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PRIMITIVITY IN FREE GROUPS AND FREE METABELIAN GROUPS

C. K. GUPTA, N. D. GUPTA AND V. A. ROMAN'KOV

ABSTRACT. Let $M_{n,c}$ denote the free n-generator metabelian nilpotent group of class c. For $m \leq n-2$, every primitive system of m elements of $M_{n,c}$ can be lifted to a primitive system of m elements of the absolutely free group F_n of rank n. The restriction on m cannot be improved.

Introduction. Let $F = \langle f_1, \ldots, f_n \rangle$ $(= F_n)$ be the free group of rank n and let $\mathbf{w} = \{w_1, \ldots, w_m\}$, $m \leq n$, be a system of words in F. The system $\mathbf{w}$ is said to be *primitive* if it can be included in some basis of F. Primitivity of a given system $\mathbf{w}$ can be algorithmically decided (Whitehead, see Lyndon and Schupp [6], p. 30), and there are some nice primitivity criteria in terms of certain properties of the $m \times n$ Jacobian matrix $J(\mathbf{w}) = (\partial w_i / \partial f_j)$, over $\mathbf{Z}F$, of the Fox derivatives $\partial/\partial f_j \colon \mathbf{Z}F \longrightarrow \mathbf{Z}F$ (Birman [3] for the case $m = n$ and Umirbaev [10] for the general case). In the free metabelian groups $M = \langle x_1, \ldots, x_n \rangle$ $(= M_n)$, the corresponding primitivity criteria for a system $\mathbf{g} = \{g_1, \ldots, g_m\}$ are due to Bachmuth [1] (for the case $m = n$) and Timoshenko [9] (for the case $m \leq n-3$) who obtained necessary and sufficient conditions for the system $\mathbf{g}$ to be included in some basis of M in terms of the $m \times m$ minors of the $m \times n$ Jacobian matrix $J(\mathbf{g}) = (\partial g_i / \partial x_j)$, over $\mathbf{Z}A$, of the induced Fox derivatives $\partial/\partial x_j \colon \mathbf{Z}F \longrightarrow \mathbf{Z}A$, where $\mathbf{Z}A$ is the group ring of the free abelian group $A = \langle a_1, \ldots, a_n \rangle$ $(= A_n)$. In these cases, the algorithmic decidability of the primitivity in M of the given system $\mathbf{g}$ then reduces to the existence of a solution of a system of linear equations over the Laurent polynomial ring $\mathbf{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$ which, in turn, can be effectively decided (Timoshenko [8]).

Let V be a fully invariant subgroup of F. We say that a system $\mathbf{w} = \{w_1, \ldots, w_m\}$, $m \leq n$, of words in F is *primitive mod* V if for some choice of words $v_1, \ldots, v_m \in V$, the corresponding system $\{w_1v_1, \ldots, w_mv_m\}$ is primitive (absolutely), or equivalently, if the system $\{w_1V, \ldots, w_mV\}$ of cosets can be extended to some basis for F/V. Now let V, U be fully invariant subgroups of F with $V \geq U$. Then we say that a system of words $\mathbf{w} = \{w_1, \ldots, w_m\}$, $m \leq n$, can be *lifted* (*via* V) *to a primitive system mod* U if and only if there exists $v_i \in V$ such that the corresponding system $\{w_1v_1, \ldots, w_mv_m\}$ is primitive mod U. Let $\gamma_c(F)$ denote the c-th term of the lower central series of F and let F'' $\big(= \gamma_2(F')\big)$ denote the second commutator subgroup of F. Our primary result in this paper is the following: if $\mathbf{w} = \{w_1, \ldots, w_m\}$, $m \leq n-2$, is a primitive system

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modulo $\gamma_{c+1}(F)F''$, $c \geq 2$, then $\mathbf{w}$ can be lifted (via $\gamma_{c+1}(F)F''$) to a primitive system of F (Theorems B & E). The restriction $m \leq n-2$ cannot be improved (Remark C).

