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A CRITERION FOR VERSALITY OF DEFORMATIONS OF TUBULAR NEIGHBORHOODS OF STRONGLY PSEUDO CONVEX BOUNDARIES

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ABSTRACT. We extend the famous Kodaira-Spencer's completeness theorem for a family of deformations of complex structures (see [12]). As an application, we show that the canonical family constructed in [9] is versal.

1. **Introduction.** The purpose of this paper is to give a criterion for versality of deformations of complex structures over a tubular neighborhood of a strongly pseudo convex boundary. In the our former paper ([9]), we constructed a canonical family of complex structures over a tubular neighborhood of the strongly pseudo convex boundary which satisfies a certain condition, *from the point of view of CR-structures*. In this paper, we give a fairly general criterion for versality *in the sense of Kuranishi*. And as an application of this criterion, we see that our family constructed in [5] is versal *in the sense of Kuranishi*. Namely, we assume: we are given a family of deformations of almost complex structures over $\bar{U}$, $\big(\phi(t), T\big)$, satisfying

o is a non-singular point of T,

$\phi(t) \in \Gamma\big(\bar{U}, T' \otimes (T''N)^*\big)$,

$\phi(t) = \sum_{\lambda=1}^{q} \beta_\lambda t_\lambda + 0(t^2)$,

where $\phi(t)$ is defined by the standard way as in [10] and $\{\beta_\lambda\}_{\lambda=1}^{q}$ generates $H^{(1)}(U, T'N)$, $T'N$-valued $\bar\partial$-cohomology, and $q = \dim_C H^{(1)}(U, T'N)$. *We note that we don't assume* $P\big(\phi(t)\big) = 0$ *for all t in T*. Under this assumption, we have:

CRITERION. Assume the above. And we assume that:

$$P\big(\phi(t)\big) = 0 \bmod \mathbf{H}^{(2)}_{T'N} P\big(\phi(t)\big).$$

Define $T' \subset T$ by $T' = \{t', t' \in T \mid P\big(\phi(t')\big) = 0\}$. Then our family, $\big(\phi(t), T\big)$ is versal *in the sense of Kuranishi*, where if $\dim_C X \geq 4$, $\mathbf{H}^{(2)}_{T'N}$ = the harmonic projection of $T'N$-valued forms of type $(0, 2)$, and if $\dim_C X = 3$, $\mathbf{H}^{(2)}_{T'N} = 1 - \bar\partial N \bar\partial^*$, where N means the Neumann operator of $T'N$-valued forms of type $(0, 1)$.

The proof will be done along the lines of [3].

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2. **A family of deformations of tubular neighborhoods of strongly pseudo convex boundaries.** Let X be a complex manifold. Let Ω be a relative compact strongly pseudo convex domain with smooth boundary $b\Omega$. We consider deformations of tubular neighborhoods of the strongly pseudo convex boundary $b\Omega$. Let (T, o) be a germ of complex analytic subspaces of (C^r, o).

DEFINITION 2.1. By a family of deformations of tubular neighborhoods of a strongly pseudo convex boundary $b\Omega$, $(\mathcal{X}, \pi, T)$, we mean that $\mathcal{X}$, T are analytic spaces, and a smooth morphism $\pi: \mathcal{X} \longrightarrow (T, o)$ satisfying: $\pi^{-1}(o)$ is a tubular neighborhood of $b\Omega$ in X.

Henceforth we use the notation $(\mathcal{X}, \pi, T)$ for a family of deformations of tubular neighborhoods of a strongly pseudo convex boundary $b\Omega$. Let $(\mathcal{X}, \pi, T)$ be a family of deformations of a tubular neighborhood of a strongly pseudo convex boundary. And we set $\pi^{-1}(o) = U$. Then we can define an element $\phi(t)$ of $\Gamma\big(\bar{U}', T'N \otimes (T''N)^*\big)$, which is parametrized by T complex analytically, by the standard way as in [10], satisfying:

$$P\big(\phi(t)\big) = 0 \text{ for } t \text{ in } T,$$

where U' is also a tubular neighborhood of $b\Omega$ and $U' \subset\subset U$.

