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DIFFERENTIAL OPERATORS ON QUANTIZED FLAG MANIFOLDS AT ROOTS OF UNITY, II

TOSHIYUKI TANISAKI

To Etsuro Date on his 60th birthday

Abstract. We formulate a Beilinson–Bernstein-type derived equivalence for a quantized enveloping algebra at a root of 1 as a conjecture. It says that there exists a derived equivalence between the category of modules over a quantized enveloping algebra at a root of 1 with fixed regular Harish-Chandra central character and the category of certain twisted D-modules on the corresponding quantized flag manifold. We show that the proof is reduced to a statement about the (derived) global sections of the ring of differential operators on the quantized flag manifold. We also give a reformulation of the conjecture in terms of the (derived) induction functor.

§0. Introduction

0.1.

Let G be a connected, simply connected simple algebraic group over $\mathbb{C}$, and let H be a maximal torus of G. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Let Q and Λ be the root lattice and the weight lattice, respectively. Let h_G be the Coxeter number of G. We fix an odd integer $\ell > h_G$, which is prime to the order of Λ/Q and prime to 3 if $\mathfrak{g}$ is of type G_2, F_4, E_6, E_7, E_8, and we consider the De Concini–Kac-type quantized enveloping algebra U_ζ at $q = \zeta = \exp(2\pi\sqrt{-1}/\ell)$.

In [20], we started the investigation of the corresponding quantized flag manifold $\mathcal{B}_\zeta$, which is a noncommutative scheme, and the category of D-modules on it. In view of a general philosophy saying that quantized objects at roots of 1 resemble ordinary objects in positive characteristics, it

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is natural to pursue an analogue of the theory of D-modules on the ordinary flag manifolds in positive characteristics due to Bezrukavnikov, Mirković, and Rumynin [6]. Along this line, we have established in [20] certain Azumaya properties of the ring of differential operators on the quantized flag manifold. The aim of the present article is to investigate an analogue of another main point of [6] about the Beilinson–Bernstein-type derived equivalence.

0.2.

We denote by $\mathcal{D}_{\mathcal{B}_\zeta,1}$ the sheaf of rings of differential operators on the quantized flag manifold $\mathcal{B}_\zeta$. More generally, for each $t \in H$ we have its twisted analogue denoted by $\mathcal{D}_{\mathcal{B}_\zeta,t}$. It is obtained as the specialization $\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[H]} \mathbb{C}$ of the universally twisted sheaf $\mathcal{D}_{\mathcal{B}_\zeta}$ with respect to the ring homomorphism $\mathbb{C}[H] \to \mathbb{C}$ corresponding to $t \in H$.

Let $\mathcal{B}$ be the ordinary flag manifold for G. Then we have a Frobenius morphism $\mathrm{Fr}: \mathcal{B}_\zeta \to \mathcal{B}$, which is a finite morphism from a noncommutative scheme to an ordinary scheme. Taking the direct images, we obtain sheaves $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}, \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}$ $(t \in H)$ of rings on $\mathcal{B}$ (in the ordinary sense). Denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})$ the category of coherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules. Let $Z_{\mathrm{Har}}(U_\zeta)$ be the Harish-Chandra center of U_ζ, and let $\mathbb{C}_t$ be the corresponding 1-dimensional $Z_{\mathrm{Har}}(U_\zeta)$-module. Denote by $\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)$ the category of finitely generated $U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t$-modules. Then we have a functor

$$(0.1) \qquad R\Gamma(\mathcal{B}, \bullet) : D^b\big(\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})\big) \to D^b\big(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)\big)$$

between derived categories. It is natural in view of [6] to conjecture that (0.1) gives an equivalence if t is regular. By imitating the argument of [6], we can show that this is true if we have

$$(0.2) \qquad R\Gamma(\mathcal{B}, \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}) \cong U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda].$$

However, we do not know how to prove (0.2) at present; hence, we can only state it as a conjecture. We have also a stronger conjecture,

$$(0.3) \qquad R\Gamma\big(\mathcal{B}, \mathrm{Fr}_*(\mathcal{D}_{\mathcal{B}_\zeta})_f\big) \cong U_{\zeta,f} \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda],$$

which is the analogue of (0.2) regarding the adjoint finite parts $(\mathcal{D}_{\mathcal{B}_\zeta})_f$, $U_{\zeta,f}$ of $\mathcal{D}_{\mathcal{B}_\zeta}$, U_ζ, respectively. We will give a reformulation of (0.3) in terms of the induction functor (see Conjecture 5.2 below). It turns out that (0.3) is equivalent to some assertions in Backelin and Kremnizer [2], [3] stated to be true under certain conditions on ℓ (see Remark 5.4 below).

It is also an interesting problem to find a formulation which works even in the case when the parameter $t \in H$ is singular. In the case of Lie algebras in positive characteristics, Bezrukavnikov, Mirković, and Rumynin in [5] have succeeded in giving a more general framework, which works even for singular parameters, using partial flag manifolds (quotients of G by parabolic subgroups). In their case, the parameter space is $\mathfrak{h}^*$, and one can associate for each $h \in \mathfrak{h}^*$ a parabolic subgroup whose Levi subgroup is the centralizer of h; however, in our case the centralizer of $t \in H$ is not necessarily a Levi subgroup of a parabolic subgroup, and hence the method in [5] cannot be directly applied to our case.

0.3.

This article has the following organization. In Section 1, we recall basic facts on quantized enveloping algebras at roots of 1 and the corresponding quantized flag manifolds. In Section 2, we investigate properties of the category of D-modules. In particular, we show that (0.2) implies (0.1) for regular t and that (0.3) implies (0.2). In Sections 3 and 4, we recall some known results on the representations of quantized enveloping algebras and the induction functor, respectively. Finally, in Section 5 we give a reformulation of (0.3) in terms of the induction functor.

§1. Quantized flag manifold

1.1. Quantized enveloping algebras

1.1.1. Let G be a connected simply connected simple algebraic group over the complex number field $\mathbb{C}$. We fix Borel subgroups B^+ and B^- such that $H = B^+ \cap B^-$ is a maximal torus of G. Set $N^+ = [B^+, B^+]$, and set $N^- = [B^-, B^-]$. We denote the Lie algebras of G, B^+, B^-, H, N^+, N^- by $\mathfrak{g}$, $\mathfrak{b}^+$, $\mathfrak{b}^-$, $\mathfrak{h}$, $\mathfrak{n}^+$, $\mathfrak{n}^-$, respectively. Let $\Delta \subset \mathfrak{h}^*$ be the root system of $(\mathfrak{g}, \mathfrak{h})$. We denote by $\Lambda \subset \mathfrak{h}^*$ and $Q \subset \mathfrak{h}^*$ the weight lattice and the root lattice, respectively. For $\lambda \in \Lambda$ we denote by θ_λ the corresponding character of H. The coordinate algebra $\mathbb{C}[H]$ of H is naturally identified with the group algebra $\mathbb{C}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}e(\lambda)$ via the correspondence $\theta_\lambda \leftrightarrow e(\lambda)$ for $\lambda \in \Lambda$. We take a system of positive roots Δ^+ such that $\mathfrak{b}^+$ is the sum of weight spaces with weights in $\Delta^+ \cup \{0\}$. Let $\{\alpha_i\}_{i \in I}$ be the set of simple roots, and let $\{\varpi_i\}_{i \in I}$ be the corresponding set of fundamental weights. We denote by Λ^+ the set of dominant integral weights. We set $Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geqq 0}\alpha_i$. Let $W \subset$

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$GL(\mathfrak{h}^*)$ be the Weyl group. For $i \in I$ we denote by $s_i \in W$ the corresponding simple reflection. We take a W-invariant symmetric bilinear form

$$(\, , \,) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$$

such that $(\alpha, \alpha) = 2$ for short roots α. For $\alpha \in \Delta$ we set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. For $i \in I$ we fix $\bar{e}_i \in \mathfrak{g}_{\alpha_i}$, $\bar{f}_i \in \mathfrak{g}_{-\alpha_i}$ such that $[\bar{e}_i, \bar{f}_i] = \alpha_i^\vee$ under the identification $\mathfrak{h} = \mathfrak{h}^*$ induced by $(\, , \,)$.

1.1.2. For $n \in \mathbb{Z}_{\geqq 0}$ we set

$$[n]_t = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}],$$
$$[n]_t! = [n]_t [n-1]_t \cdots [2]_t [1]_t \in \mathbb{Z}[t, t^{-1}].$$

We denote by $U_{\mathbb{F}}$ the quantized enveloping algebra over $\mathbb{F} = \mathbb{Q}(q^{1/|\Lambda/Q|})$ associated to $\mathfrak{g}$. Namely, $U_{\mathbb{F}}$ is the associative algebra over $\mathbb{F}$ generated by elements

$$k_\lambda \quad (\lambda \in \Lambda), \qquad e_i, f_i \quad (i \in I)$$

satisfying the relations

$$k_0 = 1, \qquad k_\lambda k_\mu = k_{\lambda+\mu} \quad (\lambda, \mu \in \Lambda),$$
$$k_\lambda e_i k_\lambda^{-1} = q^{(\lambda, \alpha_i)} e_i \quad (\lambda \in \Lambda, i \in I),$$
$$k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda, \alpha_i)} f_i \quad (\lambda \in \Lambda, i \in I),$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I),$$
$$\sum_{n=0}^{1-a_{ij}} (-1)^n e_i^{(1-a_{ij}-n)} e_j e_i^{(n)} = 0 \quad (i, j \in I, i \neq j),$$
$$\sum_{n=0}^{1-a_{ij}} (-1)^n f_i^{(1-a_{ij}-n)} f_j f_i^{(n)} = 0 \quad (i, j \in I, i \neq j),$$

where $q_i = q^{(\alpha_i, \alpha_i)/2}, k_i = k_{\alpha_i}, a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ for $i, j \in I$, and

$$e_i^{(n)} = e_i^n / [n]_{q_i}!, \qquad f_i^{(n)} = f_i^n / [n]_{q_i}!$$

for $i \in I$ and $n \in \mathbb{Z}_{\geqq 0}$. We will use the Hopf algebra structure of $U_{\mathbb{F}}$ given by

$$\begin{aligned}
&\Delta(k_\lambda) = k_\lambda \otimes k_\lambda \quad (\lambda \in \Lambda),\\
&\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \qquad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i \quad (i \in I),\\
&\varepsilon(k_\lambda) = 1, \qquad \varepsilon(e_i) = \varepsilon(f_i) = 0 \quad (\lambda \in \Lambda, i \in I),\\
&S(k_\lambda) = k_\lambda^{-1}, \qquad S(e_i) = -k_i^{-1} e_i, \qquad S(f_i) = -f_i k_i \quad (\lambda \in \Lambda, i \in I).
\end{aligned}$$

Define subalgebras $U_{\mathbb{F}}^0$, $U_{\mathbb{F}}^+$, $U_{\mathbb{F}}^-$, $U_{\mathbb{F}}^{\geqq 0}$, $U_{\mathbb{F}}^{\leqq 0}$ of $U_{\mathbb{F}}$ by

$$\begin{aligned}
&U_{\mathbb{F}}^0 = \langle k_\lambda \mid \lambda \in \Lambda \rangle, \qquad U_{\mathbb{F}}^+ = \langle e_i \mid i \in I \rangle, \qquad U_{\mathbb{F}}^- = \langle f_i \mid i \in I \rangle,\\
&U_{\mathbb{F}}^{\geqq 0} = \langle k_\lambda, e_i \mid \lambda \in \Lambda, i \in I \rangle, \qquad U_{\mathbb{F}}^{\leqq 0} = \langle k_\lambda, f_i \mid \lambda \in \Lambda, i \in I \rangle.
\end{aligned}$$

The multiplication of $U_{\mathbb{F}}$ induces isomorphisms

$$U_{\mathbb{F}} \cong U_{\mathbb{F}}^- \otimes U_{\mathbb{F}}^0 \otimes U_{\mathbb{F}}^+ \cong U_{\mathbb{F}}^+ \otimes U_{\mathbb{F}}^0 \otimes U_{\mathbb{F}}^-, \tag{1.1}$$

$$U_{\mathbb{F}}^{\geqq 0} \cong U_{\mathbb{F}}^0 \otimes U_{\mathbb{F}}^+ \cong U_{\mathbb{F}}^+ \otimes U_{\mathbb{F}}^0, \tag{1.2}$$

$$U_{\mathbb{F}}^{\leqq 0} \cong U_{\mathbb{F}}^0 \otimes U_{\mathbb{F}}^- \cong U_{\mathbb{F}}^- \otimes U_{\mathbb{F}}^0, \tag{1.3}$$

of $\mathbb{F}$-modules. The fact (1.1) is called the *triangular decomposition* of $U_{\mathbb{F}}$. For $\gamma \in Q$ we set

$$U_{\mathbb{F},\gamma}^{\pm} = \left\{ u \in U_{\mathbb{F}}^{\pm} \;\middle|\; k_\mu u k_{-\mu} = q^{(\gamma,\mu)} u \ (\mu \in \Lambda) \right\}.$$

Then we have

$$U_{\mathbb{F}}^{\pm} = \bigoplus_{\gamma \in Q^+} U_{\mathbb{F},\pm\gamma}^{\pm}.$$

For $i \in I$ we denote by T_i the automorphism of the algebra $U_{\mathbb{F}}$ given by

$$\begin{aligned}
&T_i(k_\mu) = k_{s_i\mu} (\mu \in \Lambda),\\
&T_i(e_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} e_i^{(-a_{ij}-k)} e_j e_i^{(k)} & (j \in I, j \neq i),\\ -f_i k_i & (j = i), \end{cases}\\
&T_i(f_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{k} f_i^{(k)} f_j f_i^{(-a_{ij}-k)} & (j \in I, j \neq i),\\ -k_i^{-1} e_i & (j = i) \end{cases}
\end{aligned}$$

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(see [15]). Let w_0 be the longest element of W. We fix a reduced expression

$$w_0 = s_{i_1} \cdots s_{i_N}$$

of w_0, and we set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad (1 \leqq k \leqq N).$$

Then we have $\Delta^+ = \{\beta_k \mid 1 \leqq k \leqq N\}$. For $1 \leqq k \leqq N$ we set

$$e_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(e_{i_k}), \qquad f_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(f_{i_k}).$$

Then $\{e_{\beta_N}^{m_N} \cdots e_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$ (resp., $\{f_{\beta_N}^{m_N} \cdots f_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$) is an $\mathbb{F}$-basis of $U_{\mathbb{F}}^+$ (resp., $U_{\mathbb{F}}^-$), called the *PBW (Poincaré–Birkhoff–Witt) basis* (see [14]). We have $e_{\alpha_i} = e_i$ and $f_{\alpha_i} = f_i$ for any $i \in I$.

Denote by

$$\tau : U_{\mathbb{F}}^{\geqq 0} \times U_{\mathbb{F}}^{\leqq 0} \to \mathbb{F} \tag{1.4}$$

the Drinfeld pairing. It is characterized as the unique bilinear form satisfying

$$\begin{aligned}
\tau(x, y_1 y_2) &= (\tau \otimes \tau)\big(\Delta(x), y_1 \otimes y_2\big) \quad (x \in U_{\mathbb{F}}^{\geqq 0}, y_1, y_2 \in U_{\mathbb{F}}^{\leqq 0}),\\
\tau(x_1 x_2, y) &= (\tau \otimes \tau)\big(x_2 \otimes x_1, \Delta(y)\big) \quad (x_1, x_2 \in U_{\mathbb{F}}^{\geqq 0}, y \in U_{\mathbb{F}}^{\leqq 0}),\\
\tau(k_\lambda, k_\mu) &= q^{-(\lambda,\mu)} \quad (\lambda, \mu \in \Lambda),\\
\tau(k_\lambda, f_i) &= \tau(e_i, k_\lambda) = 0 \quad (\lambda \in \Lambda, i \in I),\\
\tau(e_i, f_j) &= \delta_{ij}/(q_i^{-1} - q_i) \quad (i, j \in I)
\end{aligned}$$

(see [15], [18]). It also satisfies the following.

Lemma 1.1 ([15, Section 1.2], [18, Proposition 2.1.1]). *We have the following:*

(i) $\tau(S(x), S(y)) = \tau(x, y)$ *for* $x \in U_{\mathbb{F}}^{\geqq 0}, y \in U_{\mathbb{F}}^{\leqq 0}$;

(ii) *for* $x \in U_{\mathbb{F}}^{\geqq 0}, y \in U_{\mathbb{F}}^{\leqq 0}$ *we have*

$$yx = \sum_{(x)_2, (y)_2} \tau\big(x_{(0)}, S(y_{(0)})\big)\tau(x_{(2)}, y_{(2)}) x_{(1)} y_{(1)},$$

$$xy = \sum_{(x)_2, (y)_2} \tau(x_{(0)}, y_{(0)})\tau\big(x_{(2)}, S(y_{(2)})\big) y_{(1)} x_{(1)};$$

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(iii) $\tau(xk_\lambda, yk_\mu) = q^{-(\lambda,\mu)}\tau(x,y)$ *for* $\lambda, \mu \in \Lambda, x \in U_{\mathbb{F}}^+, y \in U_{\mathbb{F}}^-$*;*
(iv) $\tau(U_{\mathbb{F},\beta}^+, U_{\mathbb{F},-\gamma}^-) = \{0\}$ *for* $\beta, \gamma \in Q^+$ *with* $\beta \neq \gamma$*;*
(v) *for any* $\beta \in Q^+$*, the restriction* $\tau|_{U_{\mathbb{F},\beta}^+ \times U_{\mathbb{F},-\beta}^-}$ *is nondegenerate.*

We define an algebra homomorphism

$$\mathrm{ad} : U_{\mathbb{F}} \to \mathrm{End}_{\mathbb{F}}(U_{\mathbb{F}})$$

by

$$\mathrm{ad}(u)(v) = \sum_{(u)} u_{(0)} v(Su_{(1)}) \quad (u, v \in U_{\mathbb{F}}).$$

1.1.3. We fix an integer $\ell > 1$ satisfying

(a) ℓ is odd;
(b) ℓ is prime to 3 if G is of type G_2, F_4, E_6, E_7, E_8;
(c) ℓ is prime to $|\Lambda/Q|$;

and a primitive ℓth root $\zeta' \in \mathbb{C}$ of 1. Define a subring $\mathbb{A}$ of $\mathbb{F}$ by

$$\mathbb{A} = \{ f(q^{1/|\Lambda/Q|}) \mid f(x) \in \mathbb{Q}(x), f \text{ is regular at } x = \zeta' \}.$$

We set $\zeta = (\zeta')^{|\Lambda/Q|}$. We note that ζ is also a primitive ℓth root of 1 by condition (c).

We denote by $U_{\mathbb{A}}^L$, $U_{\mathbb{A}}$ the $\mathbb{A}$-forms of $U_{\mathbb{F}}$ called the *Lusztig form* and the *De Concini–Kac form*, respectively. Namely, we have

$$U_{\mathbb{A}}^L = \langle e_i^{(m)}, f_i^{(m)}, k_\lambda \mid i \in I, m \in \mathbb{Z}_{\geqq 0}, \lambda \in \Lambda \rangle_{\mathbb{A}\text{-alg}} \subset U_{\mathbb{F}},$$
$$U_{\mathbb{A}} = \langle e_i, f_i, k_\lambda \mid i \in I, \lambda \in \Lambda \rangle_{\mathbb{A}\text{-alg}} \subset U_{\mathbb{F}}.$$

We have obviously $U_{\mathbb{A}} \subset U_{\mathbb{A}}^L$. The Hopf algebra structure of $U_{\mathbb{F}}$ induces Hopf algebra structures over $\mathbb{A}$ of $U_{\mathbb{A}}^L$ and $U_{\mathbb{A}}$. We set

$$U_{\mathbb{A}}^{L,\flat} = U_{\mathbb{A}}^L \cap U_{\mathbb{F}}^\flat, \qquad U_{\mathbb{A}}^\flat = U_{\mathbb{A}} \cap U_{\mathbb{F}}^\flat \quad (\flat = +, -, 0, \geqq 0, \leqq 0),$$
$$U_{\mathbb{A},\pm\gamma}^{L,\pm} = U_{\mathbb{A}}^L \cap U_{\mathbb{F},\pm\gamma}^\pm, \qquad U_{\mathbb{A},\pm\gamma}^\pm = U_{\mathbb{A}} \cap U_{\mathbb{F},\pm\gamma}^\pm \quad (\gamma \in Q^+).$$

Then we have triangular decompositions

$$U_{\mathbb{A}}^L \cong U_{\mathbb{A}}^{L,-} \otimes_{\mathbb{A}} U_{\mathbb{A}}^{L,0} \otimes_{\mathbb{A}} U_{\mathbb{A}}^{L,+},$$
$$U_{\mathbb{A}} \cong U_{\mathbb{A}}^- \otimes_{\mathbb{A}} U_{\mathbb{A}}^0 \otimes_{\mathbb{A}} U_{\mathbb{A}}^+.$$

Moreover, we have

$$U_{\mathbb{A}}^{L,\pm} = \bigoplus_{\gamma \in Q^+} U_{\mathbb{A},\pm\gamma}^{L,\pm}, \qquad U_{\mathbb{A}}^{\pm} = \bigoplus_{\gamma \in Q^+} U_{\mathbb{A},\pm\gamma}^{\pm}.$$

The Drinfeld pairing (1.4) induces

$$({1.5}) \qquad {}^L\tau_{\mathbb{A}} : U_{\mathbb{A}}^{L,\geqq 0} \times U_{\mathbb{A}}^{\leqq 0} \to \mathbb{A}, \qquad \tau_{\mathbb{A}}^L : U_{\mathbb{A}}^{\geqq 0} \times U_{\mathbb{A}}^{L,\leqq 0} \to \mathbb{A}.$$

LEMMA 1.2. *We have* $\mathrm{ad}(U_{\mathbb{A}}^L)(U_{\mathbb{A}}) \subset U_{\mathbb{A}}$.