Primitive lifting in free metabelian groups. Let $M = M_n = \langle x_1, \ldots, x_n \rangle \cong F_n/F_n''$ be the free metabelian group of rank $n \geq 2$. Let $\varepsilon: \mathbf{Z}M \longrightarrow \mathbf{Z}$ be the augmentation map and $\Delta(M)$, the augmentation ideal of $\mathbf{Z}M$. Let A ($= A_n$) be the free abelian group generated by $a_1, \ldots, a_n$ and let $\theta: \mathbf{Z}M \longrightarrow \mathbf{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$ be the linear extension of the natural homomorphism: $M \longrightarrow A$. For $j = 1, \ldots, n$, define *induced* right partial derivative maps $\partial/\partial x_j: \mathbf{Z}M \longrightarrow \mathbf{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$ as follows (cf. [5], p. 8): write $u - \varepsilon u = (x_1 - 1)u_1 + \cdots + (x_n - 1)u_n$, $u_i \in \mathbf{Z}M$ and define $\partial u/\partial x_j = \theta u_j$. Alternately, define $\partial(u+v)/\partial x_j = \partial u/\partial x_j + \partial v/\partial x_j$, $\partial(uv)/\partial x_j = \partial u/\partial x_j \theta v + \varepsilon u \partial v/\partial x_j$, $\partial(x_j)/\partial x_j = 1$, $\partial(x_i)/\partial x_j = 0$, $i \neq j$. To each system $\mathbf{g} = \{g_1, \ldots, g_m\}$ of m elements in M there corresponds an $m \times n$ Jacobian Matrix $J(\mathbf{g}) = (\partial g_i/\partial x_j)$ of the partial derivatives. When $m = n$, we shall need the following criterion for $\mathbf{g}$ to be a basis of M (cf. [5], p. 29).

LEMMA 1 (BACHMUTH [1]). *Let $\mathbf{g} = \{g_1, \ldots, g_n\}$ be a system of elements of a free metabelian group M_n. Then $\mathbf{g}$ is a basis for M_n if and only if its Jacobian matrix $J(\mathbf{g})$ is invertible over $\mathbf{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$.*

An arbitrary system $\mathbf{g} = \{g_1, \ldots, g_m\}$ in M consists of elements of the form $x_1^{e_1} \cdots x_n^{e_n} u$, $e_i \in \mathbf{Z}$, $u \in M'$. If $\mathbf{g}$ is primitive mod M' then there exists a tame automorphism $\alpha \in \mathrm{Aut}(M)$ (i.e. α is induced by an automorphism of F_n) such that $\alpha(\mathbf{g}) = \{x_1 u_1, \ldots, x_m u_m\}$, $u_i \in M'$. A system of the form $\{x_1 u_1, \ldots, x_m u_m\}$, $u_i \in M'$, will be called an IA-*system.* Thus primitive lifting of systems in free metabelian groups reduces to primitive lifting of IA-systems of the form $\{x_1 u_1, \ldots, x_m u_m\}$, $u_i \in M'$, $m \leq n$. We say that an IA-system $\{x_1 u_1, \ldots, x_m u_m\}$, $u_i \in M'$, $m \leq n$, is IA-*primitive* if it extends to an IA-basis of M of the form $\{x_1 u_1, \ldots, x_m u_m, x_{m+1} u_{m+1}, \ldots, x_n u_n\}$, $u_i \in M'$. We shall need the following reduction lemmas.

LEMMA 2. *If an* IA-*system $\{x_1 u_1, \ldots, x_m u_m\}$, $u_i \in M_n'$, $m \leq n$, is primitive in M_n then it is* IA-*primitive.*

PROOF. Let $\{y_1, \ldots, y_m, z_{m+1}, \ldots, z_n\}$ be a basis for M, where $y_i = x_i u_i$, $i = 1, \ldots, m$; $z_j = x_1^{e_{j1}} \cdots x_n^{e_{jn}} v_j, j = m+1, \ldots, n$, $e_{j\ell} \in \mathbf{Z}$, $v_j \in M_n'$. Using Nielsen transformations of the type $z_j \longrightarrow z_j y_i^k$, $k \in \mathbf{Z}$, the basis $\{y_1, \ldots, y_m, z_{m+1}, \ldots, z_n\}$ can be transformed to a basis of the form $\{y_1, \ldots, y_m, z_{m+1}', \ldots, z_n'\}$ where z_j' are of the new form given by $z_j' = x_{m+1}^{e_{j(m+1)}} \ldots x_n^{e_{j(n)}} v_j'$, $v_j' \in M_n'$. Since modulo M_n', the subsystem $\{z_{m+1}', \ldots, z_n'\}$ generates $\{x_{m+1}, \ldots, x_n\}$, it follows that by using Nielsen transformations on $\{z_{m+1}', \ldots, z_n'\}$, the basis $\{y_1, \ldots, y_m, z_{m+1}', \ldots, z_n'\}$ can be further transformed to a basis $\{y_1, \ldots, y_m, z_{m+1}'', \ldots, z_n''\}$, where $z_j'' = x_j v_j''$, $v_j'' \in M_n'$. This completes the proof of the lemma.