3. **The notion of versality.** Let $(\mathcal{X}, \pi, T)$ be a family of deformations of a tubular neighborhood of a strongly pseudo convex boundary $b\Omega$. In this section, we recall the notion of versality (cf. [2], [3], [4]).

DEFINITION 3.1. A family of deformations of a tubular neighborhood of a strongly pseudo convex boundary $b\Omega$, $(\mathcal{X}, \pi, T)$ is called *versal* if the following holds: For any family of deformations of tubular neighborhoods of a strongly pseudo convex boundary $b\Omega$, $(\mathcal{Y}, \omega, S)$ satisfying: $o \in S$, and $\omega^{-1}(o) = V$ is an open neighborhood of $b\Omega$ in N satisfying: $\pi^{-1}(o) = U \subset\subset V$, there are a holomorphic map τ from S to T and a holomorphic map $g(s)$ from $\pi^{-1}\big(\tau(s)\big)$ to $\omega^{-1}(s)$, $g(o) =$ identity map, depending on s complex analytically and if necessary, we must shrink S sufficiently small.

4. **A criterion.** Let X be a complex manifold and let Ω be a strongly pseudo convex domain with smooth boundary $b\Omega$. We assume that we are given a family of deformations of complex structures over $\bar{U}$, $\big(T, \phi(t)\big)$, satisfying: o is a non-singular point of T,

$$\phi(t) \in \Gamma\big(\bar{U}, T'N \otimes (T''N)^*\big),$$

$$\phi(t) = \sum_{\lambda=1}^{q} \beta_\lambda t_\lambda + 0(t^2),$$

where $\{\beta_\lambda\}_{\lambda=1}^{q}$ generates $H^{(1)}(U, T'N)$, $T'N$-valued $\bar{\partial}$-cohomology at degree one and $q = \dim_C H^{(1)}(U, T'N)$, and $(t_1, \ldots, t_q)$ is a local coordinate of T at the origin.

CRITERION. Assume the above. And we assume that $P\big(\phi(t)\big) \equiv 0 \bmod \mathbf{H}^{(2)}_{T'N} P\big(\phi(t)\big)$. Then our family, $\big(\phi(t), T\big)$, is versal *in the sense of Kuranishi*, where if $\dim_C X \geq 4$,

$\mathbf{H}^{(2)}_{T'N}$ = the harmonic projection of $T'N$-valued form of type $(0,2)$, and if $\dim_C X = 3$, $\mathbf{H}^{(2)}_{T'N} = 1 - \bar\partial N \bar\partial^*$, where N means the Neumann operator of $T'N$-valued forms of type $(0,1)$.

REMARK. Here concerning the notion of a family of deformations of tubular neighborhoods of strongly pseudo convex boundaries, rigorously our $(\phi(t), T)$ should be read as $(\phi(t'), T')$, where

$$T' = \{t' \mid t' \in T, P(\phi(t')) = 0\}.$$

We show our criterion. Let $(\mathcal{N}, \omega, S)$ be an arbitrary family of deformations of a neighborhood V of $b\Omega$ satisfying: $o \in S$ and $\omega^{-1}(o) = V$, $U \subset\subset V$. We assume the following:

(4.i) o is the origin of a complex euclidean space C^r and S is an analytic subspace of a neighborhood D of o in C^r defined by $b_1(s) = \cdots = b_\ell(s) = 0$.

(4.ii) We find a finite system of open sets of $\mathcal{N}$, $\{\mathcal{U}_j\}_{j\in\Lambda}$, satisfying that there is an analytic embedding

$$\eta_j \colon \mathcal{U}_j \longrightarrow W_j \times D \text{ with } p_2 \cdot \eta_j = \omega \text{ for each } j \in \Lambda,$$

where W_j is a neighborhood of o in C^n and p_2 denotes the projection of $W_j \times D$ onto the second factor. We denote by $\zeta_j = (\zeta_j^{(1)}, \ldots, \zeta_j^{(n)})$ and $s = (s_1, \ldots, s_r)$ the coordinates of W_j and D respectively, and set $z_j^\lambda = \zeta^\lambda \cdot \eta_j|_\omega - 1_{(o)}$ for $\lambda = 1, \ldots, n$ and $U_j = \mathcal{U}_j \cap b\Omega$, where we regard ζ_j^λ as a function on $W_j \times D$,

