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ON THE STABILITY OF SOLUTIONS FOR THE $p(x)$-LAPLACIAN EQUATION AND SOME APPLICATIONS TO OPTIMISATION PROBLEMS WITH STATE CONSTRAINTS

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Abstract

We consider the stability of solutions for a family of Dirichlet problems with (p, q)-growth conditions. We apply the results obtained to show continuous dependence on a functional parameter and the existence of an optimal solution in a control problem with state constraints governed by the $p(x)$-Laplacian equation.

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1. Introduction

We shall show the stability of solutions to the following family of Dirichlet problems:

$$\begin{aligned} -\operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u_k(x)\right) &= F_u^k(x, u(x)), \quad u \in W_0^{1,p(x)}(\Omega), \\ u(x)|_{\partial\Omega} &= 0, \end{aligned} \tag{1.1}$$

where $k = 0, 1, \dots, \Omega \subset \mathbb{R}^N$ is a bounded region, $p, q \in C(\overline{\Omega})$, $1/p(x)+1/q(x) = 1$ for $x \in \Omega$ and $W_0^{1,p(x)}(\Omega)$ denotes the generalised Orlicz-Sobolev space, see [4, 5]. Let $p^- = \inf_{x\in\Omega} p(x) > N > 2$ and $p^+ = \sup_{x\in\Omega} p(x)$. Here F^k and F_u^k are Carathéodory functions with F^k being convex with respect to u on a certain interval in which F_u^k satisfies some general growth conditions. The existence of a solution is obtained by a variational method from [7]. We modify some growth assumptions and considerably simplify the proof of the main existence theorem which is now obtained as a direct consequence of a respective variational principle. Later we consider the stability of

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solutions for a family of problem type (1.1). The stability result is further applied to show that solutions to equations like (1.1) depend continuously on a functional parameter. By stability we mean conditions under which from a sequence $\{u_k\}_{k=1}^{\infty}$ being a solution to (1.1) one may choose a subsequence converging strongly in $W_0^{1,p(x)}$ to a certain $\bar{u}$ which is a solution to the problem

$$-\operatorname{div}\left(|\nabla\bar{u}(x)|^{p(x)-2}\nabla\bar{u}(x)\right) = F_u^0(x,\bar{u}(x))$$
$$u(x)\big|_{\partial\Omega} = 0,$$

where also $u_k \rightrightarrows \bar{u}$ up to a subsequence.

Further we consider an optimisation problem with state governed by a forced $p(x)$-Laplacian equation. It is shown that under some mild assumptions on the objective functional there exists an optimal solution to a certain control problem with state constraints.

Variational problems with (p,q)-growth conditions are important in applications in elastic mechanics and electro-rheological fluid dynamics (see [11, 14] and references therein) and have been studied in the last few years, see [3, 8] for existence results in the supercritical case which are based on the application of a Palais-Smale condition and a mountain pass geometry. Such an approach requires that the growth of a nonlinearity is somehow restricted which need not be the case in the present paper. It also does not allow one to prove that solutions are stable in the sense defined above.

To the best knowledge of the authors, neither the stability of solutions nor their continuous dependence on parameters for variational problems with $p(x)$-Laplacian equations have been considered so far.

The question of stability of solutions in the case when the solution itself may not be unique, as is the case in the present paper, is rarely considered in the literature. The first one to state this properly was Walczak in [12, 13]. Later Idczak [9] developed the methods from [12, 13] to consider abstract sublinear Dirichlet problems. In the second author's work [6], stability of solutions for abstract superlinear Dirichlet problems is considered. We use the ideas applied in the sources mentioned in order to prove that the family (1.1) is stable in the sense mentioned above. Once the stability is proved we are in a position to take up the question of the existence of optimal solutions to the following control problem:

$$\min J_A(u,\xi) = \int_{\Omega} f_0\big(x,u(x),\xi(x)\big)\,dx, \tag{1.2}$$

where the dynamics of the system are described by

$$\begin{aligned} -\operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right) &= F_u\big(x,u(x),\xi(x)\big), \quad u \in W_0^{1,p(x)}(\Omega), \\ u(x)\big|_{\partial\Omega} &= 0. \end{aligned} \tag{1.3}$$

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The assumptions on F and F_u will guarantee that the solution to (1.3) exists for all ξ from a certain set of functions and that the system (1.3) is stable. This together with some mild assumptions on f_0 provides the optimal solutions to problem (1.2)–(1.3).

2. Assumptions and the existence of solutions

Now we state the growth condition on F^k and modify the relevant existence results from [7]. In what follows, we denote by C_S the best Sobolev constant, that is,

$$\|u\|_{p(x)} \le C_S \|\nabla u\|_{p(x)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Since $W_0^{1,p(x)}(\Omega)$ is continuously embedded into $W_0^{1,p^-}(\Omega)$, see [5], we denote by C_1 and C_2 the following (best Sobolev) constants

$$\|\nabla u\|_{p^-} \le C_1 \|\nabla u\|_{p(x)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega), \tag{2.1}$$

$$\max_{x\in\Omega} |u(x)| \le C_2 \|\nabla u\|_{p^-} \quad \text{for all } u \in W_0^{1,p^-}(\Omega). \tag{2.2}$$

Let $\mathrm{vol}(\Omega) \le (1/p^- + 1/q^-)^{-1}$. In the existence and the stability results the following assumptions will be made.