Proof. It is sufficient to show that

$$({1.6}) \qquad \mathrm{ad}(k_\lambda)(U_{\mathbb{A}}) \subset U_{\mathbb{A}} \quad (\lambda \in \Lambda),$$

$$({1.7}) \qquad \mathrm{ad}(e_i^{(n)})(U_{\mathbb{A}}) \subset U_{\mathbb{A}} \quad (i \in I, n \in \mathbb{Z}_{\geqq 0}),$$

$$({1.8}) \qquad \mathrm{ad}(f_i^{(n)})(U_{\mathbb{A}}) \subset U_{\mathbb{A}} \quad (i \in I, n \in \mathbb{Z}_{\geqq 0}).$$

The proof of (1.6) is easy and omitted. By the formulas

$$\mathrm{ad}(x)(uv) = \sum_{(x)} \mathrm{ad}(x_{(0)})(u)\mathrm{ad}(x_{(1)})(v) \quad (x \in U_{\mathbb{A}}^L, u, v \in U_{\mathbb{A}}),$$

$$\Delta(e_i^{(n)}) = \sum_{r=0}^{n} q_i^{r(n-r)} e_i^{(n-r)} k_i^r \otimes e_i^{(r)} \quad (i \in I, n \geqq 0),$$

$$\Delta(f_i^{(n)}) = \sum_{r=0}^{n} q_i^{-r(n-r)} f_i^{(r)} \otimes k_i^{-r} f_i^{(n-r)} \quad (i \in I, n \geqq 0),$$

we have only to show that

$$({1.9}) \qquad \mathrm{ad}(e_i^{(n)})(u) \in U_{\mathbb{A}} \quad (i \in I, n \in \mathbb{Z}_{\geqq 0}, u = k_\lambda, e_j, f_j k_j),$$

$$({1.10}) \qquad \mathrm{ad}(f_i^{(n)})(u) \in U_{\mathbb{A}} \quad (i \in I, n \in \mathbb{Z}_{\geqq 0}, u = k_\lambda, e_j, f_j).$$

For $\lambda \in \Lambda$, $i, j \in I$ with $i \neq j$ and $n \in \mathbb{Z}_{>0}$, we have

$$\mathrm{ad}(e_i^{(n)})(k_\lambda) = \frac{(-1)^n q_i^{n(n-1)}}{[n]_{q_i}!} \Big(\prod_{j=0}^{n-1} (q_i^{(\lambda, \alpha_i^\vee)} - q_i^{-2j}) \Big) e_i^n k_\lambda,$$

$$\mathrm{ad}(e_i^{(n)})(e_i) = q_i^{-n(n+1)/2} (q_i - q_i^{-1})^n e_i^{n+1},$$

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$$\mathrm{ad}(e_i^{(n)})(e_j)=\begin{cases}\sum_{r=0}^n(-1)^r q_i^{r(n-1+a_{ij})}e_i^{(n-r)}e_je_i^{(r)} & (n<1-a_{ij}),\\ 0 & (n\geqq 1-a_{ij}),\end{cases}$$

$$\mathrm{ad}(e_i^{(n)})(f_ik_i)=\begin{cases}(k_i^2-1)/(q_i-q_i^{-1}) & (n=1),\\ (-1)^{n-1}q_i^{(n-1)(n+2)/2}(q_i-q_i^{-1})^{n-2}e_i^{n-1}k_i^2 & (n>1),\end{cases}$$

$$\mathrm{ad}(e_i^{(n)})(f_jk_j)=0,$$

and hence (1.9) holds. (Note that $[r]_{q_i}!$ is invertible in $\mathbb{A}$ for $r\leqq -a_{ij}$.) The proof of (1.10) is similar and omitted. □

1.1.4. Let us consider the specialization

$$\mathbb{A}\to\mathbb{C}\quad (q^{1/|\Lambda/Q|}\mapsto\zeta').$$

Note that q is mapped to $\zeta=(\zeta')^{|\Lambda/Q|}\in\mathbb{C}$, which is also a primitive ℓth root of 1. We set

$$\begin{aligned} U_\zeta^L&=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}}^L, && U_\zeta=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}},\\ U_\zeta^{L,\flat}&=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}}^{L,\flat}, && U_\zeta^{\flat}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}}^{\flat}\quad(\flat=+,-,0,\geqq 0,\leqq 0),\\ U_{\zeta,\pm\gamma}^{L,\pm}&=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A},\pm\gamma}^{L,\pm}, && U_{\zeta,\pm\gamma}^{\pm}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A},\pm\gamma}^{\pm}\quad(\gamma\in Q^+).\end{aligned}$$

Then U_ζ^L and U_ζ are Hopf algebras over $\mathbb{C}$, and we have triangular decompositions

$$U_\zeta^L\cong U_\zeta^{L,-}\otimes U_\zeta^{L,0}\otimes U_\zeta^{L,+},$$
$$U_\zeta\cong U_\zeta^{-}\otimes U_\zeta^{0}\otimes U_\zeta^{+}.$$

Moreover, we have

$$U_\zeta^{L,\pm}=\bigoplus_{\gamma\in Q^+}U_{\zeta,\pm\gamma}^{L,\pm},\qquad U_\zeta^{\pm}=\bigoplus_{\gamma\in Q^+}U_{\zeta,\pm\gamma}^{\pm}.$$

By De Concini and Kac [8, Proposition 1.7], we have the following.

LEMMA 1.3.

(i) *The set* $\{e_{\beta_N}^{m_N}\cdots e_{\beta_1}^{m_1}\mid m_1,\dots,m_N\geqq 0\}$ *(resp.,* $\{f_{\beta_N}^{m_N}\cdots f_{\beta_1}^{m_1}\mid m_1,\dots,m_N\geqq 0\}$*) forms a* $\mathbb{C}$*-basis of* U_ζ^+ *(resp.,* U_ζ^-*).*

(ii) *The set* $\{k_\lambda\mid\lambda\in\Lambda\}$ *forms a* $\mathbb{C}$*-basis of* U_ζ^0*.*

The Drinfeld pairings (1.5) induce

$$ {}^L\tau_\zeta : U_\zeta^{L,\geqq 0} \times U_\zeta^{\leqq 0} \to \mathbb{C}, \qquad \tau_\zeta^L : U_\zeta^{\geqq 0} \times U_\zeta^{L,\leqq 0} \to \mathbb{C}. \tag{1.11} $$

Moreover, we have the following (see [20, Lemma 1.5]).

Proposition 1.4. *For any $\gamma \in Q^+$, the restrictions of ${}^L\tau_\zeta$ and τ_ζ^L to*

$$ U_{\zeta,\gamma}^{L,+} \times U_{\zeta,-\gamma}^{-} \to \mathbb{C}, \qquad U_{\zeta,\gamma}^{-} \times U_{\zeta,-\gamma}^{L,-} \to \mathbb{C}, $$

respectively, are nondegenerate.

By Lemma 1.2 we have an algebra homomorphism

$$ \mathrm{ad} : U_\zeta^L \to \mathrm{End}_{\mathbb{C}}(U_\zeta). $$

In general, for a Lie algebra $\mathfrak{s}$ we denote its enveloping algebra by $U(\mathfrak{s})$. We denote by

$$ \pi : U_\zeta^L \to U(\mathfrak{g}) \tag{1.12} $$

Lusztig's Frobenius homomorphism (see [14]). Namely, π is the $\mathbb{C}$-algebra homomorphism given by

$$ \pi(e_i^{(m)}) = \begin{cases} \bar{e}_i^{(m/\ell)} & (\ell | m) \\ 0 & (\ell \not| m), \end{cases} \qquad \pi(f_i^{(m)}) = \begin{cases} \bar{f}_i^{(m/\ell)} & (\ell | m) \\ 0 & (\ell \not| m), \end{cases} \qquad \pi(k_\lambda) = 1 $$

for $i \in I$, $m \in \mathbb{Z}_{\geqq 0}$, $\lambda \in \Lambda$. Here, $\bar{e}_i^{(n)} = \bar{e}_i^n / n!$, $\bar{f}_i^{(n)} = \bar{f}_i^n / n!$ for $i \in I$ and $n \in \mathbb{Z}_{\geqq 0}$. Then π is a homomorphism of Hopf algebras.

We recall the description of the center $Z(U_\zeta)$ of the algebra U_ζ due to De Concini and Kac [8, Section 3] and De Concini and Procesi [9, Section 21]. Denote by $Z(U_{\mathbb{F}})$ the center of $U_{\mathbb{F}}$, and define a subalgebra $Z_{\mathrm{Har}}(U_\zeta)$ of $Z(U_\zeta)$ by

$$ Z_{\mathrm{Har}}(U_\zeta) = \mathrm{Im}\big(Z(U_{\mathbb{F}}) \cap U_{\mathbb{A}} \to U_\zeta\big). $$

We define a shifted action of W on the group algebra $\mathbb{C}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{C} e(\lambda)$ of Λ by

$$ w \circ e(\lambda) = \zeta^{(w\lambda - \lambda, \rho)} e(w\lambda) \quad (w \in W, \lambda \in \Lambda). \tag{1.13} $$

Let

$$ \iota : Z_{\mathrm{Har}}(U_\zeta) \to \mathbb{C}[\Lambda] \tag{1.14} $$

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be the composite of

$$Z_{\mathrm{Har}}(U_\zeta) \hookrightarrow U_\zeta \cong U_\zeta^- \otimes U_\zeta^0 \otimes U_\zeta^+ \xrightarrow{\varepsilon\otimes 1\otimes\varepsilon} U_\zeta^0 \cong \mathbb{C}[\Lambda],$$

where $U_\zeta^0 \cong \mathbb{C}[\Lambda]$ is given by $k_\lambda \leftrightarrow e(\lambda)$. Then by [8, Lemma 3.9], ι is an injective algebra homomorphism with image

$$\mathbb{C}[2\Lambda]^{W\circ} = \left\{ f \in \mathbb{C}[2\Lambda] \,\middle|\, w \circ f = f \ (\forall w \in W) \right\}.$$

In particular, we have an isomorphism

$$Z_{\mathrm{Har}}(U_\zeta) \simeq \mathbb{C}[2\Lambda]^{W\circ} \tag{1.15}$$

of $\mathbb{C}$-algebras. By [8, Section 3.1] the elements

$$e_\beta^\ell, \qquad f_\beta^\ell, \qquad k_{\ell\lambda} \quad (\beta \in \Delta^+, \lambda \in \Lambda)$$

are central in U_ζ. Let $Z_{\mathrm{Fr}}(U_\zeta)$ be the subalgebra of U_ζ generated by them. It is a Hopf subalgebra of U_ζ. Define an algebraic subgroup K of $B^+ \times B^-$ by

$$K = \left\{ (gh, g'h^{-1}) \,\middle|\, h \in H, g \in N^+, g' \in N^- \right\}.$$

By [9, Section 19.1] we have an isomorphism

$$Z_{\mathrm{Fr}}(U_\zeta) \cong \mathbb{C}[K] \tag{1.16}$$

of Hopf algebras (see also [10, Theorem 7.4]). We refer the reader to [20, Section 1.5] for the explicit description of the isomorphism (1.16). By [9], $Z(U_\zeta)$ is generated by $Z_{\mathrm{Fr}}(U_\zeta)$ and $Z_{\mathrm{Har}}(U_\zeta)$. Moreover, we have an isomorphism

$$Z(U_\zeta) \cong Z_{\mathrm{Har}}(U_\zeta) \otimes_{Z_{\mathrm{Har}}(U_\zeta)\cap Z_{\mathrm{Fr}}(U_\zeta)} Z_{\mathrm{Fr}}(U_\zeta) \quad (z_1 z_2 \leftrightarrow z_1 \otimes z_2)$$

of algebras.

1.2. Sheaves on quantized flag manifolds

1.2.1. We denote by $C_{\mathbb{F}}$ the subspace of $U_{\mathbb{F}}^* = \mathrm{Hom}_{\mathbb{F}}(U_{\mathbb{F}}, \mathbb{F})$ spanned by the matrix coefficients of finite-dimensional $U_{\mathbb{F}}$-modules of type 1 in the sense of Lusztig, and we denote by

$$\langle\ ,\ \rangle : C_{\mathbb{F}} \times U_{\mathbb{F}} \to \mathbb{F} \tag{1.17}$$

the canonical pairing. Then $C_{\mathbb{F}}$ is endowed with a Hopf algebra structure dual to $U_{\mathbb{F}}$ via (1.17). We have a $U_{\mathbb{F}}$-bimodule structure of $C_{\mathbb{F}}$ given by

$$\langle u_1 \cdot \varphi \cdot u_2, u \rangle = \langle \varphi, u_2 u u_1 \rangle \quad (\varphi \in C_{\mathbb{F}}, u, u_1, u_2 \in U_{\mathbb{F}}).$$

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Define a Λ-graded ring $A_{\mathbb{F}} = \bigoplus_{\lambda \in \Lambda^+} A_{\mathbb{F}}(\lambda)$ by

$$\begin{aligned} A_{\mathbb{F}} &= \{\varphi \in C_{\mathbb{F}} \mid \varphi \cdot f_i = 0 \ (i \in I)\}, \\ A_{\mathbb{F}}(\lambda) &= \{\varphi \in A_{\mathbb{F}} \mid \varphi \cdot k_\mu = q^{(\mu,\lambda)} \varphi \ (\mu \in \Lambda)\}. \end{aligned}$$

Note that $A_{\mathbb{F}}$ is a left $U_{\mathbb{F}}$-submodule of $C_{\mathbb{F}}$. For $\lambda \in \Lambda^+$ and $\xi \in \Lambda$, we set

$$A_{\mathbb{F}}(\lambda)_\xi = \{\varphi \in A_{\mathbb{F}}(\lambda) \mid k_\mu \cdot \varphi = q^{(\xi,\mu)} \varphi\}.$$

Then we have

$$A_{\mathbb{F}}(\lambda) = \bigoplus_{\xi \in \lambda - Q^+} A_{\mathbb{F}}(\lambda)_\xi.$$

We define $\mathbb{A}$-forms $C_{\mathbb{A}}$, $A_{\mathbb{A}}$, $A_{\mathbb{A}}(\lambda)$ $(\lambda \in \Lambda^+)$ of $C_{\mathbb{F}}$, $A_{\mathbb{F}}$, $A_{\mathbb{F}}(\lambda)$, respectively, by

$$C_{\mathbb{A}} = \{\varphi \in C_{\mathbb{F}} \mid \langle \varphi, U_{\mathbb{A}}^L \rangle \subset \mathbb{A}\}, \qquad A_{\mathbb{A}} = A_{\mathbb{F}} \cap C_{\mathbb{A}}, \qquad A_{\mathbb{A}}(\lambda) = A_{\mathbb{F}}(\lambda) \cap C_{\mathbb{A}}.$$

Then $C_{\mathbb{A}}$ is a Hopf algebra over $\mathbb{A}$, and $A_{\mathbb{A}}$ is its $\mathbb{A}$-subalgebra. Moreover, $C_{\mathbb{A}}$ is a $U_{\mathbb{A}}^L$-bimodule, and $A_{\mathbb{A}}$ is its left $U_{\mathbb{A}}^L$-submodule. We also set $A_{\mathbb{A}}(\lambda)_\xi = A_{\mathbb{F}}(\lambda)_\xi \cap A_{\mathbb{A}}$ for $\lambda \in \Lambda^+$, $\xi \in \Lambda$.

We set

$$C_\zeta = \mathbb{C} \otimes_{\mathbb{A}} C_{\mathbb{A}}, \qquad A_\zeta = \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}, \qquad A_\zeta(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}(\lambda) \quad (\lambda \in \Lambda^+).$$

Then C_ζ is a Hopf algebra over $\mathbb{C}$. Moreover, the $U_{\mathbb{F}}$-bimodule structure of $C_{\mathbb{F}}$ induces a U_ζ^L-bimodule structure of C_ζ. For $\lambda \in \Lambda^+$ and $\xi \in \Lambda$, we set $A_\zeta(\lambda)_\xi = \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}(\lambda)_\xi$. Then we have

$$A_\zeta(\lambda) = \bigoplus_{\xi \in \lambda - Q^+} A_\zeta(\lambda)_\xi.$$

We have a natural pairing

$$\langle \, , \, \rangle : C_\zeta \times U_\zeta^L \to \mathbb{C} \tag{1.18}$$

induced by (1.17).

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1.2.2. For a ring (resp., Λ-graded ring) $\mathcal{R}$ we denote by $\mathrm{Mod}(\mathcal{R})$ (resp., $\mathrm{Mod}_\Lambda(\mathcal{R})$) the category of $\mathcal{R}$-modules (resp., Λ-graded left $\mathcal{R}$-modules). Assume that we are given a homomorphism $\jmath : A \to B$ of Λ-graded rings satisfying

$$\jmath\big(A(\lambda)\big)B(\mu) = B(\mu)\jmath\big(A(\lambda)\big) \quad (\lambda, \mu \in \Lambda). \tag{1.19}$$

For $M \in \mathrm{Mod}_\Lambda(B)$, let $\mathrm{Tor}(M)$ be the subset of M consisting of $m \in M$ such that there exists $\lambda \in \Lambda^+$ satisfying $\jmath(A(\lambda+\mu))m = \{0\}$ for any $\mu \in \Lambda^+$. Then $\mathrm{Tor}(M)$ is a subobject of M in $\mathrm{Mod}_\Lambda(B)$ by (1.19). We denote by $\mathrm{Tor}_{\Lambda^+}(A,B)$ the full subcategory of $\mathrm{Mod}_\Lambda(B)$ consisting of $M \in \mathrm{Mod}_\Lambda(B)$ such that $\mathrm{Tor}(M) = M$. Note that $\mathrm{Tor}_{\Lambda^+}(A,B)$ is closed under taking subquotients and extensions in $\mathrm{Mod}_\Lambda(B)$. Let $\Sigma(A,B)$ denote the collection of morphisms f of $\mathrm{Mod}_\Lambda(B)$ such that its kernel $\mathrm{Ker}(f)$ and its cokernel $\mathrm{Coker}(f)$ belong to $\mathrm{Tor}_{\Lambda^+}(A,B)$. Then we define an abelian category $\mathcal{C}(A,B) = \mathrm{Mod}_\Lambda(B)/\mathrm{Tor}_{\Lambda^+}(A,B)$ as the localization

$$\mathcal{C}(A,B) = \Sigma(A,B)^{-1}\mathrm{Mod}_\Lambda(B)$$

of $\mathrm{Mod}_\Lambda(B)$ with respect to the multiplicative system $\Sigma(A,B)$ (see, e.g., [16] for the notion of localization of categories). We denote by

$$\omega(A,B)^* : \mathrm{Mod}_\Lambda(B) \to \mathcal{C}(A,B) \tag{1.20}$$

the canonical exact functor. It admits a right adjoint

$$\omega(A,B)_* : \mathcal{C}(A,B) \to \mathrm{Mod}_\Lambda(B), \tag{1.21}$$

which is left exact. It is known that $\omega(A,B)^* \circ \omega(A,B)_* \cong \mathrm{Id}$. By taking the degree 0 part of (1.21), we obtain a left exact functor

$$\Gamma_{(A,B)} : \mathcal{C}(A,B) \to \mathrm{Mod}\big(B(0)\big). \tag{1.22}$$

The abelian category $\mathcal{C}(A,B)$ has enough injectives, and we have the right derived functors

$$R^i\Gamma_{(A,B)} : \mathcal{C}(A,B) \to \mathrm{Mod}\big(B(0)\big) \quad (i \in \mathbb{Z}) \tag{1.23}$$

of (1.22).

