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PERTURBATION OF NODES AND POLES IN CERTAIN RATIONAL INTERPOLANTS

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Recently, E.B. Saff and A. Sharma proved a result on Walsh's type equiconvergence for certain sequences of rational interpolants having uniformly distributed poles and nodes of interpolation. Here we examine the sensitivity of this result to a slight perturbation of the poles and nodes. Some problems related to our work are also formulated.

1. Introduction

Our object in this paper is to obtain an analogue of the Saff-Sharma extension ([2], Theorem 2.3) of Walsh's equiconvergence theorem ([4], p. 153) when the nodes of interpolation and the poles of the rational interpolants are slightly perturbed. Our methods are slight variants on those of Szabados and Varga ([3], Theorem 2) who obtained similar results for complex interpolating polynomial sequences.

2. Notation and main result

Let $\omega_{j,n} (j = 1,\dots,n)$ denote the n^{th} roots of unity and let

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$G_{\rho,\sigma}(\rho,\sigma > 1)$ denote the class of infinite triangular matrices S whose n^{th} row S^n is given by $S^n = \{\sigma_{k,n}\}_{k=1}^{n}$ where

(2.1) $$|\sigma_{k,n} - \sigma\omega_{k,n}| < \rho^{-n} , \quad k = 1,\dots,n ; \; n = 1,2,3,\dots .$$

For any fixed integer $m \geq -1$, we introduce another class $F_{\rho,m}$ of triangular matrices whose n^{th} row $Z^n = \{z_{k,n'}\}_{k=1}^{n'}$, $(n' = n+m+1)$, satisfies the inequality

(2.2) $$|z_{k,n'} - \omega_{k,n'}| < \rho^{-n} , \quad k = 1,\dots,n'(=n+m+1) ; \; n = 1,2, \dots$$

When $\sigma_{k,n} - \sigma\omega_{k,n} = 0$, $(k = 1,2,\dots,n)$, we shall denote the matrix by S^* and when $z_{k,n'} = \omega_{k,n'}$, $(k = 1,2,\dots,n')$, we shall denote the matrix by Z^*. We shall denote by $\hat{S}$ the matrix whose n^{th} row $\hat{S}^n$ is given by

$$\hat{S}^n = \{\underbrace{0,\dots,0}_{m+1} , \bar{\sigma}_{1,n}^{-1} , \bar{\sigma}_{2,n}^{-1} ,\dots, \bar{\sigma}_{n,n}^{-1}\} .$$

We shall associate with the rows S^n and Z^n, the monic polynomials

(2.3) $$\gamma(z,S^n) = \prod_{k=1}^{n} (z-\sigma_{k,n}) \quad \text{and} \quad \gamma(z,Z^n) = \prod_{k=1}^{n'} (z-z_{k,n'}) .$$

Similarly, we have the monic polynomials

$$\gamma(z,S^{*n}) = z^n - \sigma^n , \; \gamma(z,Z^{*n}) = z^{n'} - 1 , \; \gamma(z,\hat{S}^{*n}) = z^{m+1}(z^n - \sigma^{-n}) .$$

If $f \in A_\rho$ (the class of functions analytic in $|z| < \rho$ but not in $|z| \leq \rho$, $\rho > 1$), $Z \in F_{\rho,m}$ and $S \in G_{\rho,\sigma}$, let $R_{n+m,n}(z,f,Z,S)$ be the rational function of the form

(2.4) $$R_{n+m,n}(z,f,Z,S) = \frac{B_{n+m,n}(z,f,Z)}{\gamma_n(z,S)} , \quad (B_{n+m,n}(z,f,Z) \in \pi_{n+m}) ,$$

which interpolates $f(z)$ in the nodes $\{z_{k,n'}\}_{k=1}^{n'}$, the zeros of $\gamma(z,Z^n)$. From this it follows that

(2.5) $$R_{n+m,n}(z,f,Z,S) = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma(t,S^n)}{\gamma(z,S^n)} \left[\frac{\gamma(t,Z^n) - \gamma(z,Z^n)}{\gamma(t,Z^n)} \right] \frac{f(t)}{t-z} \, dt ,$$

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where Γ is a circle $|t| = \rho_1$, $1 < \rho_1 < \rho$.