LEMMA 3. *Let $1 \leq m \leq p \leq n$ be fixed and assume that for each $e \geq 2$ every* IA-*system of the form $\{x_1 v_1, \ldots, x_m v_m, x_{m+1}, \ldots, x_p\}$ with $v_i \in \gamma_e(M_n)$ can be lifted to a primitive system of M_n of the form $\{x_1 v_1 w_1, \ldots, x_m v_m w_m, x_{m+1}, \ldots, x_p\}$ with $w_i \in$*

$\gamma_{e+1}(M_n)$. Then, for any $e \geq 2$, every IA-*system of the form $\{x_1u_1, \ldots, x_mu_m, x_{m+1}, \ldots, x_p\}$, with $u_i \in M'_n$, can be lifted to a primitive system of M_n of the form $\{x_1u_1w_1, \ldots, x_mu_mw_m, x_{m+1}, \ldots, x_p\}$ with $w_i \in \gamma_{e+1}(M_n)$.*

PROOF. It suffices to prove by induction on $c \geq 2$ that there is an automorphism α of M_n which transforms the given IA-system $\{x_1u_1, \ldots, x_mu_m, x_{m+1}, \ldots, x_p\}$ to an IA-system of the form $\{x_1u_{1,c}, \ldots, x_mu_{m,c}, x_{m+1}, \ldots, x_p\}$ with $u_{i,c} \in \gamma_c(M_n)$. For $c = 2$ we can choose α to be the identity automorphism. For the inductive step, let $\{x_1u_1, \ldots, x_mu_m, x_{m+1}, \ldots, x_p\}$, $u_i \in M'_n$, be already transformed to $\{x_1u_{1,c-1}, \ldots, x_mu_{m,c-1}, x_{m+1}, \ldots, x_p\}$ with $u_{i,c-1} \in \gamma_{c-1}(M_n)$ by some automorphism of M_n. By our assertion, $\{x_1u_{1,c-1}, \ldots, x_mu_{m,c-1}, x_{m+1}, \ldots, x_p\}$ can be lifted to a primitive system of M_n of the form $\{x_1u_{1,c-1}w_1, \ldots, x_mu_{m,c-1}w_m, x_{m+1}, \ldots, x_p\}$ with $w_i \in \gamma_c(M_n)$. Put $g_1 = x_1u_{1,c-1}w_1, \ldots, g_m = x_mu_{m,c-1}w_m$, $g_k = x_k$, $k = m+1, \ldots, p$. Thus there exists $\alpha \in \mathrm{Aut}(M_n)$ such that $\alpha : x_i \longrightarrow g_i$, $i = 1, \ldots, p$. Then $\alpha^{-1}(g_i) = x_i$ and for $i = 1, \ldots, m$, $\alpha^{-1}(x_iu_{i,c-1}) = \alpha^{-1}(g_iw_i^{-1}) = x_i\alpha^{-1}(w_i^{-1}) = x_iu_{i,c}$ for some $u_{i,c} \in \gamma_c(M_n)$, and $\alpha^{-1}(x_k) = x_k$ for $k = m+1, \ldots, p$. This completes the proof of the Lemma.

LEMMA 4. *If, for $1 \leq m < n$, $c \geq 2$, every* IA-*system $\{x_1v_1, x_2, \ldots, x_m\}$ with $v_1 \in \gamma_2(M_n)$ can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive system of M_n then every* IA-*system $\{x_1u_1, \ldots, x_mu_m\}$ with $u_i \in \gamma_2(M_n)$ can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive system of M_n.*