(4.iii) $\eta_j \cdot \eta_k^{-1}$ is represented by:

$$\zeta_j^\lambda = f_{jk}^\lambda(\zeta_k, s) \text{ for } \lambda = 1, \ldots, n,$$
$$s_\alpha = s_\alpha \text{ for } \alpha = 1, \ldots, r,$$

and we set

$$f_{jk}^\lambda(z_k) = f_{jk}^\lambda(z_k, o) \text{ for } \lambda = 1, \ldots, n,$$

(4.iv) $f_{ij}^\lambda\big(f_{jk}(\zeta_k, s), s\big) \equiv f_{ik}^\lambda(\zeta_k, s) \bmod b(s)$, where $\bmod b(s)$ means $\bmod\{b_\mu(s), \mu = 1, \ldots, \ell\}$ and henceforth we use this notation for brevity.

To prove the versality of the family which satisfies our condition, it suffices to show the existence of a neighborhood D' of o in D, of a family $g_i(s)$ of sections of $T'N$ over U_i which depends complex analytically on s in D' for each $i \in \Lambda$, and of a T-valued holomorphic function $\tau(s)$ on D' satisfying:

$$(4.0) \qquad \big(g_i(o)\big)^\lambda = z^\lambda \text{ for } \lambda = 1, \ldots, n, \quad \tau(o) = 0$$

$$(4.1) \qquad \big(g_i(s)\big)^\lambda - f_{ij}^\lambda\big(g_j(s), s\big) = 0 \text{ for } s \in S \text{ and } \lambda = 1, \ldots, n,$$

$$(4.2) \qquad \big(\bar\partial + \phi\big(\tau(s)\big)\big)\big(g_i(s)\big)^\lambda = 0 \text{ for } s \in S \text{ and } \lambda = 1, \ldots, n,$$

$$(4.3) \qquad h\big(\tau(s)\big) = 0 \text{ for } s \in S,$$

where $h(t) = \mathbf{H}^{(2)}_{T'N} P\big(\phi(t)\big)$, and if necessary, we must shrink S sufficiently small, and $g_i(s)$ has the expression $g_i(s) = \sum_{\lambda=1}^n \big(g_i(t)\big)^\lambda \partial/\partial z_i^\lambda$, regarded as an element of $\Gamma(U_i, T'N)$ and $\big(\bar\partial + \phi\big(\tau(s)\big)\big)\big(g_i(s)\big)^\lambda$ denotes the element $\Gamma(U_i, T'N)$ defined by the equation

$$\big(\bar\partial + \phi\big(\tau(s)\big)\big)\big(g_i(s)\big)^\lambda(X) = X\Big(\big(g_i(s)\big)^\lambda\Big) + \phi\big(\tau(s)\big)(X)\Big(\big(g_i(s)\big)^\lambda\Big), \text{ for any } X \in T'N.$$

4.1. *Construction of a formal solution.* First, we construct $\{g_i(s)\}_{i\in\Lambda}$ and $\tau(s)$ formally in s, namely we construct $\{g_i^{(\mu)}(s)\}_{i\in\Lambda}$ and $\tau^{(\mu)}(s)$ for $\mu = 1, \ldots$ satisfying:

(4.0) $\qquad (g_i^{(0)})^\lambda = z_i^\lambda$ for $\lambda = 1, \ldots, n$ and $\tau^{(0)} = 0$,

$(4.1)_\mu \qquad \left(g_i^{(\mu)}(s)\right)^\lambda - f_{ij}^\lambda\left(g_j^{(\mu)}(s), s\right) \equiv 0 \bmod \left(b_m(s), s^{\mu+1}\right), \quad m = 1, \ldots, \ell,$

$(4.2)_\mu \qquad \left(\bar\partial + \phi\left(\tau^{(\mu)}(s)\right)\right)\left(g_i^{(\mu)}(s)\right)^\lambda \equiv 0 \bmod \left(b_m(s), s^{\mu+1}\right), \quad m = 1, \ldots, \ell,$

$(4.3)_\mu \qquad h\left(\tau^{(\mu)}(s)\right) \equiv 0 \bmod \left(b_m(s), s^{\mu+1}\right), m = 1, \ldots, \ell,$