(F1) There exist positive numbers $d_0, d_1, d_2, \dots$ such that $d_k \le d_0$ for $k \in \mathbb{N}$ and, for all $k = 0, 1, 2, \dots$, $F_u^k(\cdot, \pm d_k) \in L^\infty(\Omega)$, $C_S \operatorname{esse\,sup}_{x\in\Omega} \left|F_u^k(x, \pm d_k)\right| \ge 1$ and

$$C_1 C_2 C_S \operatorname*{esse\,sup}_{x\in\Omega} \left|F_u^k(x, \pm d_k)\right| \le d_k. \tag{2.3}$$

(F2) There exists a positive number d such that for all $k = 0, 1, 2, \dots$ and $I = [-d, d]$: $F_u^k(\cdot, \pm d) \in L^\infty(\Omega)$, $F^k(x, u) : \Omega \times I \to \mathbb{R}$ are Carathéodory functions and convex in u for a.e. $x \in \Omega$, $F_u^k(x, u) : \Omega \times I \to \mathbb{R}$ are Carathéodory functions, and $F^k(x, u) := +\infty$ for $(x, u) \in \Omega \times (\mathbb{R} - I)$.

(F3) For all $k = 0, 1, 2, \dots$, $F_u^k(x, 0) \ne 0$ a.e. on Ω, $x \mapsto F^k(x, 0)$ and $x \mapsto (F^k)^*(x, 0)$ are integrable on Ω, where $(F^k)^*$ denotes the Fenchel-Young conjugate of the convex and l.s.c. function $F^k : \Omega \times \mathbb{R} \to \mathbb{R}$, see [2].

Also a functional $J_k : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ given by the formula

$$J_k(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx - \int_\Omega F^k(x, u(x))\, dx$$

and the fact that (1.1) is the Euler-Lagrange equation for this functional, see [3], will be used in our considerations. However, under the growth conditions (2.3) assumed in (F1) the functional J_k is not well defined on the whole space $W_0^{1,p(x)}(\Omega)$. We shall construct a subset of $W_0^{1,p(x)}$, denoted by X_k, on which the integral $\int_\Omega F^k(x, u(x))\, dx$ is finite and the functional J_k is bounded from below.

For any $k = 0, 1, 2, \ldots$, we consider a set X_k such that for all $u \in X_k$ the relation

$$-\operatorname{div}\left(|\nabla \tilde{u}(x)|^{p(x)-2}\nabla \tilde{u}(x)\right) = F_u^k(x, u(x)), \quad \tilde{u} \in W_0^{1,p(x)}(\Omega),$$
$$\tilde{u}(x)\big|_{\partial\Omega} = 0,$$

implies $\tilde{u} \in X_k$.

The sets X_k may be constructed as follows, compare [7]:

$$X_k = \left\{ u \in W_0^{1,p(x)}(\Omega),\ \|\nabla u\|_{L^{p(x)}(\Omega)} \le \frac{d_k}{C_1 C_2},\ u(x) \in [-d_k, d_k] \text{ a.e.} \right\}.$$

Since X_k is now weakly compact, by convexity arguments we may now prove that for each k there exists a constant η_k such that

$$\int_\Omega \left| F_u^k(x, u(x)) \right| dx \le \eta_k. \tag{2.4}$$

From [7] the following Variational Principle follows.

THEOREM 2.1. *Assume* (F1)–(F3) *hold. Assume that for every* $k = 0, 1, 2, \ldots$ *there exists* $u_k \in X_k$ *such that* $-\infty < J(u_k) = \inf_{u \in X_k} J(u) < \infty$. *Then*

$$-\operatorname{div}\left(|\nabla u_k(x)|^{p(x)-2}\nabla u_k(x)\right) = F_u^k(x, u_k(x)),$$
$$u_k(x)\big|_{\partial\Omega} = 0.$$

REMARK 1. We observe that if a solution u_k to (1.1) exists, it necessarily satisfies

$$-\operatorname{div}\left(|\nabla u_k(\cdot)|^{p(x)-2}\nabla u_k(\cdot)\right) \in L^\infty(\Omega).$$

Now we prove the existence theorem since its idea is quite different from the approach of [7].

THEOREM 2.2. *Assume* (F1)–(F3) *hold. For every* $k = 0, 1, 2, \ldots$, *there exists* $u_k \in X_k$ *such that*

$$-\operatorname{div}\left(|\nabla u_k(x)|^{p(x)-2}\nabla u_k(x)\right) = F_u^k(x, u_k(x)), \quad u_k(x)\big|_{\partial\Omega} = 0, \tag{2.5}$$
$$J_k(u_k) = \inf_{u \in X_k} J_k(u) \tag{2.6}$$

and $u_k = \lim_{n\to\infty} u_k^n$, *where* $\{u_k^n\}_{n=1}^\infty$ *is a minimising sequence for a the restriction of a functional* J_k *to the set* X_k.