PROOF. By induction on $m \geq 1$. For $m = 1$ there is nothing to prove. By the induction hypothesis $\{x_2u_2, \ldots, x_mu_m\}$ can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive system of M_n, so by Lemma 2 it can be lifted to a primitive IA-system. Thus, there is an IA-automorphism $\alpha \in \mathrm{Aut}(M_n)$ and $w_i \in \gamma_{c+1}(M_n)$ such that $\alpha : x_iu_iw_i \longrightarrow x_i$, $i = 2, \ldots, m$ and $\alpha : x_1u_1 \longrightarrow x_1v_1$, $v_1 \in \gamma_2(M_n)$. Clearly, α transforms the system $\{x_1u_1, \ldots, x_mu_m\}$ to $\{x_1v_1, x_2w_2, \ldots, x_mw_m\}$, $w_i \in \gamma_{c+1}(M_n)$. Thus the problem reduces to lifting (via $\gamma_{c+1}(M_n)$) of a system of the form $\{x_1v'_1, x_2, \ldots, x_m\}$, $v'_1 \in \gamma_2(M_n)$, to a primitive system of M_n which, by hypothesis, is the case.

As a corollary to Lemmas 3 and 4 we obtain the following important lemma.

LEMMA 5. *If, for any $c \geq 2$ and $1 \leq m < n$, every* IA-*system $\{x_1v_1, x_2, \ldots, x_m\}$, with $v_1 \in \gamma_c(M_n)$, can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive system of M_n then every* IA-*system $\{x_1u_1, \ldots, x_mu_m\}$ with $u_i \in \gamma_2(M_n)$ can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive system of M_n.*

LEMMA 6. *For each $p \in \Delta^{c-2}(M_n)$, $c \geq 3$, the system $\mathbf{g} = \{g_1, \ldots, g_n\}$ with $g_1 = x_1[x_1, x_2]^p[x_2, x_3]^{(x_2-1)p}$, $g_3 = x_3[x_1, x_2]^{-p^2}[x_2, x_3]^{p-(x_2-1)p^2}$, $g_i = x_i$, $i \neq 1, 3$, forms a basis for M_n. (Notation: $[x_i, x_j]^{g+h} = [x_i, x_j]^g[x_i, x_j]^h$).*

PROOF. By Lemma 1 it suffices to show that the Jacobian matrix $J(\mathbf{g})$ of the given system $\mathbf{g}$ is invertible over $\mathbf{Z}A$. Indeed, it is easily seen that with $\pi = \theta p$ (under $\theta : \mathbf{Z}M \longrightarrow$

$\mathbf{Z}A$), the matrix $J(\mathbf{g})$ has the form,

$$\begin{bmatrix} 1+(a_2-1)\pi & * & -(a_2-1)^2\pi & 0\dots 0 \\ 0 & 1 & 0 & 0\dots 0 \\ -(a_2-1)\pi^2 & * & 1-(a_2-1)\pi+(a_2-1)^2\pi^2 & 0\dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \quad 0 \\ 0 & 0 & 0 & 0 \quad 1 \end{bmatrix}$$

The determinant of $J(\mathbf{g})$ is easily seen to be 1, so $J(\mathbf{g})$ is invertible.

We now establish primitive lifting in M_n of a single element of $M_{n,c}$.

THEOREM A. *Let g be an arbitrary element of M_n, $n \geq 3$, such that g is primitive modulo $\gamma_{c+1}(M_n)$, $c \geq 2$. Then g can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive element of M_n.*

PROOF. Using a tame automorphism of M_n, if necessary, we may assume that g is of the form $g = x_1 u$, $u \in M_n'$. By Lemma 5 we may further assume that $u \in \gamma_c(M_n)$ and write g as:

$$g = x_1 \prod_{2\leq i\leq n} [x_1, x_i]^{p_i} \prod_{1<i<j\leq n} [x_i, x_j]^{q_{ij}}$$

where p_i, $q_{ij} \in \Delta^{c-2}(M_n)$. Define $\mathbf{h} = \{h_1, \dots, h_n\}$ with $h_1 = x_1 \prod_{1<i<j\leq n} [x_i, x_j]^{q_{ij}}$, $h_i = x_i$, $i \neq 1$. Then the Jacobian $J(\mathbf{h}) = (\partial h_i / \partial x_j)$ is of the form

$$\begin{bmatrix} 1 & * & * & \dots & * & * \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

which is clearly invertible. Thus, by Lemma 1 there is an automorphism $\beta \in \mathrm{Aut}(M_n)$ which maps h_i to x_i for all i. Modulo $\gamma_{c+1}(M_n)$, $g\beta \equiv x_1 \prod_{2\leq i\leq n} [x_1, x_i]^{p_i}$ and it suffices to prove that $g = x_1 \prod_{2\leq i\leq n} [x_1, x_i]^{p_i}$ can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive element of M_n. For each $i \neq 1$, choose $j \neq 1, i$, and consider the system $\{h_{i1}, \dots, h_{in}\}$ with

$$h_{i1} = x_1 [x_1, x_i]^{p_i} [x_i, x_j]^{(x_i-1)^{p_i}}, \quad h_{ij} = x_j [x_1, x_i]^{-p_i^2} [x_i, x_j]^{p_i (x_i-1) p_i^2},$$