If we replace Z by $\hat{S}$ in (2.4), then from a theorem of Walsh ([4], p. 224) we know that the rational function $R_{n+m,n}(z,f,\hat{S},S)$ is the best L_2-approximant to f on $|z| = 1$ over all rational functions of the form (2.4).

We shall prove

THEOREM 2.1. *Let* $\sigma > 1$ *and let the integer* $m \geq -1$ *be fixed. If* $f \in A_\rho$, $1 < \rho < \infty$, *and if* S *and* Z *are infinite triangular matrices in* $G_{\rho,\sigma}$ *and* $F_{\rho,m}$ *respectively, then*

$$\lim_{n\to\infty} \Delta(z,f,Z^n,S^n) = 0 \begin{cases} |z| < \rho^2, & \text{if } \sigma \geq \rho^2, \\ |z| \neq \sigma, & \text{if } \sigma < \rho^2, \end{cases} \tag{2.6}$$

where

$$\Delta(z,f,Z^n,S^n) := R_{n+m,n}(z,f,Z,S) - R_{n+m,n}(z,f,\hat{S},S). \tag{2.7}$$

The convergence in (2.6) is uniform and geometric on compact subsets of the region described above.

3. Representation of $\Delta(z,f,Z^n,S^n)$

The proof of Theorem 2.1 requires some estimates similar to those given by Szabados and Varga in [3]. For this purpose, we write

$$\Delta(z,f,Z^n,S^n) = \sum_{j=1}^{5} \Delta_{n,j}(z,f) \tag{3.1}$$

where

$$\begin{cases} \Delta_{n,1}(z,f) := R_{n+m,n}(z,f,Z,S) - R_{n+m,n}(z,f,Z^*,S), \\ \Delta_{n,2}(z,f) := R_{n+m,n}(z,f,Z^*,S) - R_{n+m,n}(z,f,Z^*,S^*), \\ \Delta_{n,3}(z,f) := R_{n+m,n}(z,f,Z^*,S^*) - R_{n+m,n}(z,f,\hat{S}^*,S^*), \\ \Delta_{n,4}(z,f) := R_{n+m,n}(z,f,\hat{S}^*,S^*) - R_{n+m,n}(z,f,\hat{S}^*,S), \\ \Delta_{n,5}(z,f) := R_{n+m,n}(z,f,\hat{S}^*,S) - R_{n+m,n}(z,f,\hat{S},S). \end{cases} \tag{3.2}$$

The third difference $\Delta_{n,3}(z,f)$ in (3.2) is the same as that considered by Saff and Sharma ([2], Theorem 2.3).

The lemma given below is based on the formula (2.5).

LEMMA 3.1. *If* $f \in A_\rho$, *then the differences*, $\Delta_{n,j}$, *defined in* (3.2) *have the following integral representations:*

$$\Delta_{n,1}(z,f) = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma(t,S^n)}{\gamma(z,S^n)} \left(\frac{z^{n+m+1}-1}{t^{n+m+1}-1}\right) \left(\frac{f(t)}{z-t}\right) V_{n,m}(t,z)dt ,$$

$$\Delta_{n,2}(z,f) = \frac{1}{2\pi i} \int_\Gamma \frac{t^n-\sigma^n}{z^n-\sigma^n} \left(\frac{t^{n+m+1} - z^{n+m+1}}{t^{n+m+1} - 1}\right) \left(\frac{f(t)}{t-z}\right) U_n(t,z)dt ,$$

$$\Delta_{n,4}(z,f) = \frac{1}{2\pi i} \int_\Gamma \frac{t^n-\sigma^n}{\sigma^n-z^n} \left(\frac{\gamma(t,\hat{S}^{*n}) - \gamma(z,\hat{S}^{*n})}{\gamma(t,\hat{S}^{*n})}\right) \left(\frac{f(t)}{t-z}\right) U_n(t,z)dt ,$$