$(4.4)_\mu$ $g_i^{(\mu)}(s)$ is a $\Gamma(U_i, T'N)$-valued polynomial in s of degree μ and $\tau^{(\mu)}(s)$ is a T-valued polynomial in s of the same degree satisfying that

$$g_i^{(\mu)}(s) \equiv g_i^{(\mu-1)}(s) \bmod s^\mu$$

and

$$\tau^{(\mu)}(s) \equiv \tau^{(\mu-1)}(s) \bmod s^\mu .$$

Now we construct these $\{g_i^{(\mu)}(s)\}_{i\in\Lambda}$ and $\tau^{(\mu)}(s)$ by induction on μ.

For $\mu = 0$, we set

$$(g_i^{(0)})^\lambda = z_i^\lambda \text{ for } \lambda = 1, \ldots, n \text{ and } \tau^0 = 0.$$

Suppose that $\{g_i^{(\mu-1)}(s)\}_{i\in\Lambda}$ and $\tau^{(\mu-1)}(s)$ are determined for some $\mu \geq 1$. First we define a $\Gamma(U_i \cap U_j, T'N)$-valued polynomial in s of degree μ, $\sigma_{ij}^{(\mu)}(s)$, by

$$\sigma_{ij}^{(\mu)}(s) \equiv \sum_{\lambda=1}^{n} \left\{ \left(g_i^{(\mu-1)}(s)\right)^\lambda - f_{ij}^\lambda\left(g_j^{(\mu-1)}(s), s\right)\right\} \partial/\partial z_i^\lambda \bmod s^{\mu+1}.$$

Then we set

$$g_i'|_\mu(s) = \sum_{k\in\Lambda} \rho_k \kappa_s^\mu\left(\sigma_{ki}^{(\mu)}(s)\right),$$

where $\{\rho_k\}_{k\in\Lambda}$ is a partition of unity subordinate to $\{U_k\}_{k\in\Lambda}$, and $\kappa_s^\mu(\ )$ means the μ-th polynomial part of ( ) with respect to s. Next we define $\Gamma\left(U_i, T'N \otimes (T''N)^*\right)$-valued polynomial $w_i^{(\mu)}(s)$ and $\zeta_i^{(\mu)}(s)$ of degree μ by

$$\begin{aligned} w_i^{(\mu)}(s) = -\sum_{\lambda=1}^{n} [\bar\partial\{ \left(g_i^{(\mu-1)}(s)\right)^\lambda + \left(g_i'|_\mu(s)\right)^\lambda \} \\ + \phi\left(\tau^{(\mu-1)}(s)\right)\{ \left(g_i^{(\mu-1)}(s)\right)^\lambda - (g_i^{(0)})^\lambda \}]\partial/\partial z_i^\lambda \bmod s^{\mu+1} \end{aligned}$$

and

$$\zeta_i^{(\mu)}(s) \equiv w_i^{(\mu)}(s) - \phi\left(\tau^{(\mu-1)}(s)\right)|_{U_i} \bmod s^{\mu+1}.$$

We solve $\tau_\mu^{(\sigma)}(s)$ satisfying

$$\sum_{\sigma=1}^{q} \tau_\mu^{(\sigma)}(s)\beta'_\sigma = \mathbf{H}^{(1)}_{T'N}\Big(\sum_{i\in\Lambda} \rho_i \kappa_s^\mu\left(\zeta_i^{(\mu)}(s)\right)\Big),$$

where $\beta'_\lambda = \mathbf{H}^{(1)}_{T'N}\beta_\lambda$, and $\mathbf{H}^{(1)}_{T'N}$ means the harmonic projection of $T'N$-valued form. This part is the only one different from [3].