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PROOF. We fix $k = 0, 1, 2 \dots$. We first show that J_k is bounded from below on X_k. From (2.4) it follows that

$$J_k(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx - \int_\Omega F^k(x, u(x))\, dx \geq \text{const}.$$

Now we may find a minimising sequence $\{u_k^n\}_{n=1}^\infty$ for the restriction of a functional J_k to the set X_k. Due to the properties of the set X_k this sequence may be assumed to be weakly convergent in $W_0^{1,p(x)}(\Omega)$ and therefore, up to a subsequence, strongly convergent in $L^{p^-}(\Omega)$ since $W_0^{1,p(x)}(\Omega)$ is embedded into $W_0^{1,p^-}(\Omega)$. Thus a sequence $\{u_k^n\}_{n=1}^\infty$ contains a subsequence convergent a.e. We denote this subsequence by $\{u_k^n\}_{n=1}^\infty$ and its limit by u_k. We observe that $u_k \in X_k$. Indeed, we must show that

$$\|\nabla u_k\|_{L^{p(x)}(\Omega)} \leq \frac{d_k}{C_1 C_2}, \tag{2.7}$$

and

$$u_k(x) \in [-d_k, d_k] \text{ a.e.} \tag{2.8}$$

We infer that $\|\nabla u_k^n\|_{L^{p(x)}(\Omega)} \leq d_k / C_1 C_2$ for all n and

$$\liminf_{n\to\infty} \|\nabla u_k^n\|_{L^{p(x)}(\Omega)} \geq \|\nabla u_k|_{L^{p(x)}(\Omega)}.$$

Thus (2.7) holds. By the definition of the sequence $\{u_k^n\}_{n=1}^\infty$ we also get $|u_k^n(x)| \leq d_k$. Since $\{u_k^n\}_{n=1}^\infty$ is convergent almost everywhere, we get (2.8).

We may now observe that J_k is weakly lower semicontinuous on X_k. Indeed,

$$W_0^{1,p(x)}(\Omega) \ni x \mapsto \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx \in \mathbb{R},$$

being convex and lower semicontinuous is weakly lower semicontinuous [2]. Since the limit $\lim_{n\to\infty} \int_\Omega F_u^k(x, u_k^n(x))\, dx = \int_\Omega F_u^k(x, u_k(x))\, dx$ exists we get

$$\liminf_{n\to\infty} J_k(u_k^n) \geq J_k(u_k).$$

Thus $J(u_k) = \inf_{u\in X_k} J_k(u)$. Therefore (2.6) holds.

From Theorem 2.1 it follows that (2.5) is also satisfied. □

3. Stability of solutions

We recall that

$$X_0 = \left\{ u \in W_0^{1,p(x)}(\Omega) : \|\nabla u\|_{L^{p(x)}(\Omega)} \leq \frac{d_0}{C_1 C_2},\ u(x) \in [-d_0, d_0] \text{ a.e.} \right\}$$

and observe that $X_k \subset X_0$ for all $k = 1, 2, \dots$.

THEOREM 3.1. *Assume* (F1)–(F3) *hold. Let us assume that for any* $u \in X_0$ *there exists a subsequence* $\{k_i\}_{i=1}^{\infty}$ *such that* $\lim_{i\to\infty} F_u^{k_i}(x,\bar{u}(x)) = F_u^0(x,\bar{u}(x))$ *weakly in* $L^{p(x)}(\Omega)$. *For each* $k = 0, 1, 2, \ldots$ *there exists a solution* u_k *to problem* (1.1). *There exists a subsequence* $\{u_{k_n}\}_{n=1}^{\infty}$ *of the sequence* $\{u_k\}_{k=1}^{\infty}$ *and* $\bar{u} \in W_0^{1,p(x)}$ *such that* $u_{k_n} \to \bar{u}$, *strongly in* $W_0^{1,p(x)}(\Omega)$, $u_{k_n} \rightrightarrows \bar{u}$ *and*

$$-\operatorname{div}\left(|\nabla\bar{u}(x)|^{p(x)-2}\nabla\bar{u}(x)\right) = F_u^0(x,\bar{u}(x)), \quad \bar{u}(x)\big|_{\partial\Omega} = 0.$$

PROOF. By Theorem 2.2 it follows that for each $k = 0, 1, 2, 3, \ldots$ there exists $u_k \in W_0^{1,p(x)}(\Omega)$ such that

$$-\operatorname{div}\left(|\nabla u_k(x)|^{p(x)-2}\nabla u_k(x)\right) = F_u^k(x,u_k(x)), \quad u_k(x)\big|_{\partial\Omega} = 0.$$