$h_{ik} = x_k$, $k \neq 1, j$. Then there is a tame automorphism $\tau_i \in \mathrm{Aut}(M_n)$ which maps x_1 to x_1, x_i to x_2 and x_j to x_3. This automorphism transforms the system $\{h_{i1}, \dots, h_{in}\}$ to a system of the form $\{g_1, \dots, g_n\}$ where

$$\begin{aligned} g_1 &= x_1 [x_1, x_2]^p [x_2, x_3]^{(x_2-1)^p}, \\ g_3 &= x_3 [x_1, x_2]^{-p^2} [x_2, x_3]^{p-(x_2-1)p^2}, \\ g_i &= x_i, \quad i \neq 1, 3, \ p \in \Delta^{c-2}(M_n). \end{aligned}$$

By Lemma 6 the system $\{g_1, \dots, g_n\}$ is a basis for M_n. Thus there is an automorphism $\alpha_i \in \mathrm{Aut}(M_n)$ such that $\alpha_i(x_1 [x_1, x_i]^{p_i}) \equiv x_1 \bmod \gamma_{c+1}(M_n)$. By successive applications, we obtain $\alpha_2 \dots \alpha_n (x_1 \prod_{2\leq i\leq n} [x_1, x_i]^{p_i}) \equiv x_1 \bmod \gamma_{c+1}(M_n)$. This completes the proof of the theorem.

For the general case, we prove the following.

THEOREM B. *For $n \geq 4$ and $m \leq n-2$, every primitive system $\mathbf{g} = \{g_1, \ldots, g_m\}$ mod $\gamma_{c+1}(F_n)F''$ can be lifted (via $\gamma_{c+1}(F_n)F''$) to a primitive system of F_n.*

PROOF. Note that a system $\mathbf{g} = \{g_1, \ldots, g_m\}$ is primitive mod $\gamma_c(F_n)F_n''$ if and only if there is an automorphism $\tau \in \mathrm{Aut}(F_n)$ such that $\{g_1\tau, \ldots, g_m\tau\}$ is of the form $\{x_1u_1, \ldots, x_mu_m\}$, $u_i \in F_n'$. Thus without loss of generality we can assume that $\mathbf{g} = \{x_1u_1, \ldots, x_mu_m\}$, $u_i \in F_n'$. When $n \geq 4$, every automorphism of F/F'' is tame (Bachmuth and Mochizuki [2], Roman'kov [8]). It suffices, therefore, to prove that for $m \leq n-2$ every IA-system $\mathbf{g} = \{x_1u_1, \ldots, x_mu_m\}$, $u_i \in M_n'$, can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive system of M_n. The case $m = 1$ follows from Theorem A. For $m \geq 2$, we consider an arbitrary IA-system $\mathbf{g} = \{x_1u_1, x_4u_4, \ldots, x_{m+2}u_{m+2}\}$ of m elements. By Lemma 5 we may further assume that $\mathbf{g}$ is of the form $\{x_1v_1, x_4, \ldots, x_{m+2}\}$, where $v_1 \in \gamma_c(M_n)$. As in the proof of Theorem A we may transform the system so that x_1v_1 assumes the form:

$$x_1v_1 = x_1 \prod_{2 \leq i \leq n} [x_1, x_i]^{p_i} \text{ with } p_i \in \Delta^{c-2}(M_n).$$