$$g'_\mu(s) = -\bar{\partial}^*_{T'N}N_{T'N}\Big(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu\big(\zeta_i^{(\mu)}(s)\big) - \tau_\mu(s)\Big),$$

$$\tau_\mu(s) = \sum_{\sigma=1}^{q}\tau_\mu^{(\sigma)}(s)\beta_\sigma,$$

$$\tau_\mu^{(\sigma)}(s) = \Big(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu\big(\zeta_i^{(\mu)}(s)\big), \beta'_\sigma\Big),$$

where (,) is chosen satisfying: $(\beta'_\lambda, \beta'_\mu) = \delta_{\lambda\mu}$. Then we have that $g'_\mu(s)$ is $\Gamma(U, T'N)$-valued, since $N_{T'N}$ is a C^∞ operator. Finally we set

$$g_i^{(\mu)}(s) = g_i^{(\mu-1)}(s) + g'_i|_\mu(s) + g'_\mu(s)$$

and

$$\tau^\mu(s) = \tau^{(\mu-1)}(s) + \tau_\mu(s).$$

Obviously (4.0) and $(4.4)_\mu$ are satisfied for all $\mu \geq 1$.

PROPOSITION 4.1. *For any $\mu \geq 0$,*

$(1)_\mu$ $$\big(g_i^{(\mu)}(s)\big)^\lambda - f_{ij}^\lambda\big(g_j^{(\mu)}(s), s\big) \equiv 0 \bmod \big(b(s), s^{\mu+1}\big) \textit{ for } \lambda = 1,\ldots,n,$$

$(2)_\mu$ $$\theta_i^{(\mu)}(s) - \phi\big(\tau^{(\mu)}(s)\big)|_{U_i} \equiv 0 \bmod \big(b(s), s^{\mu+1}\big) \textit{ for } \lambda = 1,\ldots,n,$$
where $\theta_i^{(\mu)}(s)$ is a $\Gamma\big(U_i, T'N\otimes(T''N)^*\big)$-valued polynomial in t of degree μ defined by:

$$\big(\bar{\partial} + \theta_i^{(\mu)}(s)\big)\big(g_i^{(\mu)}(s)\big)^\lambda \equiv 0 \bmod s^{\mu+1} \textit{ for } \lambda = 1,\ldots,n,$$

$(3)_\mu$ $$h\big(\tau^{(\mu)}(s)\big) \equiv 0 \bmod \big(b(s), s^{\mu+1}\big),$$

$(4)_\mu$ $$\sum_{k\in\Lambda}\rho_k\{\sigma_{kj}^{(\mu+1)}(s) - \sigma_{ki}^{(\mu+1)}(s)\} \equiv 0 \bmod s^{\mu+1},$$

$(5)_\mu$ $$\bar{\partial}_{T'N}\bar{\partial}^*_{T'N}N_{T'N}\{\sum_{i\in\Lambda}\rho_i w_i^{(\mu+1)}(s) - \phi\big(\tau^{(\mu)}(s)\big)\} \equiv 0 \bmod s^{\mu+1},$$

$(6)_\mu$ $$\mathbf{H}^{(1)}_{T'N}\{\sum_{i\in\Lambda}\rho_i w_i^{(\mu+1)}(s) - \phi\big(\tau^{(\mu)}(s)\big)\} \equiv 0 \bmod s^{\mu+1}.$$

PROOF. We prove this proposition by following the line in [3].

For $\mu = 0$, it is obvious. Because

$$\sigma_{ij}^{(1)}(s) \equiv 0 \bmod s, \quad w_i^{(1)}(s) \equiv 0 \bmod s$$

and

$$\phi\big(\tau^{(0)}(s)\big) = 0,$$

$(4)_0$–$(6)_0$ are also satisfied.

We suppose that $(1)_{\mu-1}$–$(6)_{\mu-1}$ are satisfied for some $\mu \geq 1$. To prove $(1)_\mu$, we recall the following lemma.

LEMMA 4.2. $\sigma_{ki}^{(\mu)}(s) - \sigma_{kj}^{(\mu)}(s) + \sigma_{ij}^{(\mu)}(s) \equiv 0 \bmod \big(b(s), s^{\mu+1}\big)$.

For the proof, see Lemma 3.2 in [3].

PROOFS OF $(1)_\mu$ AND $(4)_\mu$. The proof of this part is completely the same as in page 828 in [3]. So we omit the proof.

Next we see $(2)_\mu$ and $(3)_\mu$. For this, we must recall some lemmas.

LEMMA 4.3. $\theta_i^{(\mu)}(s) \equiv w_i^{(\mu)}(s) - \bar\partial_{T'N} g'_\mu(s)|_{U_i} \bmod \big(b(s), s^{\mu+1}\big)$.