$$\Delta_{n,5}(z,f) = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma(t,S^n)}{\gamma(z,S^n)} \left(\frac{\gamma(z,\hat{S}^{*n})}{\gamma(t,\hat{S}^{*n})}\right) \left(\frac{f(t)}{t-z}\right) W_{n,m}(t,z)dt ,$$

where

$$(3.3)\qquad \begin{cases} V_{n,m}(t,z) := \left(\dfrac{t^{n+m+1}-1}{z^{n+m+1}-1}\right) \dfrac{\gamma(z,Z^n)}{\gamma(t,Z^n)} - 1 , \\[2ex] U_n(t,z) := \dfrac{z^n-\sigma^n}{t^n-\sigma^n} \dfrac{\gamma(t,S^n)}{\gamma(z,S^n)} - 1 , \\[2ex] W_{n,m}(t,z) := \left(\dfrac{t^n-\sigma^{-n}}{z^n-\sigma^{-n}}\right) \dfrac{t^{m+1}\gamma(z,\hat{S}^n)}{z^{m+1}\gamma(t,\hat{S}^n)} - 1 . \end{cases}$$

4. Some upper bounds

LEMMA 4.1. *Let* $\rho, \sigma > 1$ *and let* m *be an integer* ≥ -1 . *Suppose* (i) $1 < |t| < \rho$, *if* $\sigma \geq \rho$ *and* (ii) $\sigma < |t| < \rho$, *if* $\sigma < \rho$. *Then we have*

$$(4.1)\qquad |V_{n,m}(t,z)| < c_1 n\rho^{-n} \quad \text{for} \quad |z| > 1 ,$$

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$$(4.2)\qquad \begin{cases} |U_n(t,z)| < c_2 n\rho^{-n} , & \\ & \text{for } |z| > 1 , |z| \neq \sigma \\ |W_{n,m}(t,z)| \le c_3 n\rho^{-n} , & \end{cases}$$

where c_1, c_2, c_3 *are positive constants independent on* n .

Proof. Using the definition of $\gamma(z,Z^n)$ we can rewrite $V_{n,m}(t,z)$ as

$$V_{n,m}(t,z) = \prod_{k=1}^{n'} \{1 + \frac{z_{k,n'} - \omega_{k,n'}}{t - z_{k,n'}}\}\{1 + \frac{\omega_{k,n'} - z_{k,n'}}{z - \omega_{k,n'}}\} - 1$$

where $\omega_{k,n'}$, $(n' = n+m+1)$, are the n'th roots of unity. Set

$$a_{n'} := \max_{1\le k\le n'} |z_{k,n'}| , \quad n = 1,2,3 \ldots$$

If we let $|t| = \rho_1$, then for sufficiently large n, $|t| = \rho_1 > a_{n'}$, and $|t-a_{n'}| \ge \rho_1 - a_{n'}$. Also, for $|z| > 1$, we have $|z-\omega_{k,n'}| > |z|-1$. This together with (2.2) gives us

$$|V_{n,m}(t,z)| \le [(1 + \frac{\rho^{-n}}{\rho_1 - a_{n'}})(1 + \frac{\rho^{-n}}{|z| - 1})]^{n'} - 1 .$$

From (2.2) we see that $a_{n'} \to 1$ as $n \to \infty$. If we set $d_0 := \min(\rho_1 - 1, |z|-1)$, then for sufficiently large n , it is easy to see that

$$|V_{n,m}(t,z)| \le (1+ \frac{\rho^{-n}}{d_0})^{2n'} -1 \le \frac{6(n+m+1)}{d_0\rho^n} .$$

This proves (4.10).

In order to prove (4.2), we observe that

$$U_n(t,z) = \prod_{k=1}^{n} \{1+ \frac{\sigma_{k,n} - \sigma\omega_{k,n}}{z - \sigma_{k,n}}\}\{1+ \frac{\sigma\omega_{k,n} - \sigma_{k,n}}{t - \sigma\omega_{k,n}}\} - 1 ,$$

and

$$W_{n,m}(t,z) = \prod_{k=1}^{n} \{1+ \frac{\bar{\sigma}_{k,n}^{-1} - \sigma^{-1}\omega_{k,n}}{t - \bar{\sigma}_{k,n}^{-1}}\}\{1+ \frac{\sigma^{-1}\omega_{k,n} - \bar{\sigma}_{k,n}^{-1}}{z - \sigma^{-1}\omega_{k,n}}\} - 1 .$$