By Lemma 6,

$$\{x_1[x_1,x_2]^{p_2}[x_2,x_3]^{(x_2-1)p_2}, x_2, x_3[x_1,x_2]^{-p_2^2}[x_2,x_3]^{p_2-(x_2-1)p_2^2}, x_4, \ldots, x_n\}$$

is a basis for M_n, which proves that $\{x_1[x_1,x_2]^{p_2}, x_4, \ldots, x_{m+2}\}$ can be lifted to a primitive system of M_n. Further, by Lemma 6, for each $i > 3$, the system $\{x_1[x_1,x_i]^{p_i}[x_i,x_3]^{(x_i-1)p_i}$, $x_2, x_3[x_1,x_i]^{-p_i^2}[x_i,x_3]^{p_i-(x_i-1)p_i^2}, x_4, \ldots, x_n\}$ is a basis for M_n and for $i = 3$, the system

$$\{x_1[x_1,x_3]^{p_3}[x_3,x_2]^{(x_3-1)p_3}, x_2[x_1,x_3]^{-p_3^2}[x_3,x_2]^{p_3-(x_3-1)p_3^2}, x_3, \ldots, x_n\}$$

is a basis for M_n. Thus there exist automorphisms $\alpha_i \in \mathrm{Aut}(M_n)$ such that with $\alpha = \alpha_2 \ldots \alpha_n$, we obtain mod $\gamma_{c+1}(M_n)$ the congruences $\alpha\left(x_1 \prod_{2\leq i \leq n}[x_1,x_i]^{p_i}\right) \equiv \alpha x_1$, $\alpha(x_i) \equiv x_i$, $i \neq 1,2,3$. Thus $\{x_1v_1, x_4, \ldots, x_{m+2})$, where $v_1 \in \gamma_c(M_n)$, can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive system of M_n and consequently, by Lemma 5, $\{g_1, g_4, \ldots, g_{m+2}\}$ can be lifted (via $\gamma_{c+1}(M_n)$) to a primitive system of M_n.

REMARK C. *For each $n \geq 3$ there exists an* IA-*system of $n-1$ elements of $M_{n,c}$ which cannot be lifted (via $\gamma_{c+1}(M_n)$) to a primitive system of M_n. Thus the restriction $m \leq n-2$ in Theorem B cannot be relaxed.*

DETAILS. Choose $g_1 = x_1[x_1,x_3,x_3]$, $g_i = x_i$, $i \neq 1,3$. We show that for any choice of $g_3 = x_3u$, $u \in M_n'$, and any choice of elements $w_i \in \gamma_4(M_n)$, $i = 1, \ldots, n$, the Jacobian matrix $J(\mathbf{g})$ of the system $\mathbf{g} = \{g_1w_1, \ldots, g_nw_n\}$ is not invertible. The matrix $J(\mathbf{g})$ has the form:

$$\begin{bmatrix} 1+(a_3-1)^2+\pi_{11} & \pi_{12} & -(a_1-1)(a_3-1)+\pi_{13} & \cdots & \pi_{1,n-1} & \pi_{1n} \\ \pi_{21} & 1+\pi_{22} & \pi_{23} & \cdots & \pi_{2,n-1} & \pi_{2n} \\ \pi_{31^*} & \pi_{32^*} & 1+\pi_{33^*} & \cdots & \pi_{3,n-1^*} & \pi_{3n^*} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \pi_{n1} & \pi_{n2} & \pi_{n3} & \cdots & \pi_{n,n-1} & 1+\pi_{nn} \end{bmatrix}$$

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where each $\pi_{ij} \in \Delta^3(A)$ and $\pi_{3i^*} \in \Delta(A)$. If $J(\mathbf{g})$ is invertible then it remains invertible under the endomorphism mapping a_3 to a_3 and a_i to 1 for each $i \neq 3$. Since, for any i, $\sum_{j=1}^n (\partial w_i/\partial x_j)(a_j - 1) = \sum_{j=1}^n \pi_{ij}(a_j - 1) = 0$, it follows that π_{i3} gets mapped to 0 under the above endomorphism. Thus the resulting matrix $J(\mathbf{g})^*$ is of the form

$$\begin{bmatrix} 1 + (a_3 - 1)^2 + \pi'_{11} & \pi'_{12} & 0 & \pi'_{1,n-1} & \pi'_{1n} \\ \pi'_{21} & 1 + \pi'_{22} & 0 & \pi'_{2,n-1} & \pi'_{2n} \\ \pi'_{31^*} & \pi'_{32^*} & 1 & \pi'_{3,n-1^*} & \pi'_{3n^*} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \pi'_{n1} & \pi'_{n2} & 0 & \pi'_{n,n-1} & 1 + \pi'_{nn} \end{bmatrix}$$