For the proof, see Lemma 3.3 in [3].

COROLLARY 4.4. $P\big(w_i^{(\mu)}(s)\big) \equiv 0 \bmod \big(b(s), s^{\mu+1}\big)$.

For the proof, see Corollary 3.4 in [3].

LEMMA 4.5. $\theta_i^{(\mu)}(s) \equiv \theta_j^{(\mu)}(s)$ *on* $U_i \cap U_j \bmod \big(b(s), s^{\mu+1}\big)$.

For the proof, see Lemma 3.5 in [3].

So we have:

COROLLARY 4.6. $w_i^{(\mu)}(s) \equiv w_j^{(\mu)}(s)$ *on* $U_i \cap U_j \bmod \big(b(s), s^{\mu+1}\big)$. *Therefore*

$$\zeta_i^{(\mu)}(s) \equiv \zeta_j^{(\mu)}(s) \text{ on } U_i \cap U_j \bmod \big(b(s), s^{\mu+1}\big).$$

And we have:

LEMMA 4.7. $h\big(\tau^{(\mu-1)}(s)\big) \equiv 0 \bmod \big(b(s), s^{\mu+1}\big)$, *where* $h(t) = \mathbf{H}^{(2)}_{T'N} P\big(\phi(t)\big)$.

For the proof, see Lemma 3.7 in [3].

LEMMA 4.8. $\bar\partial^{(1)}_{T'N}\big(\sum_{i\in\Lambda} \rho_i \zeta_i^{(\mu)}(s)\big) \equiv 0 \bmod \big(b(s), s^{\mu+1}\big)$.

For the proof, see Lemma 3.8 in [3].

LEMMA 4.9.

$$\sum_{i\in\Lambda} \rho_i \theta_i^{(\mu)}(s) - \phi\left(\tau^{(\mu)}(s)\right) \equiv 0 \bmod \big(b(s), s^{\mu+1}\big).$$

PROOF. By Lemma 4.3, we have

$$\begin{aligned}
&\sum_{i\in\Lambda} \rho_i \theta_i^{(\mu)}(s) - \phi\left(\tau^{(\mu)}(s)\right) \\
&\qquad \equiv \sum_{i\in\Lambda} \rho_i w_i^{(\mu)}(s) - \bar\partial_{T'N} g'_\mu(s) - \phi\left(\tau^{(\mu-1)}(s)\right) - \phi_1 \tau_\mu(s) \bmod \big(b(s), s^{\mu+1}\big) \\
&\qquad \equiv \sum_{i\in\Lambda} \rho_i \zeta_i^{(\mu)}(s) - \bar\partial_{T'N} \bar\partial^*_{T'N} N_{T'N} \{ \sum_{i\in\Lambda} \rho_i \kappa_s^\mu \left(\zeta_i^{(\mu)}(s)\right) - \sum_{\sigma=1}^{q} \tau_\mu^{(\sigma)}(s) \beta_\sigma \} \\
&\qquad\quad - \sum_{\sigma=1}^{q} \tau_\mu^{(\sigma)}(s) \beta_\sigma \bmod \big(b(s), s^{\mu+1}\big).
\end{aligned}$$

While by the definition of β'_σ, β_σ, there is an α_σ satisfying:

$$\beta_\sigma = \beta'_\sigma + \bar{\partial}_{T'N}\alpha_\sigma \text{ (because of } \beta'_\sigma = \mathbf{H}^{(1)}_{T'N}\beta_\sigma).$$

So

$$(1 - \bar{\partial}_{T'N}\bar{\partial}^*_{T'N}N_{T'N})\beta_\sigma = \beta'_\sigma.$$

