{2}{3}\int_0^\infty \frac{2\sin(2x)}{x}dx + \frac{2}{3}\int_0^\infty \frac{\sin x}{x}dx + \frac{1}{3}\int_0^\infty \frac{\sin x}{x}dx \\
&= \frac{\pi}{6}
\end{aligned}$$

and

$$\begin{aligned}
K_0 = \int_0^\infty \frac{g(x)}{x^4}dx &= \int_0^\infty \frac{g'(x)}{3x^3}dx = \int_0^\infty \frac{g''(x)}{6x^2}dx \\
&= \frac{1}{3}\int_0^\infty \frac{-\cos(2x) + \cos x}{x^2}dx \\
&= \frac{1}{3}\int_0^\infty \frac{2\sin(2x) - \sin x}{x}dx \\
&= \frac{2}{3}\int_0^\infty \frac{\sin(2x)}{x}dx - \frac{1}{3}\int_0^\infty \frac{\sin x}{x}dx \\
&= \frac{\pi}{6}.
\end{aligned}$$

We have thus found the values of all four collections of integrals using elementary ideas and concepts along with the familiar value of α_{-1}.

An observant reader may have noticed that there are two more collections of integrals involving the functions $(S_n(x))^2$ and $(C_n(x))^2$ with denominators involving integer powers of x. These integrals are more difficult to evaluate; details can be found in [3]. However, for completeness, we record the values of these integrals below. Using the notation $h_k = \sum_{j=1}^{k} \frac{1}{j}$ for the harmonic numbers, we find that

$$\int_0^\infty \frac{(S_n(x))^2}{x^{4n+5}}dx = \frac{2}{(4n+4)!}\left(2^{4n+2}\ln 2 - 2^{4n+2}h_{4n+4} + \sum_{k=n+1}^{2n+1}\binom{4n+4}{2k+1}h_{2k+1}\right);$$

$$\int_0^\infty \frac{(C_n(x))^2}{x^{4n+3}}dx = \frac{2}{(4n+2)!}\left(2^{4n}\ln 2 - 2^{4n}h_{4n+2} + \sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}h_{2k}\right).$$

It is rather intriguing that the values of these integrals are more complicated than those of the previous four collections.

Appendix

We want to prove that

$$\int_0^\infty \frac{S_n(x)}{x^{2n+3}}dx = \frac{(-1)^{n+1}}{(2n+2)!}\cdot\frac{\pi}{2} \quad\text{and}\quad \int_0^\infty \frac{C_n(x)}{x^{2n+2}}dx = \frac{(-1)^{n+1}}{(2n+1)!}\cdot\frac{\pi}{2},$$

where the sine equation is valid for all $n \geqslant -1$ and the cosine equation is valid for all $n \geqslant 0$. The sine case for $n = -1$ is a well-known result (referred to as α_{-1} in the body of this paper). When $n = 0$, we find that (using integration by parts and evaluating the required limits at ∞ and 0 implicitly) that

$$\begin{aligned}\int_0^\infty \frac{S_0(x)}{x^3}dx = \int_0^\infty \frac{\sin x - x}{x^3}dx &= \left[\frac{\sin x - x}{-2x^2}\right]_0^\infty + \int_0^\infty \frac{\cos x - 1}{2x^2}dx \\ &= \int_0^\infty \frac{\cos x - 1}{2x^2}dx = \frac{1}{2}\int_0^\infty \frac{C_0(x)}{x^2}dx \\ &= \left[\frac{\cos x - 1}{-2x}\right]_0^\infty - \int_0^\infty \frac{\sin x}{x}dx \\ &= -\frac{\pi}{4}.\end{aligned}$$

Note that this set of equations gives the desired result for both the S_0 integral and the C_0 integral. Now assume that the sine and cosine integral results hold for some non-negative integer n. We then have (once again using integration by parts and evaluating the required limits implicitly)

$$\begin{aligned}\int_0^\infty \frac{S_{n+1}(x)}{x^{2n+5}}dx &= \left[\frac{S_{n+1}(x)}{-(2n+4)x^{2n+4}}\right]_0^\infty + \frac{1}{2n+4}\int_0^\infty \frac{C_{n+1}(x)}{x^{2n+4}}dx \\ &= \frac{1}{2n+4}\int_0^\infty \frac{C_{n+1}(x)}{x^{2n+4}}dx \\ &= \frac{1}{2n+4}\left(\left[\frac{C_{n+1}(x)}{-(2n+3)x^{2n+3}}\right]_0^\infty + \frac{1}{2n+3}\int_0^\infty \frac{-S_n(x)}{x^{2n+3}}dx\right)\end{aligned}$$

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$$= \frac{-1}{(2n+4)(2n+3)} \int_0^\infty \frac{S_n(x)}{x^{2n+3}} dx$$

$$= \frac{-1}{(2n+4)(2n+3)} \cdot \frac{(-1)^{n+1}}{(2n+2)!} \cdot \frac{\pi}{2}$$

$$= \frac{(-1)^{n+2}}{(2n+4)!} \cdot \frac{\pi}{2}$$

and (looking inside the above equation)

$$\int_0^\infty \frac{C_{n+1}(x)}{x^{2n+4}} dx = \frac{(-1)^{n+2}}{(2n+3)!} \cdot \frac{\pi}{2}.$$

The results for both sets of integrals now follow by induction.

To evaluate the values for J_0 and L_0 directly, we use integration by parts and the value of α_{-1} to compute

$$J_0 = \int_0^\infty \frac{\sin^2 x}{x^2} dx = \left[\frac{\sin^2 x}{-x} \right]_0^\infty + \int_0^\infty \frac{\sin(2x)}{x} dx = \frac{\pi}{2}$$

and

$$L_0 = \int_0^\infty \frac{(\cos x - 1)^2}{x^2} dx = \int_0^\infty \frac{\cos^2 x - 2\cos x + 1}{x^2} dx$$

$$= \int_0^\infty \frac{2 - 2\cos x - \sin^2 x}{x^2} dx = \int_0^\infty \frac{4\sin^2(\frac{1}{2}x) - \sin^2 x}{x^2} dx$$

$$= 2\int_0^\infty \frac{\sin^2(\frac{1}{2}x)}{(\frac{1}{2}x)^2} \cdot \frac{dx}{2} - \int_0^\infty \frac{\sin^2 x}{x^2} dx = 2J_0 - J_0 = \frac{\pi}{2}.$$

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