We apply the above arguments to the case $A = B = A_\zeta$. Then $\mathrm{Tor}(M)$ for $M \in \mathrm{Mod}_\Lambda(A_\zeta)$ consists of $m \in M$ such that there exists $\lambda \in \Lambda^+$ satisfying $A_\zeta(\lambda)m = \{0\}$ (see [20, Lemma 3.4]). We set

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) = \mathcal{C}(A_\zeta, A_\zeta). \tag{1.24}$$

In this case, the natural functors (1.20), (1.21), (1.22) are simply denoted as

$$\omega^*: \mathrm{Mod}_\Lambda(A_\zeta) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}), \tag{1.25}$$

$$\omega_*: \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}_\Lambda(A_\zeta), \tag{1.26}$$

$$\Gamma: \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathbb{C}). \tag{1.27}$$

REMARK 1.5. In the terminology of noncommutative algebraic geometry, $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ is the category of *quasicoherent sheaves* on the quantized flag manifold $\mathcal{B}_\zeta$, which is a noncommutative projective scheme. The notations $\mathcal{B}_\zeta$, $\mathcal{O}_{\mathcal{B}_\zeta}$ have only symbolical meaning.

1.2.3. Using Lusztig's Frobenius homomorphism (1.12), we will relate the quantized flag manifold $\mathcal{B}_\zeta$ with the ordinary flag manifold $\mathcal{B} = B^-\backslash G$. Taking the dual Hopf algebras in (1.12), we obtain an injective homomorphism $\mathbb{C}[G] \to C_\zeta$ of Hopf algebras. Moreover, its image is contained in the center of C_ζ (see [14]). We will regard $\mathbb{C}[G]$ as a central Hopf subalgebra of C_ζ in the following. Setting

$$A_1 = \{\varphi \in \mathbb{C}[G] \mid \varphi(ng) = \varphi(g)\ (n \in N^-, g \in G)\},$$
$$A_1(\lambda) = \{\varphi \in A_1 \mid \varphi(tg) = \theta_\lambda(t)\varphi(g)\ (t \in H, g \in G)\} \quad (\lambda \in \Lambda^+),$$

we have a Λ-graded algebra

$$A_1 = \bigoplus_{\lambda \in \Lambda^+} A_1(\lambda).$$

We have a left G-module structure of A_1 given by

$$(x\varphi)(g) = \varphi(gx) \quad (\varphi \in A_1, x, g \in G).$$

In particular, A_1 is a $U(\mathfrak{g})$-module. Moreover, for each $\lambda \in \Lambda^+$, $A_1(\lambda)$ is a $U(\mathfrak{g})$-submodule of A_1 which is an irreducible highest-weight module with highest-weight λ. Regarding $\mathbb{C}[G]$ as a subalgebra of C_ζ, we have

$$A_1 = A_\zeta \cap \mathbb{C}[G], \qquad A_1(\lambda) = A_\zeta(\ell\lambda) \cap \mathbb{C}[G].$$

Since the Λ-graded algebra A_1 is the homogeneous coordinate algebra of the projective variety $\mathcal{B} = B^-\backslash G$, we have an identification

$$\mathrm{Mod}(\mathcal{O}_\mathcal{B}) = \mathcal{C}(A_1, A_1) \tag{1.28}$$

of abelian categories, where $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}})$ denotes the category of quasicoherent $\mathcal{O}_{\mathcal{B}}$-modules on the ordinary flag manifold $\mathcal{B}$. We set

$$\omega_{\mathcal{B}*} = \omega(A_1, A_1)_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathrm{Mod}_\Lambda(A_1). \tag{1.29}$$

For $\lambda \in \Lambda$, we denote by $\mathcal{O}_{\mathcal{B}}(\lambda) \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}})$ the invertible G-equivariant $\mathcal{O}_{\mathcal{B}}$-module corresponding to λ. Then under identification (1.28), we have

$$\omega_{\mathcal{B}*}M = \bigoplus_{\lambda\in\Lambda} \Gamma\big(\mathcal{B}, M \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{B}}(\lambda)\big) \quad \big(M \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}})\big),$$

where $\Gamma(\mathcal{B},) : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathbb{C}$ is the global section functor for the algebraic variety $\mathcal{B}$. In particular, the functor $\Gamma_{(A_1,A_1)} : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathrm{Mod}(\mathbb{C})$ is identified with $\Gamma(\mathcal{B},)$.

For a Λ-graded $\mathbb{C}$-algebra B, we define a new Λ-graded $\mathbb{C}$-algebra $B^{(\ell)}$ by

$$B^{(\ell)}(\lambda) = B(\ell\lambda) \quad (\lambda \in \Lambda).$$

Let

$$(\,)^{(\ell)} : \mathrm{Mod}_\Lambda(B) \to \mathrm{Mod}_\Lambda(B^{(\ell)}) \tag{1.30}$$

be the exact functor given by

$$M^{(\ell)}(\lambda) = M(\ell\lambda) \quad (\lambda \in \Lambda)$$

for $M \in \mathrm{Mod}_\Lambda(B)$.

We have the following results (see [20, Lemma 3.9]).

Lemma 1.6. *Let B be a Λ-graded $\mathbb{C}$-algebra. Assume that we are given a homomorphism $\jmath : A_\zeta \to B$ of Λ-graded $\mathbb{C}$-algebras. We denote by $\jmath' : A_1 \to B^{(\ell)}$ the induced homomorphism of Λ-graded $\mathbb{C}$-algebras. Assume that*

$$\jmath\big(A_\zeta(\lambda)\big)B(\mu) = B(\mu)\jmath\big(A_\zeta(\lambda)\big) \quad (\lambda, \mu \in \Lambda),$$
$$\jmath'\big(A_1(\lambda)\big)B^{(\ell)}(\mu) = B^{(\ell)}(\mu)\jmath'\big(A_1(\lambda)\big) \quad (\lambda, \mu \in \Lambda).$$

Then the exact functor

$$(\,)^{(\ell)} : \mathrm{Mod}_\Lambda(B) \to \mathrm{Mod}_\Lambda(B^{(\ell)})$$

induces an equivalence

$$\mathrm{Fr}_* : \mathcal{C}(A_\zeta, B) \to \mathcal{C}(A_1, B^{(\ell)}) \tag{1.31}$$

of abelian categories. Moreover, we have

$$\omega(A_1, B^{(\ell)})_* \circ \mathrm{Fr}_* = (\,)^{(\ell)} \circ \omega(A_\zeta, B)_*. \tag{1.32}$$

LEMMA 1.7. *Let F be a Λ-graded $\mathbb{C}$-algebra, and let $A_1 \to F$ be a homomorphism of Λ-graded $\mathbb{C}$-algebras. Assume that $\mathrm{Im}(A_1 \to F)$ is central in F. Regard F as an object of $\mathrm{Mod}_\Lambda(A_1)$, and consider $\omega_{\mathcal{B}}^* F \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}})$. Then the multiplication of F induces an $\mathcal{O}_{\mathcal{B}}$-algebra structure of $\omega_{\mathcal{B}}^* F$, and we have an identification*

$$\mathcal{C}(A_1, F) = \mathrm{Mod}(\omega_{\mathcal{B}}^* F) \tag{1.33}$$

of abelian categories, where $\mathrm{Mod}(\omega_{\mathcal{B}}^* F)$ denotes the category of quasicoherent $\omega_{\mathcal{B}}^* F$-modules. Moreover, under identification (1.33) we have

$$\Gamma_{(A_1,F)}(M) = \Gamma(\mathcal{B}, M) \in \mathrm{Mod}\big(F(0)\big) \quad \big(M \in \mathrm{Mod}(\omega_{\mathcal{B}}^* F)\big).$$

We define an $\mathcal{O}_{\mathcal{B}}$-algebra $\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta}$ by

$$\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta} = \omega_{\mathcal{B}}^* (A_\zeta^{(\ell)}).$$

We denote by $\mathrm{Mod}(\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta})$ the category of quasicoherent $\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta}$-modules. By Lemmas 1.6 and 1.7, we have the following.

LEMMA 1.8. *We have an equivalence*

$$\mathrm{Fr}_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_\zeta})$$

of abelian categories. Moreover, for $M \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ we have

$$R^i\Gamma(M) \simeq R^i\Gamma\big(\mathcal{B}, \mathrm{Fr}_*(M)\big),$$

where $\Gamma(\mathcal{B},) : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathrm{Mod}(\mathbb{C})$ on the right-hand side is the global section functor for $\mathcal{B}$.

§2. The category of D-modules

2.1. Ring of differential operators

2.1.1. We define a subalgebra $D_{\mathbb{F}}$ of $\mathrm{End}_{\mathbb{F}}(A_{\mathbb{F}})$ by

$$D_{\mathbb{F}} = \langle \ell_\varphi, r_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{F}}, u \in U_{\mathbb{F}}, \lambda \in \Lambda \rangle,$$

where

$$\ell_\varphi(\psi) = \varphi\psi, \qquad r_\varphi(\psi) = \psi\varphi, \qquad \partial_u(\psi) = u \cdot \psi, \qquad \sigma_\lambda(\psi) = q^{(\lambda,\mu)}\psi$$

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for $\psi \in A_{\mathbb{F}}(\mu)$. In fact, we have

$$D_{\mathbb{F}} = \langle \ell_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{F}}, u \in U_{\mathbb{F}}, \lambda \in \Lambda \rangle$$

by [20, Lemma 4.1].

We have a natural grading

$$D_{\mathbb{F}} = \bigoplus_{\lambda \in \Lambda^+} D_{\mathbb{F}}(\lambda),$$
$$D_{\mathbb{F}}(\lambda) = \{\Phi \in D_{\mathbb{F}} \mid \Phi(A_{\mathbb{F}}(\mu)) \subset A_{\mathbb{F}}(\lambda + \mu) \ (\mu \in \Lambda)\} \quad (\lambda \in \Lambda)$$

of $D_{\mathbb{F}}$. It is easily checked that

$$\partial_u \ell_\varphi = \sum_{(u)} \ell_{u_{(0)} \cdot \varphi} \partial_{u_{(1)}} \quad (u \in U_{\mathbb{F}}, \varphi \in A_{\mathbb{F}}),$$
$$\partial_u \sigma_\lambda = \sigma_\lambda \partial_u \quad (u \in U_{\mathbb{F}}, \lambda \in \Lambda),$$
$$\sigma_\lambda \ell_\varphi = q^{(\lambda,\mu)} \ell_\varphi \sigma_\lambda \quad (\lambda \in \Lambda, \varphi \in A_{\mathbb{F}}(\mu)).$$

Set

$$E_{\mathbb{F}} = A_{\mathbb{F}} \otimes U_{\mathbb{F}} \otimes \mathbb{F}[\Lambda].$$

We have a natural $\mathbb{F}$-algebra structure of $E_{\mathbb{F}}$ such that $A_{\mathbb{F}} \otimes 1 \otimes 1$, $1 \otimes U_{\mathbb{F}} \otimes 1$, $1 \otimes 1 \otimes \mathbb{F}[\Lambda]$ are subalgebras of $E_{\mathbb{F}}$ naturally isomorphic to $A_{\mathbb{F}}$, $U_{\mathbb{F}}$, $\mathbb{F}[\Lambda]$, respectively, and such that we have the relations

$$u\varphi = \sum_{(u)} (u_{(0)} \cdot \varphi) u_{(1)} \quad (u \in U_{\mathbb{F}}, \varphi \in A_{\mathbb{F}}),$$
$$ue(\lambda) = e(\lambda)u \quad (u \in U_{\mathbb{F}}, \lambda \in \Lambda),$$
$$e(\lambda)\varphi = q^{(\lambda,\mu)} \varphi e(\lambda) \quad (\lambda \in \Lambda, \varphi \in A_{\mathbb{F}}(\mu))$$

in $E_{\mathbb{F}}$. Here, we identify $A_{\mathbb{F}} \otimes 1 \otimes 1$, $1 \otimes U_{\mathbb{F}} \otimes 1$, $1 \otimes 1 \otimes \mathbb{F}[\Lambda]$ with $A_{\mathbb{F}}$, $U_{\mathbb{F}}$, $\mathbb{F}[\Lambda]$, respectively. Then we have a surjective algebra homomorphism

$$E_{\mathbb{F}} \to D_{\mathbb{F}} \tag{2.1}$$

sending $\varphi \in A_{\mathbb{F}}$, $u \in U_{\mathbb{F}}$, $e(\lambda) \in \mathbb{F}[\Lambda]$ $(\lambda \in \Lambda)$ to ℓ_φ, ∂_u, σ_λ, respectively. Moreover, $E_{\mathbb{F}}$ has an obvious Λ-grading so that (2.1) preserves the Λ-grading.

2.1.2. Set

$$D_{\mathbb{A}} = \langle \ell_\varphi, r_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{A}}, u \in U_{\mathbb{A}}, \lambda \in \Lambda \rangle_{\mathbb{A}\text{-alg}} \subset D_{\mathbb{F}},$$
$$E_{\mathbb{A}} = A_{\mathbb{A}} \otimes U_{\mathbb{A}} \otimes \mathbb{A}[\Lambda] \subset E_{\mathbb{A}}.$$

They are Λ-graded $\mathbb{A}$-subalgebras of $D_{\mathbb{F}}$ and $E_{\mathbb{F}}$, respectively. Again, we have

$$D_{\mathbb{A}} = \langle \ell_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{A}}, u \in U_{\mathbb{A}}, \lambda \in \Lambda \rangle_{\mathbb{A}\text{-alg}}$$

by [20]. In particular, we have a surjective homomorphism

$$E_{\mathbb{A}} \to D_{\mathbb{A}}$$

of Λ-graded algebras. Note that there is a canonical embedding

$$D_{\mathbb{A}} \to \mathrm{End}_{\mathbb{A}}(A_{\mathbb{A}}).$$

2.1.3. We set

$$D_\zeta = \mathbb{C} \otimes_{\mathbb{A}} D_{\mathbb{A}}, \qquad E_\zeta = \mathbb{C} \otimes_{\mathbb{A}} E_{\mathbb{A}} = A_\zeta \otimes U_\zeta \otimes \mathbb{C}[\Lambda].$$

D_ζ is a Λ-graded $\mathbb{C}$-algebra generated by elements of the form

$$\ell_\varphi, \qquad \partial_u, \qquad \sigma_\lambda \quad (\varphi \in A_\zeta, u \in U_\zeta, \lambda \in \Lambda).$$

We have a surjective homomorphism

$$E_\zeta \to D_\zeta$$

of Λ-graded $\mathbb{C}$-algebras.

LEMMA 2.1. *Let $z \in Z_{\mathrm{Har}}(U_\zeta)$, and write $\iota(z) = \sum_{\lambda \in \Lambda} c_\lambda k_{2\lambda}$ $(c_\lambda \in \mathbb{C})$. Then we have*

$$\partial_z = \sum_{\lambda \in \Lambda} c_\lambda \sigma_{2\lambda}.$$

Proof. This follows from the corresponding statement over $\mathbb{F}$, which is given in [19, Section 5.1]. □

REMARK 2.2. The natural algebra homomorphism $D_\zeta \to \mathrm{End}_{\mathbb{C}}(A_\zeta)$ is not injective.

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2.1.4. Define an $\mathcal{O}_{\mathcal{B}}$-algebra $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}$ by

$$\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta} = \omega_{\mathcal{B}}^* D_\zeta^{(\ell)}.$$

We define $ZD_\zeta^{(\ell)}$ to be the central subalgebra of $D_\zeta^{(\ell)}$ generated by the elements of the form

$$\ell_\varphi, \qquad \partial_u, \qquad \sigma_\lambda \quad \big(\varphi \in A_1, u \in Z_{\mathrm{Fr}}(U_\zeta), \lambda \in \Lambda\big),$$

and we set

$$\mathcal{Z}_\zeta = \omega_{\mathcal{B}}^* Z D_\zeta^{(\ell)}.$$

It is a central subalgebra of $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}$. Define a subvariety $\mathcal{V}$ of $\mathcal{B} \times K \times H$ by

$$\mathcal{V} = \big\{(B^- g, k, t) \in \mathcal{B} \times K \times H \;\big|\; g\kappa(k)g^{-1} \in t^{2\ell} N^-\big\},$$

where $\kappa : K \to G$ is given by $\kappa(k_1, k_2) = k_1 k_2^{-1}$. We denote by

$$p_{\mathcal{V}} : \mathcal{V} \to \mathcal{B}$$

the projection. Now we can state the main results of [20].

THEOREM 2.3 ([20, Theorem 5.2]). *The $\mathcal{O}_{\mathcal{B}}$-algebra $\mathcal{Z}_\zeta$ is naturally isomorphic to $p_{\mathcal{V}*}\mathcal{O}_{\mathcal{V}}$.*

Define an $\mathcal{O}_{\mathcal{V}}$-algebra $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}$ by

$$\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} = p_{\mathcal{V}}^{-1} \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{p_{\mathcal{V}}^{-1} p_{\mathcal{V}*}\mathcal{O}_{\mathcal{V}}} \mathcal{O}_{\mathcal{V}}.$$

THEOREM 2.4 ([20, Theorem 6.1]). *Here $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}$ is an Azumaya algebra of rank $\ell^{2|\Delta^+|}$.*

Define

$$\delta : \mathcal{V} \to K \times_{H/W} H$$

by $\delta(B^- g, k, t) = (k, t)$, where $K \to H/W$ is given by $k \mapsto [h]$, where h is an element of H conjugate to the semisimple part of $\kappa(k)$, and $H \to H/W$ is given by $t \mapsto [t^{2\ell}]$.

THEOREM 2.5 ([20, Theorem 6.10]). *For any $(k, t) \in K \times_{H/W} H$, the restriction of $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}$ to $\delta^{-1}(k, t)$ is a split Azumaya algebra.*

2.2. Category of D-modules

We define an abelian category $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ by

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta}) = \mathcal{C}(A_\zeta, D_\zeta).$$

By Lemmas 1.6 and 1.7, we have an equivalence

$$\mathrm{Fr}_* : \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}) \tag{2.2}$$

of abelian categories, where $\mathrm{Mod}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})$ denotes the category of quasicoherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}$-modules. Moreover, for $M \in \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ we have

$$R^i\Gamma_{(A_\zeta, D_\zeta)}(M) = R^i\Gamma\big(\mathcal{B}, \mathrm{Fr}_*(M)\big) \in \mathrm{Mod}\big(D_\zeta(0)\big), \tag{2.3}$$

where $\Gamma(\mathcal{B},)$ on the right-hand side is the global section functor for the ordinary flag variety $\mathcal{B}$.

For $t \in H$ we define an abelian category $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ by

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) = \mathrm{Mod}_{\Lambda,t}(D_\zeta)/\big(\mathrm{Mod}_{\Lambda,t}(D_\zeta) \cap \mathrm{Tor}_{\Lambda^+}(A_\zeta, D_\zeta)\big),$$

where $\mathrm{Mod}_{\Lambda,t}(D_\zeta)$ is the full subcategory of $\mathrm{Mod}_\Lambda(D_\zeta)$ consisting of $M \in \mathrm{Mod}_\Lambda(D_\zeta)$ so that $\sigma_\lambda|_{M(\mu)} = \theta_\lambda(t)\zeta^{(\lambda,\mu)}\,\mathrm{id}$ for any $\lambda, \mu \in \Lambda$. Then we can regard $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ as a full subcategory of $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ (see [19, Lemma 4.6]). Set

$$\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t} = \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t,$$

where $\mathbb{C}_t$ denotes the 1-dimensional $\mathbb{C}[\Lambda]$-module given by $e(\lambda) \mapsto \theta_\lambda(t)$ for $\lambda \in \Lambda$. The equivalence (2.2) induces the equivalence

$$\mathrm{Fr}_* : \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) \to \mathrm{Mod}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}), \tag{2.4}$$

where $\mathrm{Mod}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})$ denotes the category of quasicoherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules. In particular, for $M \in \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ we have

$$R^i\Gamma_{(A_\zeta, D_\zeta)}(M) = R^i\Gamma(\mathcal{B}, \mathrm{Fr}_*M) \in \mathrm{Mod}\big(D_{\zeta,t}(0)\big),$$

where $D_{\zeta,t}(0) = D_\zeta(0)/\sum_{\lambda\in\Lambda} D_\zeta(0)(\sigma_\lambda - \theta_\lambda(t))$.

2.3. Conjecture

By Lemma 2.1, the natural algebra homomorphism

$$U_\zeta \otimes_{\mathbb{C}} \mathbb{C}[\Lambda] \to D_\zeta(0)$$

factors through

$$U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] \to D_\zeta(0),$$

where $Z_{\mathrm{Har}}(U_\zeta)$ is identified with $\mathbb{C}[2\Lambda]^{W\circ}$ by (1.15). Hence, we have a natural algebra homomorphism

$$U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] \to \Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta}). \tag{2.5}$$

For $t \in H$ we denote by $\mathbb{C}_t$ the 1-dimensional $\mathbb{C}[\Lambda]$-module given by $e(\lambda)v = \theta_\lambda(t)v$ $(v \in \mathbb{C}_t)$. Then (2.5) induces an algebra homomorphism

$$U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t \to \Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta, t}), \tag{2.6}$$

where $\mathbb{C}_t$ is regarded as a $Z_{\mathrm{Har}}(U_\zeta)$-module by $Z_{\mathrm{Har}}(U_\zeta) \cong \mathbb{C}[2\Lambda]^{W\circ} \subset \mathbb{C}[\Lambda]$. Denote by h_G the Coxeter number for G.