Due to the fact that $X_k \subset X_0$ it follows that the sequence $\{\nabla u_k\}_{k=1}^{\infty}$ is bounded in $L^{p(x)}(\Omega)$. Hence there exists a weakly convergent subsequence in $W_0^{1,p(x)}(\Omega)$ and this subsequence may be assumed to be strongly convergent in $L^{p^-}(\Omega)$. We denote its limit by $\bar{u}$. Due to the growth assumptions we get

$$\operatorname*{esse\,sup}_{x\in\Omega}\left|F_u^k(x,\bar{u}(x)) - F_u^0(x,\bar{u}(x))\right| \le \frac{2d_0}{C_1C_2C_S}. \tag{3.1}$$

Let $\{k_i\}_{i=1}^{\infty}$ be such a subsequence that $\lim_{i\to\infty} F_u^{k_i}(x,\bar{u}(x)) = F_u^0(x,\bar{u}(x))$ weakly in $L^{p(x)}(\Omega)$. We denote all the resulting subsequences by the subscript k for simplicity. Moreover, by (3.1) and Remark 1 we infer that $\{-\operatorname{div}|\nabla u_k(\cdot)|^{p(x)-2}\nabla u_k(\cdot)\}_{k=1}^{\infty}$ is weakly convergent in $L^{p(x)}(\Omega)$, up to a subsequence, to a certain function $d \in L^{p(x)}(\Omega)$. Thus

$$\int_\Omega \left\langle -\operatorname{div}|\nabla u_k(x)|^{p(x)-2}\nabla u_k - \operatorname{div}|\nabla\bar{u}(x)|^{p(x)-2}\nabla\bar{u}, u_k(x) - \bar{u}(x)\right\rangle \to 0.$$

Hence and by the fact that the $p(x)$-Laplacian has the $(S)_+$ property, see [3], it follows that

$$\lim_{k\to\infty}\int_\Omega \left|\nabla u_k(x) - \nabla\bar{u}(x)\right|^{p(x)} dx = 0.$$

Indeed, it suffices to repeat the reasoning from the proof of [3, Theorem 3.1]. Thus $\{\nabla u_k\}_{k=1}^{\infty}$ is strongly convergent in $W_0^{1,p(x)}(\Omega)$.

We show that $u_{k_n} \rightrightarrows \bar{u}$, possibly up to a subsequence which we denote by $\{u_k\}_{k=1}^{\infty}$. Indeed, by (2.2)–(2.3) and by the mean value theorem it follows that the function u_k for $k = 0, 1, 2, \ldots$ has a continuous extension to $\overline{\Omega}$. Due to the construction of the set X_k and by the mean value theorem it follows that functions from a sequence $\{u_k\}_{k=1}^{\infty}$ are uniformly bounded and equicontinuous.

We will next prove that

$$-\operatorname{div}|\nabla\bar{u}(x)|^{p(x)-2}\nabla\bar{u}(x) = F_u^0(x,\bar{u}(x)).$$

By the convexity of F^k we get for any $u \in W_0^{1,p(x)}(\Omega)$

$$\int_\Omega \left\langle F_u^k(x,u_k(x)) - F_u^k(x,u(x)),\ u_k(x)-u(x)\right\rangle dx \geq 0.$$

Hence again by Theorem 2.2

$$\int_\Omega \left\langle -\operatorname{div}|\nabla u_k(x)|^{p(x)-2}\nabla u_k(x) - F_u^k(x,u(x)),\ u_k(x)-u(x)\right\rangle dx \geq 0.$$

Since $u_k \to \bar{u}$ strongly in $L^{p(x)}(\Omega)$ and $F_u^k(\cdot,u(\cdot)) \rightharpoonup F_u^0(\cdot,u(\cdot))$ weakly in $L^{p(x)}(\Omega)$ we easily get that

$$\int_\Omega \left\langle -F_u^k(x,u(x)),\ u_k(x)-u(x)\right\rangle dx \to \int_\Omega \left\langle -F_u^0(x,u(x)),\ \bar{u}(x)-u(x)\right\rangle dx.$$

Moreover

$$\begin{aligned}
&\int_\Omega \left\langle -\operatorname{div}|\nabla u_k(x)|^{p(x)-2}\,\nabla u_k(x),\ -u(x)\right\rangle dx \\
&\quad= \int_\Omega \left\langle |\nabla u_k(x)|^{p(x)-2}\,\nabla u_k(x),\ -\nabla u(x)\right\rangle dx \\
&\quad\to \int_\Omega \left\langle |\nabla\bar{u}\,(x)|^{p(x)-2}\,\nabla\bar{u}(x),\ -\nabla u(x)\right\rangle dx \\
&\quad= \int_\Omega \left\langle -\operatorname{div}|\nabla\bar{u}(x)|^{p(x)-2}\,\nabla\bar{u}(x),\ -u(x)\right\rangle dx.
\end{aligned}$$