Following closely the analysis given above for the proof of (4.1) we can deduce the relations (4.2) on using (2.1). □

LEMMA 4.2. *Under the hypotheses (i) and (ii) of Lemma 4.1, we have the following estimates for all sufficiently large* n :

$$(4.3) \qquad \left| \frac{\gamma(z,\hat{S}^n)}{\gamma(t,\hat{S}^n)} \right| \le c_3 \left|\frac{z^n-\sigma^{-n}}{t^n-\sigma^{-n}}\right| \quad \text{for} \quad |z| , |t| > \sigma^{-1} ,$$

and

$$(4.4) \qquad \left| \frac{\gamma(t,S^n)}{\gamma(z,S^n)} \right| \le c_4 \left|\frac{t^n-\sigma^n}{z^n-\sigma^n}\right| , \quad \text{for} \quad |z| , |t| \ne \sigma ,$$

where the positive constants c_3 *and* c_4 *are independent of* n .

Proof. If we write $z - \bar{\sigma}_{k,n}^{-1}$ as

$$z - \bar{\sigma}_{k,n}^{-1} = (z-\sigma^{-1}\omega_{k,n})\{1+ \frac{\sigma^{-1}\omega_{k,n} - \bar{\sigma}_{k,n}^{-1}}{z - \sigma^{-1}\omega_{k,n}}\}$$

then

$$\frac{\gamma(z,\hat{S}^n)}{z^{m+1}} := \prod_{k=1}^{n} (z-\bar{\sigma}_{k,n}^{-1}) = (z^n - \sigma^{-n}) \prod_{k=1}^{n} \{1+ \frac{\sigma^{-1}\omega_{k,n} - \bar{\sigma}_{k,n}^{-1}}{z - \sigma^{-1}\omega_{k,n}}\}$$

Similarly, we have

$$\frac{\gamma(t,\hat{S}^n)}{t^{m+1}} = (t^n - \sigma^{-n}) \prod_{k=1}^{n} \{1+ \frac{\sigma^{-1}\omega_{k,n} - \bar{\sigma}_{k,n}^{-1}}{t - \sigma^{-1}\omega_{k,n}}\} .$$

From (2.1), it is easy to see that $|\bar{\sigma}_{k,n}^{-1} - \sigma^{-1}\omega_{k,n}| < \sigma^{-1}\rho^{-n}$. Hence

$$\left| \frac{t^{m+1}\gamma(z,\hat{S}^n)}{z^{m+1}\gamma(t,\hat{S}^n)} \right| \le \left| \frac{z^n-\sigma^{-n}}{t^n-\sigma^{-n}} \right| \left\{ \frac{1+\sigma^{-1}\rho^{-n}(|z|-\sigma^{-1})^{-1}}{1-\sigma^{-1}\rho^{-n}(|t|+\sigma^{-1})^{-1}} \right\}^n ,$$

where the second factor on the right side approaches 1 as $n \to \infty$. This shows that there exists a positive constant c_3, independent of n, such that

$$\left| \frac{\gamma(z,\hat{S}^n)}{\gamma(t,\hat{S}^n)} \right| \leq c_3 \left| \frac{z^n - \sigma^{-n}}{t^n - \sigma^{-n}} \right| ,$$ for all sufficiently large n, which

is the desired estimate (4.3).

Similarly, we can prove (4.4) by repeating the above argument for $\gamma(t,S^n)$ and $\gamma(z,S^n)$. □

5. Region of convergence of $\{\Delta_{n,j}(z,f)\}_{j=1}^5$

As pointed out earlier, the sequence $\{\Delta_{n,3}(z,f)\}_1^\infty$ (compare (3.2)) is identical to the one considered by Saff and Sharma ([2], Theorem 2.3). We shall show in the following proposition that the sequences $\{\Delta_{n,j}(z,f)\}_{n=1}^\infty$, $(j = 1,2,\ldots,5)$, have the same region of convergence.