CONJECTURE 2.6. *Assume that $\ell > h_G$. The algebra homomorphism (2.5) is an isomorphism, and we have*

$$R^i\Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta}) = 0$$

for $i \neq 0$.

PROPOSITION 2.7. *Let $\ell > h_G$, and assume that Conjecture 2.6 is valid. Then for $t \in H$ we have*

$$\Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta, t}) \cong U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t$$

and

$$R^i\Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta, t}) = 0 \quad (i \neq 0).$$

Proof. Define $f : \mathcal{V} \to H$ to be the composite of the embedding $\mathcal{V} \to \mathcal{B} \times K \times H$ and the projection $\mathcal{B} \times K \times H \to H$ onto the third factor. Since $p_{\mathcal{V}}$ is an affine morphism, we have $Rp_{\mathcal{V}*}\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} = p_{\mathcal{V}*}\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} = \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta}$. Hence, we have

$$U_\zeta \otimes^L_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] = U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] \cong R\Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta}) = R\Gamma(\mathcal{V}, \tilde{\mathcal{D}}_{\mathcal{B}_\zeta}).$$

Here we use the fact that $\mathbb{C}[\Lambda]$ is a free $Z_{\mathrm{Har}}(U_\zeta)$-module (see [17]). Denote by $\mathcal{O}_t$ the $\mathcal{O}_H$-module corresponding to the $\mathbb{C}[\Lambda]$-module $\mathbb{C}_t$. Similarly, we have

$$\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t} = p_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t) = Rp_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t).$$

Since f is flat, we have $Lf^*\mathcal{O}_t = f^*\mathcal{O}_t$. Hence, by Theorem 2.4 we have

$$\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} Lf^*\mathcal{O}_t = \tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} f^*\mathcal{O}_t = \tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes_{\mathcal{O}_\mathcal{V}} f^*\mathcal{O}_t.$$

It follows that

$$\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t} = Rp_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} Lf^*\mathcal{O}_t) = Rp_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}) \otimes^L_{\mathcal{O}_H} \mathcal{O}_t.$$

Hence we have

$$\begin{aligned} R\Gamma(\mathcal{B}, \mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}) &= R\Gamma\big(H, Rf_*(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} Lf^*\mathcal{O}_t)\big) \\ &= R\Gamma(H, Rf_*\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_H} \mathcal{O}_t) = R\Gamma(H, Rf_*\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}) \otimes^L_{\mathbb{C}[\Lambda]} \mathbb{C}_t \\ &= R\Gamma(\mathcal{V}, \tilde{\mathcal{D}}_{\mathcal{B}_\zeta}) \otimes^L_{\mathbb{C}[\Lambda]} \mathbb{C}_t = U_\zeta \otimes^L_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda] \otimes^L_{\mathbb{C}[\Lambda]} \mathbb{C}_t \\ &= U_\zeta \otimes^L_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t. \end{aligned}$$

□

2.4. Derived Beilinson–Bernstein equivalence

We show that Conjecture 2.6 implies a variant of the Beilinson–Bernstein equivalence for derived categories.

Recall that we have an identification

$$Z_{\mathrm{Har}}(U_\zeta) \cong \mathbb{C}[2\Lambda]^{W\circ} \subset \mathbb{C}[2\Lambda] \subset \mathbb{C}[\Lambda].$$

Recall also that we identify $\mathbb{C}[\Lambda]$ with the coordinate algebra $\mathbb{C}[H]$ of H. Set $H^{(2)} = H/\mathrm{Ker}(H \ni t \mapsto t^2 \in H)$, and let $\pi : H \to H^{(2)}$ be the canonical homomorphism. Then we have a natural identification $\mathbb{C}[H^{(2)}] = \mathbb{C}[2\Lambda]$ so that $\pi^* : \mathbb{C}[H^{(2)}] \to \mathbb{C}[H]$ is identified with the inclusion $\mathbb{C}[2\Lambda] \subset \mathbb{C}[\Lambda]$. Denote the isomorphism $H \cong H^{(2)}$ corresponding to $\mathbb{C}[\Lambda] \ni e(\lambda) \leftrightarrow e(2\lambda) \in \mathbb{C}[2\Lambda]$ by $t \leftrightarrow t^{1/2}$. Then we have $\pi(t) = (t^2)^{1/2}$. The shifted action (1.13) of W on $\mathbb{C}[2\Lambda]$ induces an action of W on $H^{(2)}$ given by

$$w \circ t^{1/2} = \big(w(tt_{2\rho})t_{2\rho}^{-1}\big)^{1/2} \quad (w \in W, t \in H),$$

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where $t_{2\rho} \in H$ is given by $\theta_\mu(t_{2\rho}) = \zeta^{2(\mu,\rho)}$ for any $\mu \in \Lambda$ (note that $2(\mu,\rho) \in \mathbb{Z}$), and $Z_{\mathrm{Har}}(U_\zeta)$ is regarded as the coordinate algebra of the quotient variety $(W\circ)\backslash H^{(2)}$. For $t \in H$ we denote by $\chi_t : \mathbb{C}[\Lambda] \to \mathbb{C}$ the corresponding algebra homomorphism. By the above argument, we have

$$\chi_{t_1}|_{Z_{\mathrm{Har}}(U_\zeta)} = \chi_{t_2}|_{Z_{\mathrm{Har}}(U_\zeta)} \iff (t_1^2)^{1/2} \in W \circ t_2^{1/2}.$$

We say that $t \in H$ is *regular* if

$$\{w \in W \mid w \circ (t^2)^{1/2} = (t^2)^{1/2}\} = \{1\}.$$

We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})$ (resp., $\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)$) the category of coherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules (resp., finitely generated $U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t$-modules). We also denote by $\mathrm{Mod}_{\mathrm{coh},t}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})$ (resp., $\mathrm{Mod}_{f,t}(U_\zeta)$) the category of coherent $\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}$-modules (resp., finitely generated U_ζ-modules) killed by some power of the maximal ideal of $\mathbb{C}[\Lambda]$ (resp., $Z_{\mathrm{Har}}(U_\zeta)$) corresponding to $t \in H$.

THEOREM 2.8. *Let $\ell > h_G$, and assume that Conjecture 2.6 is valid. If $t \in H$ is regular, then the natural functors*

$$R\Gamma_{\hat{t}} : D^b(\mathrm{Mod}_{\mathrm{coh},t}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})) \to D^b(\mathrm{Mod}_{f,t}(U_\zeta)),$$
$$R\Gamma_t : D^b(\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})) \to D^b(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t))$$

give equivalences of derived categories.

The proof of this result is completely similar to that of the corresponding fact for Lie algebras in positive characteristics given in [6, Theorem 5.3.1]. We give below only an outline of it. First note the following.

PROPOSITION 2.9 ([7, Theorem B]). *Here U_ζ has finite homological dimension.*

The functors

$$R\Gamma_{\hat{t}} : D^b(\mathrm{Mod}_{\mathrm{coh},t}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})) \to D^b(\mathrm{Mod}_{f,t}(U_\zeta)),$$
$$R\Gamma_t : D^-(\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})) \to D^-(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t))$$

have left adjoints

$$\mathcal{L}_{\hat{t}} : D^b(\mathrm{Mod}_{f,t}(U_\zeta)) \to D^b(\mathrm{Mod}_{\mathrm{coh},t}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})),$$
$$\mathcal{L}_t : D^-(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t)) \to D^-(\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})).$$

Arguing exactly as in [6, Sections 3.3, 3.4] using Theorem 2.4 and Proposition 2.9, we obtain the following.

Proposition 2.10.

(i) *If t is regular, the adjunction morphism* $\mathrm{Id} \to R\Gamma_{\hat{t}} \circ \mathcal{L}_{\hat{t}}$ *is an isomorphism on* $D^b(\mathrm{Mod}_{f,t}(U_\zeta))$.

(ii) *For any t, the adjunction morphism* $\mathrm{Id} \to R\Gamma_t \circ \mathcal{L}_t$ *is an isomorphism on* $D^-(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}_t))$.

Arguing exactly as in [6, Section 3.5] using Theorem 2.4, Proposition 2.10, and Lemma 2.11 below, we obtain Theorem 2.8. Details are omitted.

Lemma 2.11 ([21, Section 2.4]). *The variety $\mathcal{V}$ is a symplectic manifold.*

2.5. Finite part

2.5.1. In [20, Section 4], we also introduced a quotient algebra D'_ζ of E_ζ, which is closely related to D_ζ. Let us recall its definition. Take bases $\{x_p\}_p$, $\{y_p\}_p$, $\{x^L_p\}_p$, $\{y^L_p\}_p$ of U^+_ζ, U^-_ζ, $U^{L,+}_\zeta$, $U^{L,-}_\zeta$, respectively, such that

$$\tau^L_\zeta(x_{p_1}, y^L_{p_2}) = \delta_{p_1,p_2}, \qquad {}^L\tau_\zeta(x^L_{p_1}, y_{p_2}) = \delta_{p_1,p_2}.$$

We assume that

$$x_p \in U^+_{\zeta,\beta_p}, \qquad y_p \in U^-_{\zeta,-\beta_p}, \qquad x^L_p \in U^{L,+}_{\zeta,\beta_p}, \qquad y^L_p \in U^{L,-}_{\zeta,-\beta_p}$$

for $\beta_p \in Q^+$.

For $\varphi \in A_\zeta(\lambda)_\xi$ with $\lambda \in \Lambda^+, \xi \in \Lambda$, we set

$$\Omega'_1(\varphi) = \sum_p (y^L_p \cdot \varphi) x_p \in E_{\zeta,\diamondsuit},$$

$$\Omega'_2(\varphi) = \sum_p \big((Sx^L_p) \cdot \varphi\big) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \in E_{\zeta,\diamondsuit},$$

$$\Omega'(\varphi) = \Omega'_1(\varphi) - \Omega'_2(\varphi) \in E_{\zeta,\diamondsuit}.$$

We extend Ω' to whole A_ζ by linearity. Then D'_ζ is defined by

$$D'_\zeta = E_\zeta \Big/ \sum_{\varphi \in A_\zeta} A_\zeta \Omega'(\varphi) U_\zeta \mathbb{C}[\Lambda].$$

We have a sequence

$$E_\zeta \to D'_\zeta \to D_\zeta$$

of surjective homomorphisms of Λ-graded algebras. Moreover, $D'_\zeta \to D_\zeta$ induces an isomorphism

$$\omega^* D'_\zeta \cong \omega^* D_\zeta \tag{2.7}$$

in $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ (see [20, Corollary 6.6]).

2.5.2. We set

$$U^0_{\mathbb{F},\diamondsuit}=\bigoplus_{\lambda\in\Lambda}\mathbb{F}k_{2\lambda}\subset U^0_{\mathbb{F}},\qquad U_{\mathbb{F},\diamondsuit}=S(U^-_{\mathbb{F}})U^0_{\mathbb{F},\diamondsuit}U^+_{\mathbb{F}}\subset U_{\mathbb{F}}.$$

Then we see easily the following.

LEMMA 2.12. *The subspace $U_{\mathbb{F},\diamondsuit}$ is an $\mathrm{ad}(U_{\mathbb{F}})$-stable subalgebra of $U_{\mathbb{F}}$.*

Set

$$U_{\mathbb{F},f}=\left\{u\in U_{\mathbb{F}}\;\middle|\;\dim\mathrm{ad}(U_{\mathbb{F}})(u)<\infty\right\}. \tag{2.8}$$

Then $U_{\mathbb{F},f}$ is a subalgebra of $U_{\mathbb{F}}$. Moreover, by [12] we have

$$U_{\mathbb{F},f}=\sum_{\lambda\in\Lambda^+}\mathrm{ad}(U_{\mathbb{F}})(k_{-2\lambda}), \tag{2.9}$$

and hence $U_{\mathbb{F},f}$ is a subalgebra of $U_{\mathbb{F},\diamondsuit}$. Note that $U_{\mathbb{F},\diamondsuit}$ and $U_{\mathbb{F},f}$ are not Hopf subalgebras of $U_{\mathbb{F}}$; nevertheless, they satisfy the following.

LEMMA 2.13. *We have*

$$\Delta(U_{\mathbb{F},f})\subset U_{\mathbb{F}}\otimes U_{\mathbb{F},f},\qquad \Delta(U_{\mathbb{F},\diamondsuit})\subset U_{\mathbb{F}}\otimes U_{\mathbb{F},\diamondsuit}.$$

Proof. For $u\in U_{\mathbb{F}}$ and $\lambda\in\Lambda^+$, we have

$$\begin{aligned}\Delta\big(\mathrm{ad}(u)(k_{-2\lambda})\big)&=\sum_{(u)}\Delta\big(u_{(0)}k_{-2\lambda}(Su_{(1)})\big)\\&=\sum_{(u)_3}u_{(0)}k_{-2\lambda}(Su_{(3)})\otimes u_{(1)}k_{-2\lambda}(Su_{(2)})\\&=\sum_{(u)_2}u_{(0)}k_{-2\lambda}(Su_{(2)})\otimes\mathrm{ad}(u_{(1)})(k_{-2\lambda}).\end{aligned}$$

Hence, the first formula follows from (2.9). Since $U_{\mathbb{F},\diamondsuit}$ is generated by e_i, Sf_i for $i\in I$ and $k_{2\lambda}$ for $\lambda\in\Lambda$, the second formula is a consequence of the fact that $\Delta(e_i)$, $\Delta(Sf_i)$, $\Delta(k_{2\lambda})$ belong to $U_{\mathbb{F}}\otimes U_{\mathbb{F},\diamondsuit}$. $\square$

We set

$$\begin{aligned}E_{\mathbb{F},\diamondsuit}&=A_{\mathbb{F}}\otimes U_{\mathbb{F},\diamondsuit}\otimes\mathbb{F}[\Lambda]\subset E_{\mathbb{F}},\\E_{\mathbb{F},f}&=A_{\mathbb{F}}\otimes U_{\mathbb{F},f}\otimes\mathbb{F}[\Lambda]\subset E_{\mathbb{F}}.\end{aligned}$$

By Lemma 2.13, they are subalgebras of $E_{\mathbb{F}}$.

We set

$$U^0_{\mathbb{A},\diamondsuit} = U^0_{\mathbb{F},\diamondsuit} \cap U_{\mathbb{A}} = \bigoplus_{\lambda\in\Lambda} \mathbb{A}k_{2\lambda}, \qquad U_{\mathbb{A},\diamondsuit} = U_{\mathbb{F},\diamondsuit} \cap U_{\mathbb{A}} = S(U^-_{\mathbb{A}})U^0_{\mathbb{A},\diamondsuit}U^+_{\mathbb{A}},$$
$$U_{\mathbb{A},f} = U_{\mathbb{A}} \cap U_{\mathbb{F},f},$$

and

$$E_{\mathbb{A},\diamondsuit} = E_{\mathbb{A}} \cap E_{\mathbb{F},\diamondsuit} = A_{\mathbb{A}} \otimes U_{\mathbb{A},\diamondsuit} \otimes \mathbb{A}[\Lambda] \subset E_{\mathbb{F},\diamondsuit},$$
$$E_{\mathbb{A},f} = E_{\mathbb{A}} \cap E_{\mathbb{F},f} = A_{\mathbb{A}} \otimes U_{\mathbb{A},f} \otimes \mathbb{A}[\Lambda] \subset E_{\mathbb{F},f}.$$

We also set

$$E_{\zeta,\diamondsuit} = \mathbb{C} \otimes_{\mathbb{A}} E_{\mathbb{A},\diamondsuit} = A_\zeta \otimes U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda] \subset E_\zeta,$$
$$E_{\zeta,f} = \mathbb{C} \otimes_{\mathbb{A}} E_{\mathbb{A},f} = A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \subset E_\zeta,$$

and

$$D_{\zeta,\diamondsuit} = \mathrm{Im}(E_{\zeta,\diamondsuit} \to D_\zeta), \qquad D_{\zeta,f} = \mathrm{Im}(E_{\zeta,f} \to D_\zeta),$$
$$D'_{\zeta,\diamondsuit} = \mathrm{Im}(E_{\zeta,\diamondsuit} \to D'_\zeta), \qquad D'_{\zeta,f} = \mathrm{Im}(E_{\zeta,f} \to D'_\zeta).$$

By

$$E_\zeta \cong E_{\zeta,\diamondsuit} \otimes_{U_{\zeta,\diamondsuit}} U_\zeta$$

we obtain

$$D'_{\zeta,\diamondsuit} = E_{\zeta,\diamondsuit} \Big/ \sum_{\varphi\in A_\zeta} A_\zeta \Omega'(\varphi) U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda], \tag{2.10}$$

$$D'_\zeta \cong D'_{\zeta,\diamondsuit} \otimes_{U_{\zeta,\diamondsuit}} U_\zeta. \tag{2.11}$$

2.5.3. Since U_ζ is a free $U_{\zeta,\diamondsuit}$-module, we have

$$R^i\Gamma(\omega^* D'_\zeta) \cong R^i\Gamma(\omega^* D'_{\zeta,\diamondsuit}) \otimes_{U_{\zeta,\diamondsuit}} U_\zeta$$

for any $i \in \mathbb{Z}$. Since $U_{\zeta,\diamondsuit}$ is a localization of $U_{\zeta,f}$ with respect to the Ore subset $\{k_{-2\lambda} \mid \Lambda \in \Lambda^+\}$, we have

$$R^i\Gamma(\omega^* D'_{\zeta,\diamondsuit}) \cong R^i\Gamma(\omega^* D'_{\zeta,f}) \otimes_{U_{\zeta,f}} U_{\zeta,\diamondsuit}$$

for any $i \in \mathbb{Z}$. It follows that

$$R^i\Gamma(\omega^* D'_\zeta) \cong R^i\Gamma(\omega^* D'_{\zeta,f}) \otimes_{U_{\zeta,f}} U_\zeta \tag{2.12}$$

for any $i \in \mathbb{Z}$. Note that

$$R^i\Gamma(\mathcal{B}, \mathrm{Fr}_* \mathcal{D}_{\mathcal{B}_\zeta}) \cong R^i\Gamma(\omega^* D'_\zeta)$$

by Lemma 1.8 and (2.7). Hence Conjecture 2.6 is a consequence of the following stronger conjecture.

Conjecture 2.14. *Assume that $\ell > h_G$. We have*

$$\Gamma(\omega^* D'_{\zeta,f}) \cong U_{\zeta,f} \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda],$$

and

$$R^i\Gamma(\omega^* D'_{\zeta,f}) = 0$$

for $i \neq 0$.

In the rest of this article, we give a reformulation of Conjecture 2.14 in terms of the induction functor.

§3. Representations

3.1.

For simplicity, we introduce a new notation, $\tilde{U}_{\mathbb{F}}^- = S(U_{\mathbb{F}}^-)$. Then we have $\tilde{U}_{\mathbb{F}}^- = \langle \tilde{f}_i \mid i \in I \rangle$, where $\tilde{f}_i = f_i k_i$ for $i \in I$. Moreover, setting

$$\tilde{U}_{\mathbb{F},\gamma}^- = \left\{ u \in \tilde{U}_{\mathbb{F}}^- \;\middle|\; k_\mu u k_{-\mu} = q^{(\gamma,\mu)} u \ (\mu \in \Lambda) \right\}$$

for $\gamma \in Q$, we have

$$\tilde{U}_{\mathbb{F}}^- = \bigoplus_{\gamma \in Q^+} \tilde{U}_{\mathbb{F},-\gamma}^-, \qquad \tilde{U}_{\mathbb{F},-\gamma}^- = U_{\mathbb{F},-\gamma}^- k_\gamma \quad (\gamma \in Q^+).$$

We also set

$$\tilde{U}_{\mathbb{A}} = U_{\mathbb{A}} \cap \tilde{U}_{\mathbb{F}}, \qquad \tilde{U}_{\mathbb{A},-\gamma} = U_{\mathbb{A}} \cap \tilde{U}_{\mathbb{F},-\gamma} \quad (\gamma \in Q^+),$$
$$\tilde{U}_\zeta = \mathbb{C} \otimes_{\mathbb{A}} \tilde{U}_{\mathbb{A}}, \qquad \tilde{U}_{\zeta,-\gamma} = \mathbb{C} \otimes_{\mathbb{A}} \tilde{U}_{\mathbb{A},-\gamma} \quad (\gamma \in Q^+).$$

Then we have

$$\tilde{U}_{\mathbb{A}}^- = \bigoplus_{\gamma \in Q^+} \tilde{U}_{\mathbb{A},-\gamma}^-, \qquad \tilde{U}_\zeta^- = \bigoplus_{\gamma \in Q^+} \tilde{U}_{\zeta,-\gamma}^-.$$

3.2.