Indeed, for any $f \in C_0^\infty(\Omega)$ we get

$$\begin{aligned}
\int_\Omega \langle d(x), f(x)\rangle\, dx &= \lim_{k\to\infty}\int_\Omega \left\langle -\operatorname{div}|\nabla u_k(x)|^{p(x)-2}\,\nabla u_k(x),\ f(x)\right\rangle dx \\
&= \lim_{k\to\infty}\int_\Omega \left\langle |\nabla u_k(x)|^{p(x)-2}\,\nabla u_k(x),\ \nabla f(x)\right\rangle dx \\
&= \int_\Omega \left\langle |\nabla\bar{u}(x)|^{p(x)-2}\,\nabla\bar{u}(x),\ \nabla f(x)\right\rangle dx,
\end{aligned}$$

so by the Euler-Lagrange Lemma for multiple integrals [10], we infer that

$$d(x) = -\operatorname{div}|\nabla\bar{u}(x)|^{p(x)-2}\nabla\bar{u}(x).$$

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We further observe that

$$\int_\Omega \langle - \operatorname{div} |\nabla u_k(x)|^{p(x)-2} \nabla u_k(x), u_k(x) \rangle dx$$
$$= \int_\Omega |\nabla u_k(x)|^{p(x)} dx \to \int_\Omega |\nabla \bar{u}(x)|^{p(x)} dx$$
$$= \int_\Omega \langle - \operatorname{div} |\nabla \bar{u}(x)|^{p(x)-2} \nabla \bar{u}(x), \bar{u}(x) \rangle dx.$$

Hence

$$\int_\Omega \langle - \operatorname{div} |\nabla \bar{u}(x)|^{p(x)-2} \nabla \bar{u}(x) - F_u^0(x, u(x)), \bar{u}(x) - u(x) \rangle dx \geq 0 \tag{3.2}$$

for any $u \in W_0^{1,p(x)}(\Omega)$.

Now we apply the Minty "trick", that is, we consider the points $\bar{u} + tu$, where $u \in W_0^{1,p(x)}(\Omega)$, $u(x) \in I$ a.e. and $t > 0$ such that $\bar{u}(x) + tu(x) \in I$ a.e. By (3.2) we obtain

$$\int_\Omega \langle - \operatorname{div} |\nabla \bar{u}(x)|^{p(x)-2} \nabla \bar{u}(x) - F_u^0 (x, \bar{u}(x) + tu(x)), u(x) \rangle dx \leq 0.$$

Since the function maps to $t \mapsto F^0(\cdot, \bar{u}(\cdot) + tu(\cdot))$ and is convex it follows that its derivative $t \mapsto \int_\Omega \langle F_u^0(x, \bar{u}(x) + tu(x)), u(x) \rangle dx$ is continuous for sufficiently small t. Hence

$$0 \geq \lim_{t \to 0} \int_\Omega \langle - \operatorname{div} |\nabla \bar{u}(x)|^{p(x)-2} \nabla \bar{u}(x) - F_u^0(x, \bar{u}(x) + tu(x)), u(x) \rangle dx$$
$$= \int_\Omega \langle - \operatorname{div} |\nabla \bar{u}(x)|^{p(x)-2} \nabla \bar{u}(x) - F_u^0(x, \bar{u}(x)), u(x) \rangle dx$$

for any $u \in W_0^{1,p(x)}(\Omega)$. Since $- \operatorname{div} |\nabla \bar{u}(\cdot)|^{p(x)-2} \nabla \bar{u}(\cdot) - F_u^0(\cdot, \bar{u}(\cdot)) \in L^{p(x)}(\Omega)$ we obtain that $- \operatorname{div} |\nabla \bar{u}(x)|^{p(x)-2} \nabla \bar{u}(x) = F_u^0(x, \bar{u}(x))$ a.e. □

The above theorem provides a sufficient condition for the family of problems given by (1.1) to be stable. It asserts that the problem

$$- \operatorname{div} |\nabla u(x)|^{p(x)-2} \nabla u(x) = F_u^0(x, u(x)), \quad u(x)\big|_{\partial\Omega} = 0 \tag{3.3}$$

has a solution $\bar{u}$ understood as a limit of a sequence of solutions to the family (1.1) for $k = 1, 2, \ldots$. Now problem (3.3) has yet another solution u^0 which minimises J_0 over X_0. We shall consider the question of whether $\bar{u}$ is a minimiser of J_0 over X_0. To assure this we have to make one additional assumption.