Hence

$$\begin{aligned}
&\sum_{i\in\Lambda}\rho_i\theta_i^{(\mu)}(s) - \phi\left(\tau^{(\mu)}(s)\right)\\
&\quad\equiv \sum_{i\in\Lambda}\rho_i\zeta_i^{(\mu)}(s) - \bar{\partial}_{T'N}\bar{\partial}^*_{T'N}N_{T'N}\Big(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu\left(\zeta_i^{(\mu)}(s)\right)\Big)\\
&\qquad - \sum_{\sigma=1}^{q}\Big(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu\left(\zeta_i^{(\mu)}(s)\right), \beta'_\sigma\Big)\beta'_\sigma \bmod\big(b(s), s^{\mu+1}\big)\\
&\quad\equiv \sum_{i\in\Lambda}\rho_i\zeta_i^{(\mu)}(s) - \bar{\partial}_{T'N}\bar{\partial}^*_{T'N}N_{T'N}\Big(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu\left(\zeta_i^{(\mu)}(s)\right)\Big)\\
&\qquad - \mathbf{H}^{(1)}_{T'N}\Big(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu\left(\zeta_i^{(\mu)}(s)\right)\Big) \bmod\big(b(s), s^{\mu+1}\big)\\
&\quad\equiv \sum_{i\in\Lambda}\rho_i\zeta_i^{(\mu)}(s) - \bar{\partial}_{T'N}\bar{\partial}^*_{T'N}N_{T'N}\Big(\sum_{i\in\Lambda}\rho_i\zeta_i^{(\mu)}(s)\Big)\\
&\qquad - \mathbf{H}^{(1)}_{T'N}\Big(\sum_{i\in\Lambda}\rho_i\zeta_i^{(\mu)}(s)\Big) \bmod\big(b(s), s^{\mu+1}\big) \qquad (\text{by } (5)_{\mu-1} \text{ and } (6)_{\mu-1})\\
&\quad\equiv \bar{\partial}^{(1)*}_{T'N}\bar{\partial}^{(1)}_{T'N}N_{T'N}\Big(\sum_{i\in\Lambda}\rho_i\zeta_i^{(\mu)}(s)\Big)\\
&\quad\equiv 0 \bmod\big(b(s), s^{\mu+1}\big) \qquad (\text{by Lemma 4.8}).
\end{aligned}$$

■

PROOF OF $(2)_\mu$. By Lemma 4.9 with Lemma 4.5,

$$\theta_i^{(\mu)}(s) \equiv \phi\left(\tau^{(\mu)}(s)\right)|_{U_i} \bmod\big(b(s), s^{\mu+1}\big).$$

■

PROOF OF $(3)_\mu$. Since the linear term of $h(t)$ is 0, we have

$$\begin{aligned}
h\big(\tau^{(\mu)}(s)\big) &\equiv h\big(\tau^{(\mu-1)}(s)\big) \bmod s^{\mu+1}\\
&\equiv 0 \bmod\big(b(s), s^{\mu+1}\big) \text{ (by Lemma 4.7)}
\end{aligned}$$

PROOF OF $(5)_\mu$.

$$\begin{aligned}
&\bar{\partial}_{T'N}\bar{\partial}^*_{T'N}N_{T'N}\{\sum_{i\in\Lambda}\rho_i w_i^{(\mu+1)}(s) - \phi\left(\tau^{(\mu)}(s)\right)\}\\
&\quad\equiv \bar{\partial}_{T'N}\bar{\partial}^*_{T'N}N_{T'N}\{\sum_{i\in\Lambda}\rho_i w_i^{\mu+1}(s) - \phi\left(\tau^{(\mu-1)}(s)\right) - \phi_1\tau_\mu(s)\} \bmod s^{\mu+1}\\
&\quad\equiv \bar{\partial}_{T'N}\bar{\partial}^*_{T'N}N_{T'N}\Big(\sum_{i\in\Lambda}\rho_i\zeta_i^{(\mu)}(s)\Big) - \bar{\partial}_{T'N}\bar{\partial}^*_{T'N}N_{T'N}\big(\phi_1\tau_\mu(s)\big) \bmod s^{\mu+1}\\
&\quad\equiv 0 \bmod s^{\mu+1} \text{ (by the definition of } \tau_\mu(s) \text{ and } (5)_{\mu-1}).
\end{aligned}$$

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PROOF OF $(6)_\mu$.