PROPOSITION 5.1. *Let* $\sigma > 1$, $\rho > 1$ *and an integer* $m \geq -1$ *be given. If* $Z \in F_{\rho,m}$, $S \in G_{\rho,\sigma}$ *and* $f \in A_\rho$ *then for* $j = 1,2,3,4,5$

$$(5.1) \qquad \lim_{n\to\infty} \Delta_{n,j}(z,f) = 0 \qquad \begin{cases} |z| < \rho^2 \;, & \text{if } \sigma \geq \rho^2 \;, \\ |z| \neq \sigma \;, & \text{if } \sigma < \rho^2 \;, \end{cases}$$

where $\Delta_{n,j}(z,f)$ *is the same as defined in (3.2). The convergence in (5.1) is uniform and geometric on any compact subset of the regions described above.*

Proof. For $j = 3$, (5.1) is known [2]. Next we consider the case when $j = 1$. From Lemma (3.1) we know that

$$\Delta_{n,1}(z,f) = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma(t,S^n)}{\gamma(z,S^n)} \left(\frac{z^{n+m+1}-1}{t^{n+m+1}-1}\right) \left(\frac{f(t)}{z-t}\right) V_{n,m}(t,z)dt$$

where Γ is a circle $|t| = \rho_1$, $1 < \rho_1 < \rho$. An application of (4.1) and (4.4) to the above integral shows that

(5.2) $$|\Delta_{n,1}(z,f)| \le \frac{nC}{\rho^n}\left(\frac{|z|^{n+m+1}+1}{\rho_1^{n+m+1}-1}\right)\frac{\rho_1^n+\sigma^n}{|z^n-\sigma^n|},$$

for all n sufficiently large. Here C is a positive constant independent of n. If $\sigma \ge \rho^2$ and $\tau \ge \rho$, we get

$$\overline{\lim_{n\to\infty}}\{\max_{|z|=\tau}|\Delta_{n,1}(z,f)|\}^{1/n} \le \frac{\tau}{\rho_1\rho}.$$

Letting $\rho_1 \to \rho$, we conclude that

$$\lim_{n\to\infty}\Delta_{n,1}(z,f) = 0, \quad \text{for} \quad |z| < \rho^2.$$

If $\sigma < \rho^2$, again it follows from (5.2) that

$$\lim_{n\to\infty}\Delta_{n,1}(z,f) = 0, \quad \text{for all } z \text{ with } |z| \ne \sigma.$$

This completes the proof for the sequence $\{\Delta_{n,1}(z,f)\}_1^\infty$.

We omit the proof for the remaining three cases when $j = 2,4$ and 5.

6. Proof of Theorem 2.1

From (3.1) we know that

$$\Delta(z,f,Z^n,S^n) = \sum_{j=1}^{5}\Delta_{n,j}(z,f).$$

Therefore,

$$|\Delta(z,f,Z^n,S^n)| \le \sum_{j=1}^{5}|\Delta_{n,j}(z,f)|.$$

A straightforward calculation on using Proposition 5.1 shows that

$$\lim_{n\to\infty}\Delta(z,f,Z^n,S^n) = 0 \quad \begin{cases} |z| < \rho^2, & \text{if } \sigma \ge \rho^2, \\ |z| \ne \sigma, & \text{if } \sigma < \rho^2. \end{cases}$$

This completes the proof. □

7. Some problems

It will be interesting to know about the sharpness as well as some extensions of Theorem 2.1. More precisely, we ask the following questions:

1. Suppose that S and Z are infinite triangular matrices in $G_{\rho,\sigma}$ and $F_{\rho,m}$ respectively. Let $\hat{z}$ be any point fixed outside the circle $|z| = \rho^2$. Is it possible to constuct a function $\hat{f} \in A_\rho$ so that the sequence $\{\Delta(\hat{z},\hat{f},Z^n,S^n)\}_{n=1}^{\infty}$ given by (2.7) turns out unbounded?

2. Our main result may be looked upon as a general form of Theorem 2.3 [2]. Is it possible to determine a Saff-Sharma type extension ([2], Theorem 3.3) for this result?

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