PROPOSITION 3.2. *Assume* (F1)–(F3) *hold. Let us assume that for any* $u \in X_0$ *there exists a subsequence* $\{k_i\}_{i=1}^{\infty}$ *such that* $\lim_{i\to\infty} F_u^{k_i}(x, \bar{u}(x)) = F_u^0(x, \bar{u}(x))$ *weakly in* $L^{p(x)}(\Omega)$. *For each* $k = 1, 2, \dots$ *there exists a solution* u_k *to problem* (1.1). *There exists a subsequence* $\{u_{k_n}\}_{n=1}^{\infty}$ *of the sequence* $\{u_k\}_{k=1}^{\infty}$ *and* $\bar{u} \in W_0^{1,p(x)}$ *such that* $u_{k_n} \to \bar{u}$, *strongly in* $W_0^{1,p(x)}(\Omega)$ *and* $u_{k_n} \rightrightarrows \bar{u}$, *and*

$$-\operatorname{div}\left(|\nabla \bar{u}(x)|^{p(x)-2}\nabla \bar{u}(x)\right) = F_u^0(x, \bar{u}(x)), \quad \bar{u}(x)\big|_{\partial\Omega} = 0.$$

Let moreover for all $u \in X_0$

$$F^{k_n}(x, u(x)) \xrightarrow[n\to\infty]{} F^0(x, u(x)) \quad a.e. \tag{3.4}$$

Then $\bar{u} \in X_0$

$$\lim_{n\to\infty}\left(J_{k_n}(u_{k_n}) - J_0(u_{k_n})\right) = 0, \tag{3.5}$$

$$\liminf_{n\to\infty}\left(J_{k_n}(u_{k_n}) - J_0(u^0)\right) \le 0 \tag{3.6}$$

and

$$J_0(\bar{u}) = \inf_{u\in X_0} J_0(\bar{u}). \tag{3.7}$$

PROOF. The first part follows by Theorem 3.1. We choose a sequence $\{u_{k_n}\}_{n=1}^{\infty}$ as in the proof of Theorem 3.1. It is evident that $\bar{u} \in X_0$ and $\lim_{n\to\infty} J_0(u_{k_n}) \ge J_0(\bar{u})$.

To prove (3.5) it suffices to demonstrate that

$$\lim_{n\to\infty}\int_\Omega \left(F^{k_n}(x, u_{k_n}(x)) - F^0(x, u_{k_n}(x))\right) dx = 0.$$

We have

$$\begin{aligned}\lim_{n\to\infty}\bigg(\int_\Omega &\left(F^{k_n}(x, u_{k_n}(x)) - F^{k_n}(x, \bar{u}(x))\right) dx\\ &+ \int_\Omega \left(F^{k_n}(x, \bar{u}(x)) - F^0(x, \bar{u}(x))\right) dx\\ &- \int_\Omega \left(F^0(x, u_{k_n}(x)) - F^0(x, \bar{u}(x))\right) dx\bigg) = 0.\end{aligned}$$

Now by (3.4) it follows that

$$\lim_{n\to\infty}\left(\int_\Omega \left(F^{k_n}(x, \bar{u}(x)) - F^0(x, \bar{u}(x))\right) dx\right) = 0$$

and since F^0 is continuous and $\{u_{k_n}\}_{n=1}^{\infty}$ uniformly convergent we obtain

$$\lim_{n\to\infty}\left(\int_\Omega \left(F^0(x, \bar{u}(x)) - F^0(x, u_{k_n}(x))\right) dx\right).$$

Now by convexity we get the following estimation:

$$\int_\Omega \left(F^{k_n}(x,\bar u(x)) - F^{k_n}(x,u_{k_n}(x))\right)dx$$
$$\le \max\left\{\int_\Omega \left|F_u^{k_n}(x,u_{k_n})(\bar u(x)-u_{k_n}(x))\right|dx,\ \int_\Omega \left|F_u^{k_n}(x,\bar u)(\bar u(x)-u_{k_n}(x))\right|dx\right\}.$$

We observe that by the construction of the set X_0

$$\int_\Omega \left|F_u^{k_n}(x,u_{k_n}(x))\left(\bar u(x)-u_{k_n}(x)\right)\right|dx$$
$$\le \frac{d_0}{C_1C_2C_S}\int_\Omega \left|(\bar u(x)-u_{k_n}(x))\right|dx$$
$$\le \frac{d_0}{C_1C_2C_S}\operatorname{vol}(\Omega)\left(\frac{1}{p^-}+\frac{1}{q^-}\right)\int_\Omega \left|(\bar u(x)-u_{k_n}(x))\right|^{p(x)}dx \to 0.$$

Analogously

$$\int_\Omega \left|F_u^{k_n}(x,\bar u(x))(\bar u(x)-u_{k_n}(x))\right|dx \to 0.$$

So we have (3.5).

To prove (3.6) we observe that since u_{k_n} is a minimiser for J_{k_n} it follows that

$$J_{k_n}(u_{k_n}) - J_0(u^0) \le J_{k_n}(u^0) - J_0(u^0)$$
$$= \int_\Omega \left(F^{k_n}(x,u^0(x)) - F^0(x,u^0(x))\right)dx.$$

Now by (3.4) we obtain that $\int_\Omega \left(F^{k_n}(x,u^0(x)) - F^0(x,u^0(x))\right)dx \to 0$. Hence we establish (3.6).