For $\lambda \in \Lambda$, we define an algebra homomorphism $\chi_\lambda : U^0_{\mathbb{F}} \to \mathbb{F}$ by $\chi_\lambda(k_\mu) = q^{(\lambda,\mu)}$ $(\mu \in \Lambda)$. For $M \in \mathrm{Mod}(U_{\mathbb{F}})$ and $\lambda \in \Lambda$, we set

$$M_\lambda = \{ m \in M \mid hm = \chi_\lambda(h)m \ (h \in U^0_{\mathbb{F}}) \}.$$

For $\lambda \in \Lambda$, we define $M_{+,\mathbb{F}}(\lambda), M_{-,\mathbb{F}}(\lambda) \in \mathrm{Mod}(U_{\mathbb{F}})$ by

$$M_{+,\mathbb{F}}(\lambda) = U_{\mathbb{F}} \Big/ \sum_{y \in U^-_{\mathbb{F}}} U_{\mathbb{F}}\big(y - \varepsilon(y)\big) + \sum_{h \in U^0_{\mathbb{F}}} U_{\mathbb{F}}\big(h - \chi_\lambda(h)\big),$$

$$M_{-,\mathbb{F}}(\lambda) = U_{\mathbb{F}} \Big/ \sum_{x \in U^+_{\mathbb{F}}} U_{\mathbb{F}}\big(x - \varepsilon(x)\big) + \sum_{h \in U^0_{\mathbb{F}}} U_{\mathbb{F}}\big(h - \chi_\lambda(h)\big),$$

where $M_{+,\mathbb{F}}(\lambda)$ is a lowest-weight module with lowest-weight λ, and $M_{-,\mathbb{F}}(\lambda)$ is a highest-weight module with highest-weight λ. We have isomorphisms

$$M_{+,\mathbb{F}}(\lambda) \cong U^+_{\mathbb{F}} \quad (\overline{u} \leftrightarrow u), \qquad M_{-,\mathbb{F}}(\lambda) \cong U^-_{\mathbb{F}} \quad (\overline{u} \leftrightarrow u)$$

of $\mathbb{F}$-modules. Moreover, we have weight-space decompositions

$$M_{+,\mathbb{F}}(\lambda) = \bigoplus_{\mu \in \lambda + Q^+} M_{+,\mathbb{F}}(\lambda)_\mu, \qquad M_{-,\mathbb{F}}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} M_{-,\mathbb{F}}(\lambda)_\mu.$$

For $\lambda \in \Lambda^+$ we define $L_{+,\mathbb{F}}(-\lambda), L_{-,\mathbb{F}}(\lambda) \in \mathrm{Mod}_f(U_{\mathbb{F}})$ by

$$\begin{aligned} L_{+,\mathbb{F}}(-\lambda) = U_{\mathbb{F}} \Big/ & \sum_{y \in U^-_{\mathbb{F}}} U_{\mathbb{F}}\big(y - \varepsilon(y)\big) \\ & + \sum_{h \in U^0_{\mathbb{F}}} U_{\mathbb{F}}\big(h - \chi_{-\lambda}(h)\big) + \sum_{i \in I} U_{\mathbb{F}} e_i^{((\lambda,\alpha_i^\vee)+1)}, \\ L_{-,\mathbb{F}}(\lambda) = U_{\mathbb{F}} \Big/ & \sum_{x \in U^+_{\mathbb{F}}} U_{\mathbb{F}}\big(x - \varepsilon(x)\big) \\ & + \sum_{h \in U^0_{\mathbb{F}}} U_{\mathbb{F}}\big(h - \chi_{\lambda}(h)\big) + \sum_{i \in I} U_{\mathbb{F}} f_i^{((\lambda,\alpha_i^\vee)+1)}. \end{aligned}$$

While $L_{+,\mathbb{F}}(-\lambda)$ is a finite-dimensional irreducible lowest-weight module with lowest-weight $-\lambda$, here $L_{-,\mathbb{F}}(\lambda)$ is a finite-dimensional irreducible highest-weight module with highest-weight λ. We have weight-space decompositions

$$L_{+,\mathbb{F}}(-\lambda)=\bigoplus_{\mu\in-\lambda+Q^+}L_{+,\mathbb{F}}(-\lambda)_\mu,\qquad L_{-,\mathbb{F}}(\lambda)=\bigoplus_{\mu\in\lambda-Q^+}L_{-,\mathbb{F}}(\lambda)_\mu.$$

For $\lambda\in\Lambda^+$ we have isomorphisms

$$L_{+,\mathbb{F}}(-\lambda)\cong U_{\mathbb{F}}^{L,+}\Big/\sum_{i\in I}U_{\mathbb{F}}^{L,+}e_i^{((\lambda,\alpha_i^\vee)+1)}\quad(\overline{u}\leftrightarrow\overline{u}),$$
$$L_{-,\mathbb{F}}(\lambda)\cong \tilde{U}_{\mathbb{F}}^{L,-}\Big/\sum_{i\in I}\tilde{U}_{\mathbb{F}}^{L,-}\tilde{f}_i^{((\lambda,\alpha_i^\vee)+1)}\quad(\overline{u}\leftrightarrow\overline{u})$$

of vector spaces (see [13]).

Let M be a $U_{\mathbb{F}}$-module with weight-space decomposition $M=\bigoplus_{\mu\in\Lambda}M_\mu$ such that $\dim M_\mu<\infty$ for any $\mu\in\Lambda$. We define a $U_{\mathbb{F}}$-module $M^\bigstar$ by

$$M^\bigstar=\bigoplus_{\mu\in\Lambda}M_\mu^*\subset M^*=\mathrm{Hom}_{\mathbb{F}}(M,\mathbb{F}),$$

where the action of $U_{\mathbb{F}}$ is given by

$$\langle um^*,m\rangle=\big\langle m^*,(Su)m\big\rangle\quad(u\in U_{\mathbb{F}},m^*\in M^\bigstar,m\in M).$$

Here $\langle\,,\,\rangle:M^\bigstar\times M\to\mathbb{F}$ is the natural pairing.

We set

$$M_{\pm,\mathbb{F}}^*(\lambda)=\big(M_{\mp,\mathbb{F}}(-\lambda)\big)^\bigstar\quad(\lambda\in\Lambda),$$
$$L_{\pm,\mathbb{F}}^*(\mp\lambda)=\big(L_{\mp,\mathbb{F}}(\pm\lambda)\big)^\bigstar\quad(\lambda\in\Lambda^+).$$

Since $L_{\mp,\mathbb{F}}(\pm\lambda)$ is irreducible, we have

$$L_{\pm,\mathbb{F}}^*(\mp\lambda)\cong L_{\pm,\mathbb{F}}(\mp\lambda)\quad(\lambda\in\Lambda^+).$$

We define isomorphisms

$$\Phi_\lambda:U_{\mathbb{F}}^+\to M_{+,\mathbb{F}}^*(\lambda),\qquad \Psi_\lambda:\tilde{U}_{\mathbb{F}}^-\to M_{-,\mathbb{F}}^*(\lambda)\tag{3.1}$$

of vector spaces by

$$\big\langle\Phi_\lambda(x),\overline{v}\big\rangle=\tau(x,v)\quad(x\in U_{\mathbb{F}}^+,v\in\tilde{U}_{\mathbb{F}}^-),$$
$$\big\langle\Psi_\lambda(y),\overline{Su}\big\rangle=\tau(u,y)\quad(y\in\tilde{U}_{\mathbb{F}}^-,u\in U_{\mathbb{F}}^+).$$

Lemma 3.1.

(i) *The* $U_{\mathbb{F}}$*-module structure of* $M^*_{+,\mathbb{F}}(\lambda)$ *is given by*

$$h\cdot\Phi_\lambda(x)=\chi_{\lambda+\gamma}(h)\Phi_\lambda(x)\quad(x\in U^+_{\mathbb{F},\gamma},h\in U^0_{\mathbb{F}}),\tag{3.2}$$

$$v\cdot\Phi_\lambda(x)=\sum_{(x)}\tau(x_{(0)},Sv)\Phi_\lambda(x_{(1)})\quad(x\in U^+_{\mathbb{F}},v\in U^-_{\mathbb{F}}),\tag{3.3}$$

$$u\cdot\Phi_\lambda(x)=\Phi_\lambda\big(k_{-\lambda}\big(\mathrm{ad}(u)(k_\lambda xk_\lambda)\big)k_{-\lambda}\big)\quad(x\in U^+_{\mathbb{F}},u\in U^+_{\mathbb{F}}).\tag{3.4}$$

(ii) *The* $U_{\mathbb{F}}$*-module structure of* $M^*_{-,\mathbb{F}}(\lambda)$ *is given by*

$$h\cdot\Psi_\lambda(y)=\chi_{\lambda-\gamma}(h)\Psi_\lambda(y)\quad(y\in\tilde{U}^-_{\mathbb{F},-\gamma},h\in U^0_{\mathbb{F}}),\tag{3.5}$$

$$u\cdot\Psi_\lambda(y)=\sum_{(y)}\tau(u,y_{(0)})\Psi_\lambda(y_{(1)})\quad(y\in\tilde{U}^-_{\mathbb{F}},u\in U^+_{\mathbb{F}}),\tag{3.6}$$

$$v\cdot\Psi_\lambda(y)=\Psi_\lambda\big(k_\lambda\big(\mathrm{ad}(v)(k_{-\lambda}yk_{-\lambda})\big)k_\lambda\big)\quad(y\in\tilde{U}^-_{\mathbb{F}},v\in U^-_{\mathbb{F}}).\tag{3.7}$$

Proof. We will prove only (i). The proof of (ii) is similar and omitted. Note that for $x\in U^+_{\mathbb{F}}$, $a\in U_{\mathbb{F}}$, and $v\in\tilde{U}^-_{\mathbb{F}}$, we have

$$\langle a\cdot\Phi_\lambda(x),\overline{v}\rangle=\langle\Phi_\lambda(x),\overline{(Sa)v}\rangle.$$

Let us show (3.2). For $v\in\tilde{U}^-_{\mathbb{F},-\delta}$, we have

$$\begin{aligned}\langle h\cdot\Phi_\lambda(x),\overline{v}\rangle&=\langle\Phi_\lambda(x),\overline{(Sh)v}\rangle=\delta_{\gamma,\delta}\langle\Phi_\lambda(x),\overline{(Sh)v}\rangle\\&=\delta_{\gamma,\delta}\chi_{\lambda+\gamma}(h)\langle\Phi_\lambda(x),\overline{v}\rangle=\chi_{\lambda+\gamma}(h)\langle\Phi_\lambda(x),\overline{v}\rangle.\end{aligned}$$

Hence, (3.2) holds. Let us next show (3.3). For $v\in\tilde{U}^-_{\mathbb{F}}$, we have

$$\begin{aligned}\langle y\cdot\Phi_\lambda(x),\overline{v}\rangle&=\langle\Phi_\lambda(x),\overline{(Sy)v}\rangle=\tau\big(x,(Sy)v\big)=\sum_{(x)}\tau(x_{(0)},Sy)\tau(x_{(1)},v)\\&=\Big\langle\Phi_\lambda\Big(\sum_{(x)}\tau(x_{(0)},Sy)x_{(1)}\Big),\overline{v}\Big\rangle.\end{aligned}$$

Hence, (3.3) also holds. Let us finally show (3.4). We may assume that $u\in U^+_{\mathbb{F},\beta}$ for some $\beta\in Q^+$. Then we can write

$$\Delta u=\sum_j u_jk_{\beta'_j}\otimes u'_j\quad(\beta_j,\beta'_j\in Q^+,\beta_j+\beta'_j=\beta,u_j\in U^+_{\mathbb{F},\beta_j},u'_j\in U^+_{\mathbb{F},\beta'_j}).$$

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For $v \in \tilde{U}_{\mathbb{F}}^-$, we have

$$\begin{aligned}
\langle u \cdot \Phi_\lambda(x), \overline{v} \rangle &= \langle \Phi_\lambda(x), \overline{(Su)v} \rangle \\
&= \sum_{(u)_2,(v)_2} \tau(Su_{(2)}, v_{(0)}) \tau(Su_{(0)}, Sv_{(2)}) \langle \Phi_\lambda(x), \overline{v_{(1)}(Su_{(1)})} \rangle \\
&= \sum_{j,(v)_2} \tau(Su'_j, v_{(0)}) \tau(u_j k_{\beta'_j}, v_{(2)}) \langle \Phi_\lambda(x), \overline{v_{(1)}(Sk_{\beta'_j})} \rangle \\
&= \sum_{j,(v)_2} q^{(\lambda, \beta'_j - \beta_j)} \tau(Su'_j, v_{(0)}) \tau(u_j k_{\beta'_j}, v_{(2)}) \langle \Phi_\lambda(x), \overline{v_{(1)} k_{-\beta_j}} \rangle \\
&= \sum_{j,(v)_2} q^{(\lambda, \beta'_j - \beta_j)} \tau(Su'_j, v_{(0)}) \tau(u_j k_{\beta'_j}, v_{(2)}) \tau(x, v_{(1)} k_{-\beta_j}) \\
&= \sum_{j,(v)_2} q^{(\lambda, \beta'_j - \beta_j)} \tau(Su'_j, v_{(0)}) \tau(x, v_{(1)}) \tau(u_j k_{\beta'_j}, v_{(2)}) \\
&= \sum_{j} q^{(\lambda, \beta'_j - \beta_j)} \tau\big(u_j k_{\beta'_j} x (Su'_j), v\big) \\
&= \big\langle \Phi_\lambda\big(k_{-\lambda}\big(\mathrm{ad}(u)(k_\lambda x k_\lambda)\big) k_{-\lambda}\big), \overline{v} \big\rangle.
\end{aligned}$$

Here, we have used Lemma 1.1. Note also that $\Delta \tilde{U}_{\mathbb{F}}^- \subset \sum_{\gamma \in Q^+} \tilde{U}_{\mathbb{F}}^- k_\gamma \otimes \tilde{U}_{\mathbb{F},-\gamma}^-$, and hence we have $\Delta_2 \tilde{U}_{\mathbb{F}}^- \subset \sum_{\gamma,\delta \in Q^+} \tilde{U}_{\mathbb{F}}^- k_{\gamma+\delta} \otimes \tilde{U}_{\mathbb{F},-\gamma}^- k_\delta \otimes \tilde{U}_{\mathbb{F},-\delta}^-$. Thus, (3.4) is proved. $\square$

For $\lambda \in \Lambda$ we denote by $\mathbb{F}_\lambda^{\geqq 0} = \mathbb{F} 1_\lambda^{\geqq 0}$ (resp., $\mathbb{F}_\lambda^{\leqq 0} = \mathbb{F} 1_\lambda^{\leqq 0}$) the 1-dimensional $U_{\mathbb{F}}^{\geqq 0}$-module(resp., $U_{\mathbb{F}}^{\leqq 0}$-module) such that $h 1_\lambda^{\geqq 0} = \chi_\lambda(h) 1_\lambda^{\geqq 0}$, $u 1_\lambda^{\geqq 0} = \varepsilon(u) 1_\lambda^{\geqq 0}$ for $h \in U_{\mathbb{F}}^0$ and $u \in U_{\mathbb{F}}^+$ (resp., $h 1_\lambda^{\leqq 0} = \chi_\lambda(h) 1_\lambda^{\leqq 0}$, $u 1_\lambda^{\leqq 0} = \varepsilon(u) 1_\lambda^{\leqq 0}$ for $h \in U_{\mathbb{F}}^0$ and $u \in U_{\mathbb{F}}^-$).

Note that for any $\lambda \in \Lambda$, $k_{-2\lambda} U_{\mathbb{F}}^+$ (resp., $\tilde{U}_{\mathbb{F}}^- k_{-2\lambda}$) is $\mathrm{ad}(U_{\mathbb{F}}^{\geqq 0})$-stable (resp., $\mathrm{ad}(U_{\mathbb{F}}^{\leqq 0})$-stable). We see easily from Lemma 3.1 the following.

LEMMA 3.2. *Let* $\lambda \in \Lambda$.

(i) *The linear map*

$$k_{-2\lambda} U_{\mathbb{F}}^+ \to M_{+,\mathbb{F}}^*(-\lambda) \otimes \mathbb{F}_\lambda^{\geqq 0} \quad \big(k_{-\lambda} x k_{-\lambda} \mapsto \Phi_{-\lambda}(x) \otimes 1_\lambda^{\geqq 0}\big)$$

is an isomorphism of $U_{\mathbb{F}}^{\geqq 0}$*-modules, where* $k_{-2\lambda} U_{\mathbb{F}}^+$ *is regarded as a* $U_{\mathbb{F}}^{\leqq 0}$*-module by the adjoint action.*

(ii) *The linear map*

$$\tilde{U}_{\mathbb{F}}^{-}k_{-2\lambda} \to \mathbb{F}_{-\lambda}^{\leqq 0} \otimes M_{-,\mathbb{F}}^{*}(\lambda) \quad \bigl(k_{-\lambda}yk_{-\lambda} \mapsto 1_{-\lambda}^{\leqq 0} \otimes \Psi_{\lambda}(y)\bigr)$$

is an isomorphism of $U_{\mathbb{F}}^{\leqq 0}$-modules, where $\tilde{U}_{\mathbb{F}}^{-}k_{-2\lambda}$ is regarded as a $U_{\mathbb{F}}^{\leqq 0}$-module by the adjoint action.

We have an injective $U_{\mathbb{F}}$-homomorphism

$$L_{\pm,\mathbb{F}}^{*}(\mp\lambda) \to M_{\pm,\mathbb{F}}^{*}(\mp\lambda) \quad (\lambda \in \Lambda^{+}) \tag{3.8}$$

induced by the natural homomorphism $M_{\pm,\mathbb{F}}(\mp\lambda) \to L_{\pm,\mathbb{F}}(\mp\lambda)$. For $\lambda \in \Lambda^{+}$ we define subspaces $U_{\mathbb{F}}^{+}(\lambda)$, $\tilde{U}_{\mathbb{F}}^{-}(\lambda)$ of $U_{\mathbb{F}}^{+}$, $\tilde{U}_{\mathbb{F}}^{-}$, respectively, by

$$U_{\mathbb{F}}^{+}(\lambda) = \Phi_{-\lambda}^{-1}\bigl(L_{+,\mathbb{F}}^{*}(-\lambda)\bigr), \qquad \tilde{U}_{\mathbb{F}}^{-}(\lambda) = \Psi_{\lambda}^{-1}\bigl(L_{-,\mathbb{F}}^{*}(\lambda)\bigr).$$

LEMMA 3.3.

(i) *For $\lambda, \mu \in \Lambda^{+}$ we have*

$$U_{\mathbb{F}}^{+}(\lambda) \subset U_{\mathbb{F}}^{+}(\lambda+\mu), \qquad \tilde{U}_{\mathbb{F}}^{-}(\lambda) \subset \tilde{U}_{\mathbb{F}}^{-}(\lambda+\mu).$$

(ii) *We have*

$$U_{\mathbb{F}}^{+} = \sum_{\lambda \in \Lambda^{+}} U_{\mathbb{F}}^{+}(\lambda), \qquad \tilde{U}_{\mathbb{F}}^{-} = \sum_{\lambda \in \Lambda^{+}} \tilde{U}_{\mathbb{F}}^{-}(\lambda).$$

Proof. We will prove only the statements for $U_{\mathbb{F}}^{+}$. By definition, we have $U_{\mathbb{F}}^{+}(\lambda) = \{x \in U_{\mathbb{F}}^{+} \mid \tau(x, I_{\lambda}) = \{0\}\}$, where $I_{\lambda} = \sum_{i \in I} \tilde{U}_{\mathbb{F}}^{-} \tilde{f}_{i}^{((\lambda,\alpha_{i}^{\vee})+1)}$.

Hence, (i) is a consequence of $I_{\lambda} \supset I_{\lambda+\mu}$ for $\lambda, \mu \in \Lambda^{+}$. To show (ii) it is sufficient to show that for any $\beta \in Q^{+}$ there exists some $\lambda \in \Lambda^{+}$ such that $U_{\mathbb{F},\beta}^{+} \subset U_{\mathbb{F}}^{+}(\lambda)$. Set $m = \mathrm{ht}(\beta)$. If $\lambda \in \Lambda^{+}$ satisfies $(\lambda, \alpha_{i}^{\vee}) \geqq m$ for any $i \in I$, then we have $I_{\lambda} \subset \bigoplus_{\gamma \in Q^{+}, \mathrm{ht}(\gamma) > m} \tilde{U}_{\mathbb{F},-\gamma}^{-}$. From this we obtain $\tau(U_{\mathbb{F},\beta}^{+}, I_{\lambda}) = \{0\}$, and hence $U_{\mathbb{F},\beta}^{+} \subset U_{\mathbb{F}}^{+}(\lambda)$. ∎

LEMMA 3.4. *For $\lambda \in \Lambda^{+}$, we have*

$$\tilde{U}_{\mathbb{F}}^{-}(\lambda)k_{-2\lambda} \subset U_{\mathbb{F},f}, \qquad k_{-2\lambda}U_{\mathbb{F}}^{+}(\lambda) \subset U_{\mathbb{F},f}.$$

Proof. By Lemma 3.2, we have an isomorphism

$$k_{-2\lambda}U_{\mathbb{F}}^{+}(\lambda) \to L_{+,\mathbb{F}}^{*}(-\lambda) \otimes \mathbb{F}_{\lambda}^{\geqq 0} \quad \bigl(k_{-\lambda}xk_{-\lambda} \mapsto \Phi_{-\lambda}(x) \otimes 1_{\lambda}^{\geqq 0}\bigr)$$

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of $U_{\mathbb{F}}^{\geqq 0}$-modules. We have $L_{+,\mathbb{F}}^*(-\lambda) \cong L_{+,\mathbb{F}}(-\lambda)$, and hence $L_{+,\mathbb{F}}^*(-\lambda) \otimes \mathbb{F}_\lambda^{\geqq 0}$ is generated by $\Phi_{-\lambda}(1) \otimes 1_\lambda^{\geqq 0}$ as a $U_{\mathbb{F}}^{\geqq 0}$-module. It follows that

$$k_{-2\lambda} U_{\mathbb{F}}^+(\lambda) = \mathrm{ad}(U_{\mathbb{F}}^{\geqq 0})(k_{-2\lambda}) \subset U_{\mathbb{F},f}$$

by (2.9). The proof of $\tilde{U}_{\mathbb{F}}^-(\lambda) k_{-2\lambda} \subset U_{\mathbb{F},f}$ is similar. $\square$

3.3.