where $\pi'_{ij} \in \Delta^3\langle a_3 \rangle$ and $\pi'_{3i^*} \in \Delta\langle a_3 \rangle$. The determinant of $J(\mathbf{g})^*$ is of the form $1 + (a_3 - 1)^2 + (a_3 - 1)^3\pi$ and if it is invertible then we must have $1 + (a_3 - 1)^2 + (a_3 - 1)^3\pi = a_3^k$ for some $k \in \mathbf{Z}$. Working modulo $\Delta^2\langle a_3 \rangle$ shows that k must be zero, so that $(a_3 - 1)^2 + (a_3 - 1)^3\pi = 0$ which, however, is not possible in the cyclic group ring $\mathbf{Z}\langle a_3 \rangle$.

Primitive lifting in F_3 of a single element. Let $g = x_1 u, u \in \gamma_c(M_3)$. By Theorem A, g can be lifted (via $\gamma_{c+1}(M_3)$) to a primitive element of M_3. Since M_3 admits wild automorphisms (Chein [4]), lifting g to a primitive element of F_3 does not follow instantly as was the case for $n \geq 4$. For simplicity of notation we let $M = M_3$ be generated by x, y, z. In preparation we first prove,

THEOREM D. *Every* IA-*element of the form $g = x[y,z]^{p(x,y,z)}$ can be lifted (via F_3'') to a primitive element of F_3.*

PROOF. The proof consists in exhibiting a tame automorphism of M which maps x to $x[y,z]^{p(x,y,z)}$. For each $i,j,k \in \mathbf{Z}$, consider the tame automorphisms α_{jk} and β_i of M given by $\alpha_{jk} = \{x \to x[y,z]^{y^j z^k}, y \to y, z \to z\}$, $\beta_i = \{x \to x, y \to x^{-i} y x^i, z \to x^{-i} z x^i\}$, and define the tame automorphism $\delta_{ijk} = \beta_i^{-1}\alpha_{jk}\beta_i$. It is easy to see that each δ_{ijk} is of the form $\delta_{ijk} = \{x \to x[y,z]^{x^i y^j z^k}, y \to y^u, z \to z^u\}$, $u \in M'$. If $\delta_{i'j'k'}$ is also of the form $\delta_{i'j'k'} = \{x \to x[y,z]^{x^{i'} y^{j'} z^{k'}}, y \to y^{u'}, z \to z^{u'}\}$, $u' \in M'$, then we see that $\delta_{ijk}\delta_{i'j'k'} = \{x \to x[y,z]^{x^i y^j z^k + x^{i'} y^{j'} z^{k'}}, y \to y^{u'''}, z \to z^{u'''}\}$. Since $p(x,y,z)$ is a $\mathbf{Z}$-linear sum of group elements of the form $x^i y^j z^k$, $i,j,k \in \mathbf{Z}$, it follows that there is a tame automorphism $\mu \in gp\{\delta_{ij}, i,j,k \in \mathbf{Z}\}$ which has the form $\mu = \{x \to x[y,z]^{p(x,y,z)}, y \to y^w, z \to z^w\}$, $w = w(x,y,z) \in M'$. This completes the proof of the theorem.

We can now prove the following main result of this section.

THEOREM E. *Every primitive element of $M_{3,c}, c \geq 2$, can be lifted (via $\gamma_{c+1}(M)F''$) to a primitive element of F_3*

PROOF. We may assume that $c \geq 3$ (the case $c = 2$ being trivial) and by Lemma 5 that the given primitive element g has the form $g = xu$, $u \in \gamma_c(M)$. Since u is of the form

$$u = [y,z]^{q(x,y,z)}[x,y,z]^{p(x,y,z)}[x,z,z]^{p'(x,y,z)}[x,y,z]^{p''(x,y,z)},$$

with $q(x,y,z) \in \Delta^{c-2}(M)$ and $p(x,y,z)$, $p'(x,y,z)$, $p''(x,y,z) \in \Delta^{c-3}(M)$, it suffices to prove that each of the elements of the form $x[y,z]^{q(x,y,z)}$, $x[x,y,y]^{p(x,y,z)}$, $x[x,y,z]^{p(x,y,z)}$, with $q(x,y,z) \in \Delta^{c-2}(M)$ and $p(x,y,z) \in \Delta^{c-3}(M)$ can be lifted (via $\gamma_{c+1}(M)$) to primitive elements of F_3.