We now prove (3.7). By Theorem 2.2 it follows that there exists $u^0 \in X_0$ and a sequence $\{u_n^0\}_{n=1}^\infty$ such that $u_n^0 \rightharpoonup u^0$ in $W_0^{1,p(x)}(\Omega)$, $\liminf_{n\to\infty} J_0(u_n^0) \ge J_0(u^0)$ and $J_0(u^0) = \inf_{u\in X_0} J_0(u)$. We suppose that $J_0(u^0) < J_0(\bar u)$. Then there exists $\varepsilon > 0$ (fixed) such that $J_0(u^0) < J_0(\bar u) - \varepsilon$. We observe that

$$J_{k_n}(u_{k_n}) - J_0(u^0) > J_{k_n}(u_{k_n}) - J_0(\bar u) + \varepsilon$$
$$= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_{k_n}(x)|^{p(x)} - |\nabla \bar u(x)|^{p(x)}\right)dx$$
$$- \int_\Omega \left(F^{k_n}(x,u_{k_n}(x)) - F^0(x,\bar u(x))\right)dx + \varepsilon \to \varepsilon.$$

Thus $\liminf_{n\to\infty}\left(J_{k_n}(u_{k_n}) - J_0(u^0)\right) > \varepsilon$. Thus we have obtained a contradiction with (3.6) and $\bar u$ is a minimiser of J_0 over X_0. □

4. The continuous dependence on parameters

Now we shall apply stability results to investigate a continuous dependence on parameters for the Dirichlet problem

$$-\operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right) = F_u(x, u(x), \xi(x)), \quad u(x)\big|_{\partial\Omega} = 0, \tag{4.1}$$

where $\xi : \Omega \to \mathbb{R}^m$ is a functional parameter from a set

$$L_M = \{\xi : \Omega \to \mathbb{R}^m : \xi \text{ is measurable }, \xi(x) \in M \text{ a.e.}\}$$

and $M \subset \mathbb{R}^m$ is a given bounded set.

We assume that the following assumptions hold.

(Fp1) There exist numbers $d_1 > d > 0$ such that $F_u(\cdot, \pm d, \xi(\cdot)) \in L^\infty(\Omega)$ and $F_u(\cdot, \pm d_1, \xi(\cdot)) \in L^\infty(\Omega)$ for all $\xi \in L_M$ and C_S esse $\sup_{x\in\Omega} |F_u(x, \pm d, \xi(x))| \geq 1$.

(Fp2) Let $I = [-d, d]$ and $I_1 = [-d_1, d_1]$. Now $F : \Omega \times I_1 \times U$ is a Carathéodory function convex in u for a.e. $x \in \Omega$ such that $F(x, u, \xi) = +\infty$ for $(x, u, \xi) \in \Omega \times (\mathbb{R}\backslash I_1) \times U$ and $F_u : \Omega \times I_1 \times U$ is a Carathéodory function; for all $\xi \in L_M$

$$C_1 C_2 C_S \operatorname{esse}\sup_{x\in\Omega} |F_u(x, \pm d, \xi(x))| \leq d. \tag{4.2}$$

(Fp3) The function $F_u(x, 0, \xi(x)) \neq 0$, for a.e. $x \in \Omega$, $x \mapsto |F(x, 0, \xi(x))|$ and $x \mapsto |F^*(x, 0, \xi(x))|$ are integrable for all $\xi \in L_M$.

We put

$$X = \left\{u \in W_0^{1,p(x)}(\Omega) : \|u\|_{L^{p(x)}(\Omega)} \leq \frac{d}{C_1 C_2},\ u(x) \in I \text{ a.e.}\right\}.$$

Below we shall consider a sequence $\{\xi_k\}_{k=1}^\infty$, $\xi_k \in L_M$ such that $\xi_k \to \bar{\xi}$ in $L^{p(x)}(\Omega)$. We put $F_u^k(\cdot, u(\cdot)) = F_u(\cdot, u(\cdot), \xi_k(\cdot))$ and $F_u^0(\cdot, u(\cdot)) = F_u(\cdot, u(\cdot), \bar{\xi}(\cdot))$. Thus J_k reads as

$$J_k(u) = \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}\,dx - \int_\Omega F(x, u(x), \xi_k(x))\,dx$$

and

$$J_0(u) = \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}\,dx - \int_\Omega F(x, u(x), \bar{\xi}(x))\,dx, \tag{4.3}$$

with $X_k = X$ for all $k = 0, 1, 2, \ldots$.

THEOREM 4.1. *Assume that* (Fp1)–(Fp3) *hold and that* $\{\xi_k\}_{k=1}^\infty$, $\xi_k \in L_M$, *is a sequence such that* $\xi_k \to \bar{\xi}$ *in* $L^{p(x)}(\Omega)$. *For each* $k = 0, 1, 2, \ldots$ *there exists a* *solution* u_k *to problem* (4.1). *There exists a subsequence* $\{u_{k_n}\}_{n=1}^{\infty}$ *of the sequence* $\{u_k\}_{k=1}^{\infty}$ *and* $\bar{u} \in X$ *such that* $u_{k_n} \to \bar{u}$, *strongly in* $W_0^{1,p(x)}(\Omega)$, $u_{k_n} \rightrightarrows \bar{u}$,

$$-\operatorname{div}(|\nabla \bar{u}(x)|^{p(x)-2}\nabla \bar{u}(x)) = F_u^0(x, \bar{u}(x), \bar{\xi}(x)), \quad \bar{u}(x)\big|_{\partial\Omega} = 0,$$

and $J_0(\bar{u}) = \inf_{u \in X} J_0(\bar{u})$.