It is well known that, for $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$, there exists $h \in U_{\mathbb{A}}^{L,0}$ such that $\chi_\lambda(h) = 1$ and $\chi_\mu(h) = 0$. In particular, we have $\chi_\lambda \neq \chi_\mu$ (see, e.g., [20, Lemma 2.3]).

For $M \in \mathrm{Mod}(U_{\mathbb{A}}^L)$ and $\lambda \in \Lambda$, we set

$$M_\lambda = \left\{ m \in M \,\middle|\, hm = \chi_\lambda(h) m \ (h \in U_{\mathbb{A}}^{L,0}) \right\}.$$

For $\lambda \in \Lambda$, we define $M_{+,\mathbb{A}}(\lambda), M_{-,\mathbb{A}}(\lambda) \in \mathrm{Mod}(U_{\mathbb{A}}^L)$ by

$$M_{+,\mathbb{A}}(\lambda) = U_{\mathbb{A}}^L \Big/ \sum_{y \in U_{\mathbb{A}}^{L,-}} U_{\mathbb{A}}^L \big(y - \varepsilon(y)\big) + \sum_{h \in U_{\mathbb{A}}^{L,0}} U_{\mathbb{A}}^L \big(h - \chi_\lambda(h)\big),$$

$$M_{-,\mathbb{A}}(\lambda) = U_{\mathbb{A}}^L \Big/ \sum_{x \in U_{\mathbb{A}}^{L,+}} U_{\mathbb{A}}^L \big(x - \varepsilon(x)\big) + \sum_{h \in U_{\mathbb{A}}^{L,0}} U_{\mathbb{A}}^L \big(h - \chi_\lambda(h)\big).$$

By the triangular decomposition we have isomorphisms

$$M_{+,\mathbb{A}}(\lambda) \cong U_{\mathbb{A}}^{L,+} \quad (\overline{u} \leftrightarrow u), \qquad M_{-,\mathbb{A}}(\lambda) \cong U_{\mathbb{A}}^{L,-} \quad (\overline{u} \leftrightarrow u)$$

of $\mathbb{A}$-modules. In particular, $M_{\pm,\mathbb{A}}(\lambda)$ is a free $\mathbb{A}$-module, and we have $\mathbb{F} \otimes_{\mathbb{A}} M_{\pm,\mathbb{A}}(\lambda) \cong M_{\pm,\mathbb{F}}(\lambda)$. Moreover, we have weight-space decompositions

$$M_{+,\mathbb{A}}(\lambda) = \bigoplus_{\mu \in \lambda + Q^+} M_{+,\mathbb{A}}(\lambda)_\mu, \qquad M_{-,\mathbb{A}}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} M_{-,\mathbb{A}}(\lambda)_\mu.$$

For $\lambda \in \Lambda^+$, we define $L_{+,\mathbb{A}}(-\lambda) \in \mathrm{Mod}(U_{\mathbb{A}}^L)$ (resp., $L_{-,\mathbb{A}}(\lambda) \in \mathrm{Mod}(U_{\mathbb{A}}^L)$) to be the $U_{\mathbb{A}}^L$-submodule of $L_{+,\mathbb{F}}(-\lambda)$ (resp., $L_{-,\mathbb{F}}(\lambda)$) generated by $\overline{1} \in L_{+,\mathbb{F}}(-\lambda)$ (resp., $\overline{1} \in L_{-,\mathbb{F}}(\lambda)$). By definition, $L_{\pm,\mathbb{A}}(\mp\lambda)$ is a free $\mathbb{A}$-module, and we have $\mathbb{F} \otimes_{\mathbb{A}} L_{\pm,\mathbb{A}}(\mp\lambda) \cong L_{\pm,\mathbb{F}}(\mp\lambda)$. Moreover, we have weight-space decompositions

$$L_{+,\mathbb{A}}(-\lambda) = \bigoplus_{\mu \in -\lambda + Q^+} L_{+,\mathbb{A}}(-\lambda)_\mu, \qquad L_{-,\mathbb{A}}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} L_{-,\mathbb{A}}(\lambda)_\mu.$$

The canonical surjective $U_{\mathbb{F}}$-homomorphism $M_{\pm,\mathbb{F}}(\mp\lambda) \to L_{\pm,\mathbb{F}}(\mp\lambda)$ induces a surjective $U_{\mathbb{A}}^L$-homomorphism

$$M_{\pm,\mathbb{A}}(\mp\lambda) \to L_{\pm,\mathbb{A}}(\mp\lambda) \quad (\lambda \in \Lambda^+). \tag{3.9}$$

Note that (3.9) is a split epimorphism of $\mathbb{A}$-modules since $\mathbb{A}$ is a PID (Principal Ideal Domain), and note that $M_{\pm,\mathbb{A}}(\mp\lambda)_\mu$, $L_{\pm,\mathbb{A}}(\mp\lambda)_\mu$ are torsion-free finitely generated $\mathbb{A}$-modules for each $\mu \in \Lambda$.

Let M be a $U_{\mathbb{A}}^L$-module with weight-space decomposition $M = \bigoplus_{\mu\in\Lambda} M_\mu$ such that M_μ is a free $\mathbb{A}$-module of finite rank for any $\mu \in \Lambda$. We define a $U_{\mathbb{A}}^L$-module $M^\bigstar$ by

$$M^\bigstar = \bigoplus_{\mu\in\Lambda} \mathrm{Hom}_{\mathbb{A}}(M_\mu, \mathbb{A}) \subset \mathrm{Hom}_{\mathbb{A}}(M, \mathbb{A}),$$

where the action of $U_{\mathbb{A}}^L$ is given by

$$\langle um^*, m\rangle = \big\langle m^*, (Su)m\big\rangle \quad (u \in U_{\mathbb{A}}^L, m^* \in M^\bigstar, m \in M).$$

Here $\langle\,,\,\rangle : M^\bigstar \times M \to \mathbb{A}$ is the natural pairing.

We set

$$M^*_{\pm,\mathbb{A}}(\lambda) = \big(M_{\mp,\mathbb{A}}(-\lambda)\big)^\bigstar \quad (\lambda \in \Lambda),$$

$$L^*_{\pm,\mathbb{A}}(\mp\lambda) = \big(L_{\mp,\mathbb{A}}(\pm\lambda)\big)^\bigstar \quad (\lambda \in \Lambda^+).$$

Then $M^*_{\pm,\mathbb{A}}(\lambda)$ for $\lambda \in \Lambda$ and $L^*_{\pm,\mathbb{A}}(\mp\lambda)$ for $\lambda \in \Lambda^+$ are free $\mathbb{A}$-modules satisfying

$$\mathbb{F} \otimes_{\mathbb{A}} M^*_{\pm,\mathbb{A}}(\lambda) \cong M^*_{\pm,\mathbb{F}}(\lambda), \qquad \mathbb{F} \otimes_{\mathbb{A}} L^*_{\pm,\mathbb{A}}(\mp\lambda) \cong L^*_{\pm,\mathbb{F}}(\mp\lambda).$$

Moreover, we can identify $M^*_{\pm,\mathbb{A}}(\lambda)$ and $L^*_{\pm,\mathbb{A}}(\mp\lambda)$ with $\mathbb{A}$-submodules of $M^*_{\pm,\mathbb{F}}(\lambda)$ and $L^*_{\pm,\mathbb{F}}(\mp\lambda)$, respectively. Under this identification we have

$$L^*_{\pm,\mathbb{A}}(\mp\lambda) = L^*_{\pm,\mathbb{F}}(\mp\lambda) \cap M^*_{\pm,\mathbb{A}}(\mp\lambda) \quad (\lambda \in \Lambda^+). \tag{3.10}$$

In particular, the $U_{\mathbb{A}}^L$-homomorphism

$$L^*_{\pm,\mathbb{A}}(\mp\lambda) \to M^*_{\pm,\mathbb{A}}(\mp\lambda) \quad (\lambda \in \Lambda^+) \tag{3.11}$$

is a split monomorphism of $\mathbb{A}$-modules.

By abuse of notation we write

$$\Phi_\lambda : U_{\mathbb{A}}^+ \to M^*_{+,\mathbb{A}}(\lambda), \qquad \Psi_\lambda : \tilde{U}_{\mathbb{A}}^- \to M^*_{-,\mathbb{A}}(\lambda) \tag{3.12}$$

for the isomorphisms of $\mathbb{A}$-modules induced by (3.1). By Lemma 3.1 we have the following.

LEMMA 3.5.

(i) *The* $U_{\mathbb{A}}^L$*-module structure of* $M_{+,\mathbb{A}}^*(\lambda)$ *is given by*

$$h \cdot \Phi_\lambda(x) = \chi_{\lambda+\gamma}(h)\Phi_\lambda(x) \quad (x \in U_{\mathbb{A},\gamma}^+, h \in U_{\mathbb{A}}^{L,0}), \tag{3.13}$$

$$v \cdot \Phi_\lambda(x) = \sum_{(x)} \tau_{\mathbb{A}}^L(x_{(0)}, Sv)\Phi_\lambda(x_{(1)}) \quad (x \in U_{\mathbb{A}}^+, v \in U_{\mathbb{A}}^{L,-}), \tag{3.14}$$

$$u \cdot \Phi_\lambda(x) = \Phi_\lambda\big(k_{-\lambda}\big(\mathrm{ad}(u)(k_\lambda x k_\lambda)\big)k_{-\lambda}\big) \quad (x \in U_{\mathbb{A}}^+, u \in U_{\mathbb{A}}^{L,+}). \tag{3.15}$$

(ii) *The* $U_{\mathbb{A}}^L$*-module structure of* $M_{-,\mathbb{A}}^*(\lambda)$ *is given by*

$$h \cdot \Psi_\lambda(y) = \chi_{\lambda-\gamma}(h)\Psi_\lambda(y) \quad (y \in \tilde{U}_{\mathbb{A},-\gamma}^-, h \in U_{\mathbb{A}}^{L,0}), \tag{3.16}$$

$$u \cdot \Psi_\lambda(y) = \sum_{(y)} {}^L\tau_{\mathbb{A}}(u, y_{(0)})\Psi_\lambda(y_{(1)}) \quad (y \in \tilde{U}_{\mathbb{A}}^-, u \in U_{\mathbb{A}}^{L,+}), \tag{3.17}$$

$$v \cdot \Psi_\lambda(y) = \Psi_\lambda\big(k_\lambda\big(\mathrm{ad}(v)(k_{-\lambda} y k_{-\lambda})\big)k_\lambda\big) \quad (y \in \tilde{U}_{\mathbb{A}}^-, v \in U_{\mathbb{A}}^{L,-}). \tag{3.18}$$

For $\lambda \in \Lambda^+$ we define $\mathbb{A}$-submodules $U_{\mathbb{A}}^+(\lambda)$, $\tilde{U}_{\mathbb{A}}^-(\lambda)$ of $U_{\mathbb{A}}^+$, $\tilde{U}_{\mathbb{A}}^-$, respectively, by

$$U_{\mathbb{A}}^+(\lambda) = \Phi_{-\lambda}^{-1}\big(L_{+,\mathbb{A}}^*(-\lambda)\big), \qquad \tilde{U}_{\mathbb{A}}^-(\lambda) = \Psi_\lambda^{-1}\big(L_{-,\mathbb{A}}^*(\lambda)\big).$$

The embeddings

$$U_{\mathbb{A}}^+(\lambda) \hookrightarrow U_{\mathbb{A}}^+, \qquad \tilde{U}_{\mathbb{A}}^-(\lambda) \hookrightarrow \tilde{U}_{\mathbb{A}}^- \quad (\lambda \in \Lambda^+) \tag{3.19}$$

are split monomorphisms of $\mathbb{A}$-modules. By (3.10), we have

$$U_{\mathbb{A}}^+(\lambda) = U_{\mathbb{F}}^+(\lambda) \cap U_{\mathbb{A}}^+, \qquad \tilde{U}_{\mathbb{A}}^-(\lambda) = \tilde{U}_{\mathbb{F}}^-(\lambda) \cap \tilde{U}_{\mathbb{A}}^- \quad (\lambda \in \Lambda^+). \tag{3.20}$$

In particular, we have

$$U_{\mathbb{A}}^+(\lambda) \subset U_{\mathbb{A}}^+(\lambda+\mu), \qquad \tilde{U}_{\mathbb{A}}^-(\lambda) \subset \tilde{U}_{\mathbb{A}}^-(\lambda+\mu) \quad (\lambda, \mu \in \Lambda^+), \tag{3.21}$$

$$U_{\mathbb{A}}^+ = \sum_{\lambda \in \Lambda^+} U_{\mathbb{A}}^+(\lambda), \qquad \tilde{U}_{\mathbb{A}}^- = \sum_{\lambda \in \Lambda^+} \tilde{U}_{\mathbb{A}}^-(\lambda), \tag{3.22}$$

$$\tilde{U}_{\mathbb{A}}^-(\lambda)k_{-2\lambda} \subset U_{\mathbb{A},f}, \qquad k_{-2\lambda}U_{\mathbb{A}}^+(\lambda) \subset U_{\mathbb{A},f} \quad (\lambda \in \Lambda^+) \tag{3.23}$$

by Lemmas 3.3 and 3.4.

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3.4.

Let $\lambda \in \Lambda$. By abuse of notation we also denote by $\chi_\lambda : U_\zeta^{L,0} \to \mathbb{C}$ the $\mathbb{C}$-algebra homomorphism induced by $\chi_\lambda : U_{\mathbb{A}}^{L,0} \to \mathbb{A}$. Then $\{\chi_\lambda\}_{\lambda \in \Lambda}$ is a linearly independent subset of the $\mathbb{C}$-module $\mathrm{Hom}_{\mathbb{C}}(U_\zeta^{L,0}, \mathbb{C})$. For $M \in \mathrm{Mod}(U_\zeta^L)$ and $\lambda \in \Lambda$, we set

$$M_\lambda = \{m \in M \mid hm = \chi_\lambda(h)m \ (h \in U_\zeta^{L,0})\}.$$

For $\lambda \in \Lambda$ we set

$$M_{\pm,\zeta}(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} M_{\pm,\mathbb{A}}(\lambda), \qquad M^*_{\pm,\zeta}(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} M^*_{\pm,\mathbb{A}}(\lambda).$$

For $\lambda \in \Lambda^+$ we set

$$L_{\pm,\zeta}(\mp\lambda) = \mathbb{C} \otimes_{\mathbb{A}} L_{\pm,\mathbb{A}}(\mp\lambda), \qquad L^*_{\pm,\zeta}(\mp\lambda) = \mathbb{C} \otimes_{\mathbb{A}} L^*_{\pm,\mathbb{A}}(\mp\lambda).$$

We have canonical U_ζ^L-homomorphisms

$$M_{\pm,\zeta}(\mp\lambda) \to L_{\pm,\zeta}(\mp\lambda) \quad (\lambda \in \Lambda^+), \tag{3.24}$$

$$L^*_{\pm,\zeta}(\mp\lambda) \to M^*_{\pm,\zeta}(\mp\lambda) \quad (\lambda \in \Lambda^+). \tag{3.25}$$

Note that (3.24) is surjective and that (3.25) is injective.

For any $\lambda \in \Lambda^+$ we have an isomorphism

$$A_\zeta(\lambda) \cong L^*_{-,\zeta}(\lambda) \tag{3.26}$$

of U_ζ^L-modules (see, e.g., [11, Chapter 9], [20, Section 3.1]).

Let $\lambda \in \Lambda$. By abuse of notation we also denote by

$$\Phi_\lambda : U_\zeta^+ \to M^*_{+,\zeta}(\lambda), \qquad \Psi_\lambda : \tilde{U}_\zeta^- \to M^*_{-,\zeta}(\lambda)$$

the isomorphisms of $\mathbb{C}$-modules given by

$$\langle \Phi_\lambda(x), \overline{v} \rangle = \tau_\zeta^L(x, v) \quad (x \in U_\zeta^+, v \in \tilde{U}_\zeta^{L,-}),$$
$$\langle \Psi_\lambda(y), \overline{Su} \rangle = {}^L\tau_\zeta(u, y) \quad (y \in \tilde{U}_\zeta^-, u \in U_\zeta^{L,+}).$$

By Lemma 3.5, we have the following.

LEMMA 3.6.

(i) *The U_ζ^L-module structure of $M_{+,\zeta}^*(\lambda)$ is given by*

$$h\cdot\Phi_\lambda(x)=\chi_{\lambda+\gamma}(h)\Phi_\lambda(x)\quad(x\in U_{\zeta,\gamma}^+,h\in U_\zeta^{L,0}),\tag{3.27}$$

$$v\cdot\Phi_\lambda(x)=\sum_{(x)}\tau_\zeta^L(x_{(0)},Sv)\Phi_\lambda(x_{(1)})\quad(x\in U_\zeta^+,v\in U_\zeta^{L,-}),\tag{3.28}$$

$$u\cdot\Phi_\lambda(x)=\Phi_\lambda\big(k_{-\lambda}\big(\mathrm{ad}(u)(k_\lambda xk_\lambda)\big)k_{-\lambda}\big)\quad(x\in U_\zeta^+,u\in U_\zeta^{L,+}).\tag{3.29}$$

(ii) *The U_ζ^L-module structure of $M_{-,\zeta}^*(\lambda)$ is given by*

$$h\cdot\Psi_\lambda(y)=\chi_{\lambda-\gamma}(h)\Psi_\lambda(y)\quad(y\in\tilde U_{\zeta,-\gamma}^-,h\in U_\zeta^{L,0}),\tag{3.30}$$

$$u\cdot\Psi_\lambda(y)=\sum_{(y)}{}^L\tau_\zeta(u,y_{(0)})\Psi_\lambda(y_{(1)})\quad(y\in\tilde U_\zeta^-,u\in U_\zeta^{L,+}),\tag{3.31}$$

$$v\cdot\Psi_\lambda(y)=\Psi_\lambda\big(k_\lambda\big(\mathrm{ad}(v)(k_{-\lambda}yk_{-\lambda})\big)k_\lambda\big)\quad(y\in\tilde U_\zeta^-,v\in U_\zeta^{L,-}).\tag{3.32}$$

For $\lambda\in\Lambda^+$, we set

$$U_\zeta^+(\lambda)=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}}^+(\lambda),$$
$$\tilde U_\zeta^-(\lambda)=\mathbb{C}\otimes_{\mathbb{A}}\tilde U_{\mathbb{A}}^-(\lambda).$$

Then $U_\zeta^+(\lambda)$ and $\tilde U_\zeta^-(\lambda)$ are the $\mathbb{C}$-submodules of U_ζ^+ and $\tilde U_\zeta^-$, respectively, satisfying $\Phi_{-\lambda}(U_\zeta^+(\lambda))=L_{+,\zeta}^*(-\lambda)$ and $\Psi_\lambda(\tilde U_\zeta^-(\lambda))=L_{-,\zeta}^*(\lambda)$. We have linear isomorphisms

$$\Phi_{-\lambda}:U_\zeta^+(\lambda)\to L_{+,\zeta}^*(-\lambda),\qquad\Psi_\lambda:\tilde U_\zeta^-(\lambda)\to L_{-,\zeta}^*(\lambda)\quad(\lambda\in\Lambda^+).\tag{3.33}$$

By (3.21), (3.22), and (3.23), we have

$$U_\zeta^+(\lambda)\subset U_\zeta^+(\lambda+\mu),\qquad\tilde U_\zeta^-(\lambda)\subset\tilde U_\zeta^-(\lambda+\mu)\quad(\lambda,\mu\in\Lambda^+),\tag{3.34}$$

$$U_\zeta^+=\sum_{\lambda\in\Lambda^+}U_\zeta^+(\lambda),\qquad\tilde U_\zeta^-=\sum_{\lambda\in\Lambda^+}\tilde U_\zeta^-(\lambda),\tag{3.35}$$

$$\tilde U_\zeta^-(\lambda)k_{-2\lambda}\subset U_{\zeta,f},\qquad k_{-2\lambda}U_\zeta^+(\lambda)\subset U_{\mathbb{A},f}\quad(\lambda\in\Lambda^+).\tag{3.36}$$

By (3.35) and (3.36), we can easily see the following.