Primitive lifting of $x[y,z]^{q(x,y,z)}$ (mod $\gamma_{c+1}(M)$) follows from Theorem D. For primitive lifting of $x[x,y,y]^{p(x,y,z)}$ (mod $\gamma_{c+1}(M)$) we only need to establish a tame automorphism of M which maps x to $x[x,y,y]^{p(x,y,z)}$ (mod $\gamma_{c+1}(M)$). Indeed, for the given $p(x,y,z) \in \Delta^{c-3}(M)$ we choose, using proof of Theorem D, a tame automorphism μ of M given by $\mu = \{x \longrightarrow x[y,z]^{-p(x,y,z)},\ y \longrightarrow y^w,\ z \longrightarrow z^w\}$, $w = w(x,y,z) \in M'$ and a tame automorphism $\lambda = \{x \longrightarrow x,\ y \longrightarrow y,\ z \longrightarrow z[x,y]\}$ of M. Then modulo $\gamma_{c+1}(M)$ we observe that

$$\begin{aligned}\mu(x) &= x[y,z]^{-p(x,y,z)},\\ \lambda\big(\mu(x)\big) &= x\big[y,z[x,y]\big]^{-p(x,y,z)} \equiv x[x,y,y]^{p(x,y,z)}[y,z]^{-p(x,y,z)},\\ \mu^{-1}\Big(\lambda\big(\mu(x)\big)\Big) &\equiv x[x,y,y]^{p(x,y,z)}.\end{aligned}$$

Also, $\mu^{-1}\Big(\lambda\big(\mu(y)\big)\Big) \equiv y$ (mod M'), $\mu^{-1}\Big(\lambda\big(\mu(z)\big)\Big) \equiv z$ (mod M'). Thus $u\lambda\mu^{-1}$ has the required form:

$$\mu\lambda\mu^{-1} = \{x \longrightarrow x[x,y,y]^{p(x,y,z)},\ y \longrightarrow yu,\ z \longrightarrow zv\}$$

modulo $\gamma_{c+1}(M)$.

For primitive lifting of $x[x,y,z]^{p(x,y,z)}$, we choose $\mu = \{x \longrightarrow x[y,z]^{-p(x,y,z)},\ y \longrightarrow y^w,\ z \longrightarrow z^w\}$, $w = w(x,y,z) \in M'$, as before and choose $\rho = \{x \longrightarrow x,\ y \longrightarrow y[y,x],\ z \longrightarrow z\}$. Then, modulo $\gamma_{c+1}(M)$, $\mu\rho\mu^{-1}$ has the required form $\mu\rho\mu^{-1} = \{x \longrightarrow x[x,y,z]^{p(x,y,z)},\ y \longrightarrow yu,\ z \longrightarrow zv\}$. This completes the proof of Theorem E.

Concluding Remarks. Since every IA-automorphism of M_2 is inner (Bachmuth [1]), $g = x_1u$ can be lifted to a primitive element of M_2 if and only if u is of the form $[x_1, v]$. Thus, for $c \geq 3$, not every primitive element of $M_{2,c}$ can be lifted to a basis of M_2.

The existence of non-tame automorphisms of M_3 was first shown by Chein [4]. Specifically, the automorphism $\{x \longrightarrow x[y,z,x,x],\ y \longrightarrow y,\ z \longrightarrow z\}$ of M_3 cannot be lifted to an automorphism of the free group F_3. It is easily seen that every endomorphism in M_3 of the form $\{x \longrightarrow x[y,z]^{p(x,y,z)},\ y \longrightarrow y,\ z \longrightarrow z\}$ is an automorphism of M_3. So, for each $p(x,y,z) \in \mathbf{Z}M_3$, the element $x[y,z]^{p(x,y,z)}$ is primitive in M_3 and we call it a *Chein element* of M_3. By Theorem D, it follows that every Chein element of M_3 can be lifted to a primitive element of F_3. It is natural to ask: can every primitive element of M_3 be lifted to a primitive element of F_3? Finally, by Timoshenko's results primitivity in M_n, $n \geq 4$, is algorithmically decidable. We conclude by asking: is primitivity in M_3 algorithmically decidable?

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University of Manitoba
Winnipeg, Manitoba R3T 2N2

University of Manitoba
Winnipeg, Manitoba R3T 2N2

Kompleksny Otdel
Pr. Mira 19-a
Omsk 644050
USSR