PROOF. By (4.2) it follows that for all $u \in X$ we have

$$F_u(\cdot, u(\cdot), \xi_k(\cdot)) \underset{k\to\infty}{\longrightarrow} F_u(\cdot, u(\cdot), \bar{\xi}(\cdot)) \quad \text{weakly in } L^{p(x)}(\Omega).$$

Hence Theorem 3.1 applies with $F_u^k(\cdot, u(\cdot)) = F_u(\cdot, u(\cdot), \xi_k(\cdot))$. To prove the last assertion we should show that $F(x, u(x), \xi_k(x)) \to F(x, u(x), \bar{\xi}(\bar{x}))$ a.e. and use Proposition 3.2. This follows since $\{\xi_k\}_{k=1}^{\infty}$ is convergent almost everywhere and F is continuous with respect to the third variable. □

5. Applications to optimisation

Now we apply the results on continuous dependence on parameters in order to show that there exists an optimal solution to the optimisation problem (1.2)–(1.3). We assume that f_0 and F satisfy (F1)–(F3) and that the following assumptions also hold:

(Fp4) Let $\lambda > 0$ be fixed and let $\xi : \Omega \to \mathbb{R}^m$ be a functional parameter from a set

$$L_M = \left\{u : [0, \pi] \to \mathbb{R}^m \mid u \text{ is Lipschitz with respect to } \lambda,\ u(t) \in M \text{ a.e.}\right\}.$$

Such a definition of L_M provides that any bounded sequence of functions in L_M contains a subsequence strongly convergent in $L^{p(x)}(\Omega)$.

(Fp5) $f_0 : \Omega \times I_1 \times M$ is measurable with respect to the first variable and continuous with respect to the two last variables. It is also quasi-convex with respect to ξ. Moreover there exists a function $\psi \in L^1(\Omega)$ such that for all $u \in I_1$ and all $\xi \in M$

$$|f_0(x, u, \xi)| \le \psi(x) \quad \text{for a.e. } x \in \Omega.$$

Now we easily obtain that any sequence in L_M contains a subsequence that is weakly convergent in $L^{p(x)}(\Omega)$. We consider a set A consisting of all these pairs (u, ξ) where $u \in X$ is a solution to (1.3) corresponding to $\xi \in L_M$ and which minimises the relevant action functional. By m we denote the infimum of J_A over A. We have the following theorem.

THEOREM 5.1. *Let us assume that* (Fp1)–(Fp5) *all hold. There exists a pair* $(\bar{u}, \bar{\xi})$ *in* A *such that* $J_A(\bar{u}, \bar{\xi}) = m$.

PROOF. We first observe that the set A is relatively weakly compact in $W_0^{1,p(x)}(\Omega) \times L^{p(x)}(\Omega)$ and J_A is bounded from below on A. Thus there exists a minimising sequence $\{u_k, \xi_k\}_{k=1}^{\infty}$ for the problem (1.2)–(1.3). We may assume that a sequence $\{u_k, \xi_k\}_{k=1}^{\infty}$ is weakly convergent to a certain pair $(\bar{u}, \bar{\xi}) \in W_0^{1,p(x)}(\Omega) \times L^{p(x)}(\Omega)$. From Theorem 4.1 it follows that there exists a subsequence u_{k_n} such that $u_{k_n} \to \bar{u}$, strongly in $W_0^{1,p(x)}(\Omega)$, $u_{k_n} \rightrightarrows \bar{u}$, and that $\bar{u}$ is a solution to the problem (1.3) corresponding to $\bar{\xi}$ and which minimises the relevant action functional, see (4.3). Thus $(\bar{u}, \bar{\xi}) \in A$. Since f_0 is a Carathéodory function, since it is quasi-convex with respect to ξ and since $u_{k_n} \rightrightarrows \bar{u}$, it follows using [1, Theorem 10.9.vi] that, for a subsequence,

$$\liminf_{n\to\infty} J_A(u_{k_n}, \xi_{k_n}) \geq J_A(\bar{u}, \bar{\xi}) = m. \qquad \square$$

REMARK 2. We observe that the optimal solution $\bar{u}$ has some qualitative properties, for example, $\|\bar{u}\|_{L^{p(x)}(\Omega)} \leq \frac{d}{C_1 C_2}$, $\bar{u}(x) \in I$ a.e. and $-\operatorname{div}(|\nabla \bar{u}(\cdot)|^{p(x)-2}\nabla \bar{u}(\cdot)) \in L^{\infty}(\Omega)$.

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