LEMMA 3.7. *For any $u \in U_\zeta$ there exists some $\lambda \in \Lambda^+$ such that $uk_{-2\lambda} \in U_{\zeta,f}$.*

§4. Induction functor

We set

$$C_\zeta^{\leqq 0} = C_\zeta / I, \quad I = \{\varphi \in C_\zeta \mid \langle \varphi, U_\zeta^{L,\leqq 0} \rangle = \{0\}\}.$$

Then $C_\zeta^{\leqq 0}$ is a Hopf algebra, and we have a Hopf pairing

$$\langle\ ,\ \rangle : C_\zeta^{\leqq 0} \times U_\zeta^{L,\leqq 0} \to \mathbb{C}.$$

We have a canonical Hopf algebra homomorphism

$$\mathrm{res} : C_\zeta \to C_\zeta^{\leqq 0}.$$

Following Backelin and Kremnizer [2, Section 3], we define abelian categories $\mathcal{M}_\zeta$ and $\mathcal{M}_\zeta^{\mathrm{eq}}$ as follows.

An object of $\mathcal{M}_\zeta$ is a triplet (M, α, β) with

(1) M a vector space over $\mathbb{C}$,
(2) $\alpha : C_\zeta \otimes M \to M$ a left C_ζ-module structure of M,
(3) $\beta : M \to C_\zeta^{\leqq 0} \otimes M$ a left $C_\zeta^{\leqq 0}$-comodule structure of M

such that β is a morphism of C_ζ-modules. (Or, equivalently, α is a morphism of $C_\zeta^{\leqq 0}$-comodules.) A morphism from (M, α, β) to (M', α', β') is a linear map $\varphi : M \to M'$ which is a morphism of C_ζ-modules as well as that of $C_\zeta^{\leqq 0}$-comodules.

An object of $\mathcal{M}_\zeta^{\mathrm{eq}}$ is a quadruple $(M, \alpha, \beta, \gamma)$ with

(1) M a vector space over $\mathbb{C}$,
(2) $\alpha : C_\zeta \otimes M \to M$ a left C_ζ-module structure of M,
(3) $\beta : M \to C_\zeta^{\leqq 0} \otimes M$ a left $C_\zeta^{\leqq 0}$-comodule structure of M,
(4) $\gamma : M \to M \otimes C_\zeta$ a right C_ζ-comodule structure of M

subject to the conditions that $(M, \alpha, \beta) \in \mathcal{M}_\zeta$, that β and γ commute with each other, and that γ is a homomorphism of left C_ζ-modules. A morphism from $(M, \alpha, \beta, \gamma)$ to $(M', \alpha', \beta', \gamma')$ is a linear map $\varphi : M \to M'$ which is compatible with the left C_ζ-module structure, the left $C_\zeta^{\leqq 0}$-comodule structure, and the right C_ζ-comodule structure.

For a coalgebra $\mathcal{C}$ we denote by $\mathrm{Comod}(\mathcal{C})$ (resp., $\mathrm{Comod}^r(\mathcal{C})$) the category of left $\mathcal{C}$-comodules (resp., right $\mathcal{C}$-comodules). We define functors

$$\Xi : \mathcal{M}_\zeta^{\mathrm{eq}} \to \mathrm{Comod}(C_\zeta^{\leqq 0}),$$
$$\Upsilon : \mathrm{Comod}(C_\zeta^{\leqq 0}) \to \mathcal{M}_\zeta^{\mathrm{eq}}$$

by

$$\Xi(M) = \{M \in M \mid \gamma(m) = m \otimes 1\},$$
$$\Upsilon(L) = C_\zeta \otimes L.$$

By Backelin and Kremnizer [2, Section 3.5], we have the following.

PROPOSITION 4.1. *The functor* $\Xi : \mathcal{M}_\zeta^{\mathrm{eq}} \to \mathrm{Comod}(C_\zeta^{\leqq 0})$ *gives an equivalence of categories, and its quasi-inverse is given by* Υ.

REMARK 4.2. For $M \in \mathcal{M}_\zeta^{\mathrm{eq}}$ we have an isomorphism

$$\Xi(M) \cong \mathbb{C} \otimes_{C_\zeta} M$$

of vector spaces by Proposition 4.1. Here $C_\zeta \to \mathbb{C}$ is given by ε.

For $\lambda \in \Lambda$ we define $\chi_\lambda^{\leqq 0} \in C_\zeta^{\leqq 0} \subset \mathrm{Hom}_{\mathbb{C}}(U_\zeta^{L,\leqq 0}, \mathbb{C})$ by

$$\chi_\lambda^{\leqq 0}(hu) = \chi_\lambda(h)\varepsilon(u) \quad (h \in U_\zeta^{L,0}, u \in U_\zeta^{L,-}).$$

We define left exact functors

$$\omega_{\mathcal{M}*} : \mathcal{M}_\zeta \to \mathrm{Mod}_\Lambda(A_\zeta), \tag{4.1}$$
$$\Gamma_{\mathcal{M}} : \mathcal{M}_\zeta \to \mathrm{Mod}(\mathbb{C}) \tag{4.2}$$

by

$$\omega_{\mathcal{M}*}(M) = \bigoplus_{\lambda \in \Lambda} (\omega_{\mathcal{M}*}(M))(\lambda) \subset M,$$
$$(\omega_{\mathcal{M}*}(M))(\lambda) = \{m \in M \mid \beta(m) = \chi_\lambda^{\leqq 0} \otimes m\},$$
$$\Gamma_{\mathcal{M}}(M) = (\omega_{\mathcal{M}*}(M))(0).$$

We denote by $\mathrm{Mod}_\Lambda^{\mathrm{eq}}(A_\zeta)$ the category consisting of $N \in \mathrm{Mod}_\Lambda(A_\zeta)$ equipped with a right C_ζ-comodule structure $\gamma : N \to N \otimes C_\zeta$ such that

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$\gamma(N(\lambda)) \subset N(\lambda) \otimes C_\zeta$ for any $\lambda \in \Lambda$ and $\gamma(\varphi n) = \Delta(\varphi)\gamma(n)$ for any $\varphi \in A_\zeta$ and $n \in N$. (Note that $\Delta(A_\zeta(\lambda)) \subset A_\zeta(\lambda) \otimes C_\zeta$.) By definition, (4.1) and (4.2) induce left exact functors

$$\omega^{\mathrm{eq}}_{\mathcal{M}*} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta), \tag{4.3}$$

$$\Gamma^{\mathrm{eq}}_{\mathcal{M}} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Comod}^r(C_\zeta). \tag{4.4}$$

We also define a left exact functor

$$\mathrm{Ind} : \mathrm{Comod}(C^{\leqq 0}_\zeta) \to \mathrm{Comod}^r(C_\zeta) \tag{4.5}$$

by $\mathrm{Ind} = \Gamma^{\mathrm{eq}}_{\mathcal{M}} \circ \Upsilon$.

The abelian categories $\mathcal{M}_\zeta$, $\mathcal{M}^{\mathrm{eq}}_\zeta$, $\mathrm{Comod}^r(C_\zeta)$ have enough injectives, and the forgetful functor $\mathcal{M}^{\mathrm{eq}}_\zeta \to \mathcal{M}_\zeta$ sends injective objects to $\Gamma_{\mathcal{M}}$-acyclic objects (see [2, Section 3.4]). Hence, we have the following.

LEMMA 4.3. *We have*

$$\mathrm{For} \circ R^i\Gamma^{\mathrm{eq}}_{\mathcal{M}} = R^i\Gamma_{\mathcal{M}} \circ \mathrm{For} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Mod}(\mathbb{C}),$$
$$R^i\mathrm{Ind} \circ \Xi = R^i\Gamma^{\mathrm{eq}}_{\mathcal{M}} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Comod}^r(C_\zeta)$$

for any i, where $\mathrm{For} : \mathrm{Comod}^r(C_\zeta) \to \mathrm{Mod}(\mathbb{C})$ *and* $\mathrm{For} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathcal{M}_\zeta$ *are forgetful functors.*

We define an exact functor

$$\mathrm{res} : \mathrm{Comod}^r(C_\zeta) \to \mathrm{Comod}(C^{\leqq 0}_\zeta) \tag{4.6}$$

as follows. For $V \in \mathrm{Comod}^r(C_\zeta)$ with right C_ζ-comodule structure $\beta : V \to V \otimes C_\zeta$, we have $\mathrm{res}(V) = V$ as a $\mathbb{C}$-module, and the left $C^{\leqq 0}_\zeta$-comodule structure $\mathrm{res}(V) \to C^{\leqq 0}_\zeta \otimes \mathrm{res}(V)$ of $\mathrm{res}(V)$ is given by

$$\beta(v) = \sum_k v_k \otimes \varphi_k \quad \Longrightarrow \quad \gamma(v) = \sum_k \mathrm{res}(S^{-1}\varphi_k) \otimes v_k.$$

The following fact is standard.

LEMMA 4.4. *For $V \in \mathrm{Comod}^r(C_\zeta)$, $M \in \mathrm{Comod}(C^{\leqq 0}_\zeta)$, we have an isomorphism*

$$F : \mathrm{Ind}(M) \otimes V \to \mathrm{Ind}\big(\mathrm{res}(V) \otimes M\big)$$

of right C_ζ-comodules given by

$$F\Big(\Big(\sum_i \varphi_i \otimes m_i\Big) \otimes v\Big) = \sum_{i,(v)} \varphi_i v_{(1)} \otimes v_{(0)} \otimes m_i,$$

where we write the right C_ζ-comodule structure of V by

$$V \ni v \mapsto \sum_{(v)} v_{(0)} \otimes v_{(1)} \in V \otimes C_\zeta.$$

For $\lambda \in \Lambda$ we denote by $\mathbb{C}_\lambda^{\leqq 0} = \mathbb{C}1_\lambda^{\leqq 0}$ the object of $\mathrm{Comod}(C_\zeta^{\leqq 0})$ corresponding to the 1-dimensional right $U_\zeta^{L,\leqq 0}$-module given by $1_\lambda^{\leqq 0} u = \chi_\lambda^{\leqq 0}(u) 1_\lambda^{\leqq 0}$ for $u \in U_\zeta^{L,\leqq 0}$. By definition, we have an isomorphism

$$\mathrm{Ind}(\mathbb{C}_{-\lambda}^{\leqq 0}) \cong A_\zeta(\lambda) \quad (\lambda \in \Lambda^+)$$

of right C_ζ-comodules.

Let $N \in \mathrm{Mod}_\Lambda(A_\zeta)$. Then $C_\zeta \otimes_{A_\zeta} N$ turns out to be an object of $\mathcal{M}_\zeta$ by

$$\alpha\big(f \otimes (f' \otimes n)\big) = ff' \otimes n \quad (f, f' \in C_\zeta, n \in N),$$
$$\beta(f \otimes n) = \sum_{(f)} \mathrm{res}(f_{(0)}) \chi_\lambda \otimes (f_{(1)} \otimes n) \quad \big(f \in C_\zeta, n \in N(\lambda)\big).$$

Hence, we have a functor $\mathrm{Mod}_\Lambda(A_\zeta) \to \mathcal{M}_\zeta$ sending N to $C_\zeta \otimes_{A_\zeta} N$.

LEMMA 4.5. *The functor* $\mathrm{Mod}_\Lambda(A_\zeta) \to \mathcal{M}_\zeta$ *as above induces a functor*

$$\Phi : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}_\zeta.$$

Proof. It is sufficient to show that $C_\zeta \otimes_{A_\zeta} A_\zeta / A_\zeta(\lambda + \Lambda^+) = \{0\}$ for any $\lambda \in \Lambda$. Hence, we have only to show that $C_\zeta A_\zeta(\lambda) = C_\zeta$ for any $\lambda \in \Lambda^+$. Take $\varphi \in A_\zeta(\lambda)$ such that $\varepsilon(\varphi) = 1$. We have $\Delta(A_\zeta(\lambda)) \subset A_\zeta(\lambda) \otimes C_\zeta$, and hence we can write $\Delta(\varphi) = \sum_i \varphi_i \otimes \varphi'_i$ with $\varphi_i \in A_\zeta(\lambda)$, $\varphi'_i \in C_\zeta$. Then we have $C_\zeta A_\zeta(\lambda) \ni \sum_i (S^{-1}\varphi'_i)\varphi_i = 1$. □

We set

$$\Psi = \omega^* \circ \omega_{\mathcal{M}*} : \mathcal{M}_\zeta \to \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}).$$

Backelin and Kremnizer [2, Section 3.3] obtained the following result using a result of Artin and Zhang [1, Theorem 4.5].

Proposition 4.6. *The functor* $\Phi : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}_\zeta$ *gives an equivalence of categories, and its quasi-inverse is given by* Ψ. *Moreover, we have an identification*

$$\omega_{\mathcal{M}*} \circ \Phi = \omega_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}_\Lambda(A_\zeta)$$

of functors.

Hence we have the following.

Lemma 4.7. *We have*

$$R^i\Gamma = R^i\Gamma_{\mathcal{M}} \circ \Phi : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathbb{C})$$

for any i.

We set

$$\mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) = \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta)/\mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta) \cap \mathrm{Tor}_{\Lambda^+}(A_\zeta).$$

Let $N \in \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta)$. We denote the right C_ζ-comodule structure of N by $\gamma' : N \to N \otimes C_\zeta$. Then we have a right C_ζ-comodule structure $\gamma : C_\zeta \otimes_{A_\zeta} N \to (C_\zeta \otimes_{A_\zeta} N) \otimes C_\zeta$ of $C_\zeta \otimes_{A_\zeta} N$ given by

$$\gamma'(n) = \sum_k n_k \otimes \varphi_k \quad \Longrightarrow \quad \gamma(f \otimes n) = \sum_{k,(f)} (f_{(0)} \otimes n_k) \otimes f_{(1)}\varphi_k.$$

This gives a functor $\mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta) \to \mathcal{M}^{\mathrm{eq}}_\zeta$. Hence, by Lemma 4.5 we have a functor

$$\Phi^{\mathrm{eq}} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}^{\mathrm{eq}}_\zeta \tag{4.7}$$

induced by Φ. Let $M \in \mathcal{M}^{\mathrm{eq}}_\zeta$. The right C_ζ-comodule structure of M restricts to that of $\omega_{\mathcal{M}*}M$ so that $\omega_{\mathcal{M}*}M \in \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta)$. Hence, we have a functor

$$\Psi^{\mathrm{eq}} : \mathcal{M}^{\mathrm{eq}}_\zeta \to \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \tag{4.8}$$

induced by Ψ. By Proposition 4.6, we have the following.

Proposition 4.8. *The functor* $\Phi^{\mathrm{eq}} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}^{\mathrm{eq}}_\zeta$ *gives an equivalence of categories, and its quasi-inverse is given by* Ψ^{eq}.

By Proposition 4.8 we see that (4.1) and (4.2) induce

$$\omega^{\mathrm{eq}}_* = \omega^{\mathrm{eq}}_{\mathcal{M}*} \circ \Phi^{\mathrm{eq}} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}^{\mathrm{eq}}_\Lambda(A_\zeta), \tag{4.9}$$

$$\Gamma^{\mathrm{eq}} = \Gamma^{\mathrm{eq}}_{\mathcal{M}} \circ \Phi^{\mathrm{eq}} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Comod}^r(C_\zeta). \tag{4.10}$$

By Lemma 4.3, we have the following.

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LEMMA 4.9. *We have*

$$\mathrm{For} \circ R^i\Gamma^{\mathrm{eq}} = R^i\Gamma \circ \mathrm{For} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathbb{C})$$

for any i, *where* $\mathrm{For} : \mathrm{Comod}^r(C_\zeta) \to \mathrm{Mod}(\mathbb{C})$ *and* $\mathrm{For} : \mathrm{Mod}^{\mathrm{eq}}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ *are forgetful functors.*

§5. Reformulation of Conjecture 2.14

5.1. Adjoint action of U_ζ^L on D'_ζ

Define a left $U_{\mathbb{F}}$-module structure of $E_{\mathbb{F}}$ by

$$\mathrm{ad}(u)(P) = \sum_{(u)} u_{(0)} P(Su_{(1)}) \quad (u \in U_{\mathbb{F}}, P \in E_{\mathbb{F}}).$$

Then we have

$$\begin{aligned}
\mathrm{ad}(u)(P_1P_2) &= \sum_{(u)} \mathrm{ad}(u_{(0)})(P_1)\mathrm{ad}(u_{(1)})(P_2) \quad (P_1, P_2 \in E_{\mathbb{F}}),\\
\mathrm{ad}(u)(\varphi) &= u \cdot \varphi \quad (\varphi \in A_{\mathbb{F}} \subset E_{\mathbb{F}}),\\
\mathrm{ad}(u)(v) &= \sum_{(u)} u_{(0)} v(Su_{(1)}) \quad (v \in U_{\mathbb{F}} \subset E_{\mathbb{F}}),\\
\mathrm{ad}(u)\big(e(\lambda)\big) &= \varepsilon(u)e(\lambda) \quad \big(\lambda \in \Lambda, e(\lambda) \in \mathbb{F}[\Lambda] \subset E_{\mathbb{F}}\big)
\end{aligned}$$

for $u \in U_{\mathbb{F}}$. We see from [20, Lemma 4.2] that this induces a left $U_{\mathbb{F}}$-module structure of $D'_{\mathbb{F}}$. Moreover, the $U_{\mathbb{F}}$-module structures of $E_{\mathbb{F}}$ and $D'_{\mathbb{F}}$ induce $U_{\mathbb{A}}^L$-module structures of $E_{\mathbb{A}}$, $D'_{\mathbb{A}}$, $E_{\mathbb{A},\diamondsuit}$, $D'_{\mathbb{A},\diamondsuit}$, $E_{\mathbb{A},f}$, and $D'_{\mathbb{A},f}$ by Lemmas 1.2 and 2.12. Hence, by specialization we obtain U_ζ^L-module structures of E_ζ, D'_ζ, $E_{\zeta,\diamondsuit}$, $D'_{\zeta,\diamondsuit}$, $E_{\zeta,f}$, and $D'_{\zeta,f}$ also denoted by ad.

5.2.

We will regard $E_{\zeta,f}, D'_{\zeta,f} \in \mathrm{Mod}_\Lambda(A_\zeta)$ as objects of $\mathrm{Mod}_\Lambda^{\mathrm{eq}}(A_\zeta)$ by the right C_ζ-comodule structures induced from the left U_ζ^L-module structures

$$(u, P) \mapsto \mathrm{ad}(u)(P) \quad (u \in U_\zeta^L, P \in E_{\zeta,f} \text{ or } D'_{\zeta,f}).$$

Then for

$$(\Xi \circ \Phi^{\mathrm{eq}})(\omega^* D'_{\zeta,f}) \in \mathrm{Comod}(C_\zeta^{\leqq 0})$$

we have

$$R^i\Gamma(\omega^* D'_{\zeta,f}) = R^i\mathrm{Ind}\big((\Xi \circ \Phi^{\mathrm{eq}})(\omega^* D'_{\zeta,f})\big)$$

by Lemmas 4.3 and 4.9 and by (4.10).

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Define a right $(U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda])$-module V by

$$V = \big(U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]\big)/\mathcal{I},$$

where

$$\mathcal{I} = \big(\tilde{U}_\zeta^- \cap \mathrm{Ker}(\varepsilon)\big)U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda] + \sum_{\lambda\in\Lambda}\big(k_{2\lambda} - e(2\lambda)\big)U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda].$$

By the triangular decomposition $\tilde{U}_\zeta^- \otimes U_{\zeta,\diamondsuit}^0 \otimes U_\zeta^+ \cong U_{\zeta,\diamondsuit}$ we have

$$V \cong U_\zeta^+ \otimes \mathbb{C}[\Lambda]$$

as a vector space. Define a right action of $U_\zeta^{L,\leqq 0}$ on $U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]$ by

$$\big(u \otimes e(\lambda)\big) \star v = \mathrm{ad}(Sv)(u) \otimes e(\lambda) \quad (u \in U_{\zeta,\diamondsuit}, \lambda \in \Lambda, v \in U_\zeta^{L\leqq 0}).$$

It induces a right action of $U_\zeta^{L,\leqq 0}$ on V. Moreover, we see easily that this right $U_\zeta^{L,\leqq 0}$-module structure gives a left $C_\zeta^{\leqq 0}$-comodule structure of V.

Proposition 5.1. *We have*

$$(\Xi \circ \Phi^{\mathrm{eq}})(\omega^* D'_{\zeta,f}) \cong V$$

as a left $C_\zeta^{\leqq 0}$*-comodule.*

The proof is given in Section 5.3.