$$\begin{aligned}\mathbf{H}^{(1)}_{T'N}\{ \sum_{i\in\Lambda} \rho_i w_i^{\mu+1}(s) - \phi\left(\tau^{(\mu)}(s)\right)\} \\ \equiv \mathbf{H}^{(1)}_{T'N}\{ \sum_{i\in\Lambda} \rho_i \zeta_i^{(\mu+1)}(s) - \bar\partial_{T'N}\bar\partial^*_{T'N}N_{T'N}\Big(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu\left(\zeta_i^{(\mu)}(s)\right) - \phi_1\tau_\mu(s)\Big) \\ - \sum_{\sigma=1}^{q}\Big(\big(\sum_{i\in\Lambda}\rho_i\kappa_s^\mu\left(\zeta_i^{(\mu)}(s)\right), \beta'_\sigma\big)\beta_\sigma\} \\ \equiv 0 \bmod s^{\mu+1} \text{ (by } (6)_{\mu-1}).\end{aligned}$$

■

By Proposition 4.1, we have $(4.1)_\mu$ and $(4.3)_\mu$ for any $\mu \geq 0$. From $(2)_\mu$ in Proposition 4.1, we have that for any $\mu \geq 0$,

$$(4.2)_\mu \qquad \left(\bar\partial + \phi\left(\tau^{(\mu)}(s)\right)\right)\left(g_i^{(\mu)}(s)\right)^\lambda \equiv 0 \bmod \left(b(s), s^{\mu+1}\right) \text{ for } \lambda = 1,\dots n.$$

This completes the inductive construction of $g_i^{(\mu)}(s)$ and $\tau^{(\mu)}(s)$.

4.2. *Convergence of the formal power series* We see that the formal power series $g_i(s) = \lim_{\mu\to\infty} g_i^{(\mu)}(s)$ and $\tau(s) = \lim_{\mu\to\infty}\tau^{(\mu)}(s)$ converges with respect to $\| \ \|'_{(0,m)}$-norm and $| \ |$-norm respectively where $m \geq n+2$ and $| \ |$ denotes the euclidean norm on the finite dimensional vector space $\mathcal{H}$, where $\mathcal{H}$ is generated by $\beta_1,\dots,\beta_{q-1},\beta_q$ (for the definition of $\| \ \|'_{(0,m)}$-norm, see [9], and we can identify T and $\mathcal{H}$ locally at the origin).

To prove that $\{g_i^{(\mu)}(s)\}_{i\in\Lambda}$ and $\tau^{(\mu)}(s)$ converge, it suffices to show the following estimates; for all $\mu \geq 1$,

$$(4.5)_\mu \qquad \|g_i^{(\mu)}(s) - g_i^{(0)}\|'_{(0,m)} << A(s),$$

$$(4.6)_\mu \qquad |\tau^{(\mu)}(s)| << A(s),$$

where $A(s)$ is defined by:

$$A(s) = (b/16c)\sum_{\mu=1}^{\infty}(c^\mu/\mu^2)(s_1+\cdots+s_r)^\mu$$

by the complete same way as in [8]. As $\phi(t)$ is holomorphic in t and $f_{ij}(\zeta_j, s)$ is holomorphic in (ζ_j, s), we may assume the following:

$$(4.v) \qquad \|\phi(t)\|_{(0,m)} << (b_0/c_0)\sum_{\mu=1}^{\infty} c_0^\mu (t_1+\cdots+t_r)^\mu,$$

$$(4.vi) \qquad \begin{aligned}\|f_{ij}^\lambda(z_j+x, s)f_{ij}^\lambda(z_j) - \sum_{\nu=1}^{n}(\partial f_{ij}^\lambda/\partial z_j^\lambda)(z_j)x^\nu - \sum_{\alpha=1}^{r}(\partial f_{ij}^\lambda/\partial s_\alpha)(z_j,0)s_\alpha\|'_{(0,m)} \\ << (b_0/c_0)\sum_{\mu=2}^{\infty} c^\mu(x_1+\cdots+x_n+s_1+\cdots+s_r)^\mu \text{ for } \lambda = 1,\dots,n.\end{aligned}$$

However the proof of this part is the same as in [3]. So we omit this. Hence we have that $g_i(s)$ is a $\Gamma'_{(0,m)}(U_i, T'N)$-valued holomorphic function and $\tau(s)$ is a T-valued holomorphic function on some neighborhood D' of o in D. so we have our criterion. ■

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