It follows from Proposition 5.1 that Conjecture 2.14 is equivalent to the following conjecture.

Conjecture 5.2. *Assume that* $\ell > h_G$. *We have*

$$\mathrm{Ind}(V) \cong U_{\zeta,f} \otimes_{Z_{\mathrm{Har}}(U_\zeta)} \mathbb{C}[\Lambda],$$

and

$$R^i\mathrm{Ind}(V) = 0$$

for $i \neq 0$.

Remark 5.3. We can show that

$$U_{\zeta,f} \cong (C_\zeta)_{\mathrm{ad}}, \qquad V \cong {}_{\mathrm{ad}}(C_\zeta^{\leqq 0}) \otimes_{\mathbb{C}[2\Lambda]} \mathbb{C}[\Lambda],$$

where $(C_\zeta)_{\mathrm{ad}}$ (resp., ${}_{\mathrm{ad}}(C_\zeta^{\leqq 0})$) is given by the right (resp., left) adjoint coaction of C_ζ (resp., $C_\zeta^{\leqq 0}$) on itself. Hence, Conjecture 5.2 is equivalent to

$$R\,\mathrm{Ind}\big({}_{\mathrm{ad}}(C_\zeta^{\leqq 0})\big) \cong (C_\zeta)_{\mathrm{ad}} \otimes_{\mathbb{C}[2\Lambda]^W} \mathbb{C}[2\Lambda].$$

The corresponding statement for $q = 1$ is

$$R\,\mathrm{Ind}\big({}_{\mathrm{ad}}\mathbb{C}[B^-]\big) \cong \mathbb{C}[G]_{\mathrm{ad}} \otimes_{\mathbb{C}[H/W]} \mathbb{C}[H].$$

We can prove this by a geometric method.

REMARK 5.4.† A proof of Conjecture 5.2, when ℓ is a prime greater than the Coxeter number, is given by Backelin and Kremnizer in [3, Proposition 3.25]; however, in a more recent article they admit that there are gaps in [3] (see [4, Version 3, Section 1.1.2]) and propose different proofs. But it is likely that problems still remain in the new proofs given in [4], as explained below.

The proof in [4, Versions 1 and 2] is wrong because all positive roots are assumed there to be dominant (see [4, Version 2, proof of Theorem 2.1]).

Another proof given in [4, Version 3] also has problems. In Step (b) of [4, Version 3, proof of Theorem 2.2.1], the authors compare certain weight multiplicities $a_{q,\mu}$ and $b_{q,\mu}$. But since those multiplicities are infinite, the argument there should be modified using multiplicities as U_q-modules. Let us assume for simplicity that q is generic and try to modify the original argument by replacing $a_{q,\mu}, b_{q,\mu}, b'_{q,\mu}$ with their counterparts as multiplicities of U_q-modules. This even fails since $a_{1,\mu}$ (resp., $b'_{1,\mu}$) is the dimension of the 0-weight space of the irreducible module (resp., Verma module) with highest-weight μ. We also point out that the reason that U_q^λ is an integral domain is not given in Step (a).

Note that the arguments in [4, Version 3, proof of Theorem 2.2.1] are partially similar to those in the earlier manuscripts (see [2, Proposition 4.8], [3, Proposition 3.25]). The main difference is that [4, Version 3] relies on a B_q-stable filtration with 1-dimensional subquotients instead of the Joseph–Letzter filtration used in [2] and [3]. For us, the original argument in [2] and [3] for generic q using the Joseph–Letzter filtration is not comprehensible either. In the notation of [2, proof of Proposition 4.8], the validity of the formula $m_j(1) = \tilde{n}_j(1)$ is not clear to us since the Joseph–Letzter filtration does not induce at $q = 1$ the ordinary filtration for enveloping algebras and differential operators in general.

†This remark is added at the editor's request.

5.3.

We will give a proof of Proposition 5.1 in the rest of this article. By Remark 4.2, we have

$$(\Xi \circ \Phi^{\mathrm{eq}})(\omega^* D'_{\zeta,f}) \cong \mathbb{C} \otimes_{A_\zeta} D'_{\zeta,f}$$

as a vector space, where $A_\zeta \to \mathbb{C}$ is given by ε. Note that

$$\mathbb{C} \otimes_{A_\zeta} E_{\zeta,\diamondsuit} \cong U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda].$$

We first show the following.

Lemma 5.5. *We have*

$$\mathbb{C} \otimes_{A_\zeta} D'_{\zeta,\diamondsuit} \cong V.$$

Proof. By (2.10) we obtain

$$\mathbb{C} \otimes_{A_\zeta} D'_{\zeta,\diamondsuit} \cong \big(U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]\big) \Big/ \sum_{\varphi \in A_\zeta} \big(1 \otimes \Omega'(\varphi)\big)\big(U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]\big),$$

where $1 \otimes \Omega'(\varphi)$ is the image of $\Omega'(\varphi)$ in $\mathbb{C} \otimes_{A_\zeta} E_{\zeta,\diamondsuit} = U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]$. Note that $\varepsilon(A_\zeta(\lambda)_\xi) = \{0\}$ for $\lambda \in \Lambda^+$, $\xi \in \Lambda$ with $\lambda \neq \xi$, and that $\varepsilon(A_\zeta(\lambda)_\lambda) = \mathbb{C}$ for $\lambda \in \Lambda^+$. Hence, for $\varphi \in A_\zeta(\lambda)_\xi$ with $\lambda \in \Lambda^+$, $\xi \in \Lambda$ we have

$$1 \otimes \Omega'_1(\varphi) = \begin{cases} 0 & (\lambda \neq \xi), \\ \varepsilon(\varphi) & (\lambda = \xi). \end{cases}$$

Let us also compute $1 \otimes \Omega'_2(\varphi)$. Let

$$\tilde{\Psi}_\lambda : \tilde{U}^-_\zeta(\lambda) \to A_\zeta(\lambda)$$

be the composite of the linear isomorphism $\Psi_\lambda : \tilde{U}^-_\zeta(\lambda) \to L^*_{-,\zeta}(\lambda)$ (see (3.33)) and an isomorphism $f : L^*_{-,\zeta}(\lambda) \to A_\zeta(\lambda)$ of U^L_ζ-modules. We have $\tilde{\Psi}_\lambda(\tilde{U}^-_\zeta(\lambda)_{-(\lambda-\xi)}) = A_\zeta(\lambda)_\xi$ for any $\xi \in \Lambda$. Hence, we may assume that $\varepsilon = \varepsilon \circ \tilde{\Psi}_\lambda$ on $\tilde{U}^-_\zeta(\lambda)$. Let $\varphi \in A_\zeta(\lambda)_\xi$, and take $v \in \tilde{U}^-_\zeta(\lambda)_{-(\lambda-\xi)}$ satisfying $\tilde{\Psi}_\lambda(v) = \varphi$. Then we have

$$\begin{aligned}\sum_p (Sx^L_p) \cdot \varphi \otimes y_p k_{\beta_p} &= \sum_p f\big((Sx^L_p) \cdot \Psi_\lambda(v)\big) \otimes y_p k_{\beta_p} \\ &= \sum_p \zeta^{-(\beta_p,\xi)} f\big((Sx^L_p) k_{\beta_p} \cdot \Psi_\lambda(v)\big) \otimes y_p k_{\beta_p}\end{aligned}$$

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$$= \sum_{p,(v)} \zeta^{-(\beta_p,\xi)} f\big({}^L\tau_\zeta\big((Sx_p^L)k_{\beta_p}, v_{(0)}\big)\Psi_\lambda(v_{(1)})\big) \otimes y_p k_{\beta_p}$$

$$= \sum_{p,(v)} \zeta^{-(\beta_p,\xi)\,L}\tau_\zeta\big((Sx_p^L)k_{\beta_p}, v_{(0)}\big)\tilde{\Psi}_\lambda(v_{(1)}) \otimes y_p k_{\beta_p},$$

and hence

$$\begin{aligned}
1 \otimes \Omega_2'(\varphi) &= \sum_p \varepsilon\big((Sx_p^L)\cdot\varphi\big) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_{p,(v)} \zeta^{-(\beta_p,\xi)\,L}\tau_\zeta\big((Sx_p^L)k_{\beta_p}, v_{(0)}\big)\varepsilon(v_{(1)}) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_p \zeta^{-(\beta_p,\xi)\,L}\tau_\zeta\big((Sx_p^L)k_{\beta_p}, v\big) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_p \zeta^{-(\beta_p,\xi)\,L}\tau_\zeta(k_{-\beta_p}x_p^L, S^{-1}v) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_p \zeta^{-(\beta_p,\xi)-(\beta_p,\beta_p)\,L}\tau_\zeta(x_p^L, S^{-1}v) y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \\
&= \sum_p \zeta^{-(\lambda-\xi,\lambda)\,L}\tau_\zeta(x_p^L, S^{-1}v) y_p k_{\lambda-\xi} k_{2\xi} e(-2\lambda) \\
&= \zeta^{-(\lambda-\xi,\lambda)}(S^{-1}v) k_{\lambda-\xi} k_{2\xi} e(-2\lambda).
\end{aligned}$$

(Note that $(S^{-1}v)k_{\lambda-\xi} \in \tilde{U}_\zeta^-(\lambda)_{-(\lambda-\xi)}$.) It follows that

$$1 \otimes \Omega'(\varphi) = \begin{cases} -\zeta^{-(\lambda-\xi,\lambda)}(S^{-1}v)k_{\lambda-\xi}k_{2\xi}e(-2\lambda) & (\lambda \neq \xi), \\ \varepsilon(\varphi)(1 - k_{2\lambda}e(-2\lambda)) & (\lambda = \xi). \end{cases}$$

Hence, we have

$$\begin{aligned}
&\sum_{\substack{\lambda\in\Lambda^+,\\ \gamma\in Q^+}} \sum_{\varphi\in A_\zeta(\lambda)_{\lambda-\gamma}} \big(1\otimes\Omega'(\varphi)\big)\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big) \\
&\quad = \sum_{\substack{\lambda\in\Lambda^+,\\ \gamma\in Q^+\setminus\{0\}}} \tilde{U}_\zeta^-(\lambda)_{-\gamma}\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big) + \sum_{\lambda\in\Lambda^+}\big(1-k_{2\lambda}e(-2\lambda)\big)\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big) \\
&\quad = \big(\tilde{U}_\zeta^-\cap\mathrm{Ker}(\varepsilon)\big)\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big) + \sum_{\lambda\in\Lambda}\big(k_{2\lambda}-e(2\lambda)\big)\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big)
\end{aligned}$$

by (3.35). $\square$

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LEMMA 5.6. *We have*

$$\mathbb{C}\otimes_{A_\zeta} D'_{\zeta,f}\cong V.$$

Proof. We need to show that the canonical homomorphism $\mathbb{C}\otimes_{A_\zeta} D'_{\zeta,f}\to \mathbb{C}\otimes_{A_\zeta} D'_{\zeta,\diamondsuit}$ is bijective. The surjectivity is a consequence of (3.35) and (3.36). Let us give a proof of the injectivity. Set

$$\mathcal{K}=A_\zeta U_{\zeta,f}\mathbb{C}[\Lambda]\cap\sum_{\varphi\in A_\zeta}A_\zeta\Omega'(\varphi)U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda]\subset A_\zeta\otimes U_{\zeta,f}\otimes\mathbb{C}[\Lambda].$$

Then it is sufficient to show that the natural map

$$\mathbb{C}\otimes_{A_\zeta}\big((A_\zeta\otimes U_{\zeta,f}\otimes\mathbb{C}[\Lambda])/\mathcal{K}\big)\to\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big)/\mathcal{I}$$

is injective. Let $F: A_\zeta\otimes U_{\zeta,f}\otimes\mathbb{C}[\Lambda]\to U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]$ be the natural map. Then it is sufficient to show that

$$\mathcal{I}\cap\big(U_{\zeta,f}\otimes\mathbb{C}[\Lambda]\big)\subset F(\mathcal{K}). \tag{5.1}$$

Indeed, assume that (5.1) holds. Denote by

$$\begin{aligned}p: A_\zeta\otimes U_{\zeta,f}\otimes\mathbb{C}[\Lambda]&\to\mathbb{C}\otimes_{A_\zeta}\big((A_\zeta\otimes U_{\zeta,f}\otimes\mathbb{C}[\Lambda])/\mathcal{K}\big),\\ \pi: U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]&\to\big(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]\big)/\mathcal{I}\end{aligned}$$

the natural maps. We have to show that $\mathrm{Ker}(\pi\circ F)\subset\mathrm{Ker}(p)$. Take $x\in\mathrm{Ker}(\pi\circ F)$. Then $F(x)\in\mathcal{I}\cap(U_{\zeta,f}\otimes\mathbb{C}[\Lambda])$. Hence, by (5.1) there exists some $v\in\mathcal{K}$ such that $F(x)=F(v)$. Then $p(x)=p(x-v)+p(v)=p(x-v)$. Hence, we may assume that $F(x)=0$ from the beginning. Note that p factors through

$$p': A_\zeta\otimes U_{\zeta,f}\otimes\mathbb{C}[\Lambda]\to\mathbb{C}\otimes_{A_\zeta}\big(A_\zeta\otimes U_{\zeta,f}\otimes\mathbb{C}[\Lambda]\big)\big(=U_{\zeta,f}\otimes\mathbb{C}[\Lambda]\big).$$

By $F(x)=0$ we have $p'(x)=0$, and hence $p(x)=0$, as desired.

It remains to show (5.1). Let $\lambda\in\Lambda^+$, and let $\varphi\in A_\zeta(\lambda)_\lambda$. Then we have

$$\Omega'_1(\varphi)=\sum_p(y_p^L\cdot\varphi)x_p\in A_\zeta U_\zeta^+,\qquad \Omega'_2(\varphi)=\varphi k_{2\lambda}e(-2\lambda).$$

Let us show that

$$\Omega'_1(\varphi)=\sum_p(y_p^L\cdot\varphi)x_p\in A_\zeta U_\zeta^+(\lambda). \tag{5.2}$$

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This is equivalent to

$$\sum_p (y_p^L \cdot \varphi) \otimes \Phi_{-\lambda}(x_p) \in A_\zeta \otimes L^*_{+,\zeta}(-\lambda).$$

This follows from

$$\begin{aligned}\sum_p \langle \Phi_{-\lambda}(x_p), \overline{u f_i^{((\lambda,\alpha_i^\vee)+1)}} \rangle y_p^L \cdot \varphi &= \sum_p \tau_\zeta^L(x_p, u f_i^{((\lambda,\alpha_i^\vee)+1)}) y_p^L \cdot \varphi \\ &= (u f_i^{((\lambda,\alpha_i^\vee)+1)}) \cdot \varphi = 0\end{aligned}$$

for $u \in U_\zeta^{L,-}, i \in I$. Thus, (5.2) is verified. Hence, we have

$$\Omega'(\varphi) k_{-2\lambda} \in \mathcal{K}.$$

It follows that

$$F(\mathcal{K}) \supset \big(k_{-2\lambda} - e(-2\lambda)\big) U_{\zeta,f} \mathbb{C}[\Lambda] \quad (\lambda \in \Lambda^+). \tag{5.3}$$

Now let $u \in \mathcal{I} \cap (U_{\zeta,f} \otimes \mathbb{C}[\Lambda])$. If we can show that $k_{-2\mu} u \in F(\mathcal{K})$ for some $\mu \in \Lambda^+$, then we obtain

$$u = e(2\mu)\big(e(-2\mu) - k_{-2\mu}\big) u + e(2\mu) k_{-2\mu} u \in F(\mathcal{K})$$

by (5.3). Hence, it is sufficient to show that for any $u \in \mathcal{I}$ there exists some $\mu \in \Lambda^+$ such that $k_{-2\mu} u \in F(\mathcal{K})$. We may assume that there exists $\nu \in Q$ such that $k_{-2\mu} u = \zeta^{(\mu,\nu)} u k_{-2\mu}$ for any $\mu \in \Lambda$. Therefore, we have only to show that for any $u \in \mathcal{I}$ there exists some $\mu \in \Lambda^+$ such that $u k_{-2\mu} \in F(\mathcal{K})$. By Lemma 5.5 we can take $\varphi_i \in A_\zeta$, $x_i \in U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]$ $(i = 1, \ldots, N)$ such that

$$u = 1 \otimes \sum_{i=1}^N \Omega'(\varphi_i) x_i.$$

By Lemma 3.7 we can take $\mu \in \Lambda^+$ such that $\Omega'(\varphi_i) x_i k_{-2\mu} \in A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ for any i. Then we have

$$u k_{-2\mu} = \sum_{i=1}^N F\big(\Omega'(\varphi_i) x_i k_{-2\mu}\big) \in F(\mathcal{K}).$$

□

By Lemma 5.6 we obtain an isomorphism

$$(\Xi \circ \Phi^{\mathrm{eq}})(\omega^* D'_{\zeta,f}) \cong V$$

of vector spaces. We need to show that it is in fact an isomorphism of left $C_\zeta^{\leqq 0}$-comodules. This is a consequence of the corresponding fact for $E_{\zeta,f}$. Note that we have

$$\mathbb{C} \otimes_{A_\zeta} E_{\zeta,f} \cong U_{\zeta,f} \otimes \mathbb{C}[\Lambda],$$

and hence we have an isomorphism

$$(\Xi \circ \Phi^{\mathrm{eq}})(\omega^* E_{\zeta,f}) \cong U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \tag{5.4}$$

of vector spaces. Hence, we have only to show the following.

Lemma 5.7. *Under identification (5.4), the left $C_\zeta^{\leqq 0}$-comodule structure of $U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ is associated to the right $U_\zeta^{L,\leqq 0}$-module structure given by*

$$\bigl(u \otimes e(\lambda)\bigr) \cdot v = \mathrm{ad}(Sv)(u) \otimes e(\lambda) \quad (u \in U_{\zeta,f}, \lambda \in \Lambda, v \in U_\zeta^{L,\leqq 0}).$$

Proof. Note that the left $C_\zeta^{\leqq 0}$-comodule structure of $U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ is given by

$$U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \cong \Xi\bigl(C_\zeta \otimes \bigl(U_{\zeta,f} \otimes \mathbb{C}[\Lambda]\bigr)\bigr),$$

where $C_\zeta \otimes (U_{\zeta,f} \otimes \mathbb{C}[\Lambda])$ is regarded as a left $C_\zeta^{\leqq 0}$-comodule by the tensor product of C_ζ (with left $C_\zeta^{\leqq 0}$-comodule structure $(\mathrm{res} \otimes 1) \circ \Delta : C_\zeta \to C_\zeta^{\leqq 0} \otimes C_\zeta$) and $U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ with trivial left $C_\zeta^{\leqq 0}$-comodule structure. Hence, it is sufficient to show that for a right C_ζ-comodule M the right $U_\zeta^{L,\leqq 0}$-module structure of

$$M \cong \Xi(C_\zeta \otimes M) \in \mathrm{Comod}(C_\zeta^{\leqq 0})$$

is given by

$$m \cdot v = (Sv) \cdot m \quad (m \in M, v \in U_\zeta^{L,\leqq 0}).$$

Denote by M^{triv} the trivial right C_ζ-comodule which coincides with M as a vector space. We denote by $M \ni m \leftrightarrow \overline{m} \in M^{\mathrm{triv}}$ the canonical linear isomorphism. We have $C_\zeta \otimes M^{\mathrm{triv}} \in \mathrm{Comod}^r(C_\zeta)$ as the tensor product of $C_\zeta \in \mathrm{Comod}^r(C_\zeta)$ and $M^{\mathrm{triv}} \in \mathrm{Comod}^r(C_\zeta)$. We can also define a left $C_\zeta^{\leqq 0}$-comodule structure of $C_\zeta \otimes M^{\mathrm{triv}}$ as the tensor product of the left

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$C_\zeta^{\leqq 0}$-comodules C_ζ and M^{triv}, where the left $C_\zeta^{\leqq 0}$-comodule structure of M^{triv} is given by the right $U_\zeta^{L,\leqq 0}$-module structure

$$\overline{m} \cdot v = \overline{(Sv) \cdot m} \quad (m \in M, v \in U_\zeta^{L,\leqq 0}).$$

Then we have a linear isomorphism

$$C_\zeta \otimes M \ni \varphi \otimes m \mapsto \sum_{(m)} \varphi m_{(1)} \otimes \overline{m_{(0)}} \in C_\zeta \otimes M^{\mathrm{triv}}$$

preserving the right C_ζ-comodule structures and the left $C_\zeta^{\leqq 0}$-comodule structures. It follows that

$$\Xi(C_\zeta \otimes M) \cong \Xi(C_\zeta \otimes M^{\mathrm{triv}}) = M^{\mathrm{triv}} \in \mathrm{Comod}(C_\zeta^{\leqq 0}). \qquad \square$$

The proof of Proposition 5.1 is complete.

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Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka
558-8585 Japan

tanisaki@sci.osaka-cu.ac.jp