CURVES WITH ZERO DERIVATIVE IN F-SPACES

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1. Introduction. Let X be an F-space (complete metric linear space) and suppose $g:[0,1]\to X$ is a continuous map. Suppose that g has zero derivative on $[0,1]$, i.e.

$$g'(t)=\lim_{h\to 0}\frac{1}{h}(g(t+h)-g(t))=0$$

for $0\le t\le 1$ (we take the left and right derivatives at the end points). Then, if X is locally convex or even if it merely possesses a separating family of continuous linear functionals, we can conclude that g is constant by using the Mean Value Theorem. If however $X^*=\{0\}$ then it may happen that g is not constant; for example, let $X=L_p(0,1)$ $(0\le p<1)$ and $g(t)=1_{[0,t]}$ $(0\le t\le 1)$ (the characteristic function of $[0,t]$). This example is due to Rolewicz [**6**], [**7**; p. 116].

The aim of this note is to substantiate a conjecture of Rolewicz [**7**, p. 116] that every F-space X with trivial dual admits a non-constant curve $g:[0,1]\to X$ with zero derivative. In fact we shall show, given any two points $x_0, x_1\in X$, there exists a map $g:[0,1]\to X$ with $g(0)=x_0$, $g(1)=x_1$ and

$$\lim_{|t-s|\to 0}\frac{g(t)-g(s)}{t-s}=0\quad\text{uniformly for}\quad 0\le s,t\le 1.$$

To establish this result we shall need to study X-valued martingales. Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $[0,1)$ and let $\mathcal{F}_n$ $(n\ge 0)$ be an increasing family of *finite* sub-algebras of $\mathcal{B}$. Then a sequence of functions $u_n:[0,1)\to X$ is an X-valued F_n-martingale if each u_n is F_n-measurable and for $n\ge m$ we have $\mathcal{E}(u_n\mid\mathcal{F}_m)=u_m$. Here the definition of conditional expectation is the standard one with respect to Lebesgue measure λ and there are no integration problems since each u_n is finitely-valued.

It is easy to show that every F-space X with trivial dual contains a non-constant martingale $\{u_n,\mathcal{F}_n\}$ which converges to zero uniformly. However we shall need to consider *dyadic martingales*. Let $D_{n,k}=[(k-1)/2^n, k/2^n)$ $(1\le k\le 2^n, 0\le n<\infty)$. Then, for $n\ge 0$, let $\mathcal{B}_n$ be the sub-algebra of $\mathcal{B}$ generated by the sets $\{D_{n,k}:1\le k\le 2^n\}$. A dyadic martingale is simply a $\mathcal{B}_n$-martingale. The main point of the argument will be to show that we can find non-zero dyadic martingales which converge uniformly to zero.

We note here a connection with the recent work of Roberts [**4**], [**5**] on the existence of compact convex sets without extreme points. Indeed, in a needlepoint space (see [**5**]) it would be easy to show that there are non-zero dyadic martingales which converge uniformly to zero. However there are F-spaces with trivial dual which contain no needlepoints [**2**].

As usual an F-norm on a (real) vector space X is a map $x \to \|x\|$ such that

$$\|x\| > 0 \quad \text{if} \quad x \neq 0, \tag{1.0.1}$$

$$\|x+y\| \leq \|x\| + \|y\| \qquad (x, y \in X), \tag{1.0.2}$$

$$\|tx\| \leq \|x\| \qquad (|t| \leq 1), \tag{1.0.3}$$

$$\lim_{t \to 0} \|tx\| = 0 \qquad (x \in X). \tag{1.0.4}$$

The F-norm is said to be *strictly concave* if, for each $x \in X$ with $x \neq 0$, the map $t \to \|tx\|$ is strictly concave on $[0, \infty)$, i.e.

if $0 \leq s < t < \infty$ and $0 < a, b < 1$ with $a + b = 1$ then, if $x \neq 0$,

$$\|(as+bt)x\| > a\,\|sx\| + b\,\|tx\|. \tag{1.0.5}$$

Every F-space can be equipped with an (equivalent) F-norm which is strictly concave. This follows from the results of Bessaga, Petczyński and Rolewicz [**1**]. We may give X an F-norm $\|\cdot\|_0$ so that the map $t \to \|tx\|_0$ is concave and strictly increasing for each $x \neq 0$. Now define $\|x\| = \|x\|_0^{1/2}$.

2. Preliminary finite-dimensional results. Suppose N is a positive integer. We consider the space $\mathbb{R}^N$ with the natural co-ordinatewise partial ordering (i.e. $x \geq y$ if and only if $x_i \geq y_i$ for $1 \leq i \leq N$). We shall denote by $(e_k : 1 \leq k \leq N)$ the natural basis elements of $\mathbb{R}^N$. We shall use the idea of $\mathbb{R}^N$-valued submartingales and supermartingales; these have obvious meaning with respect to the ordering defined above. In addition, standard scalar convergence theorems can be applied co-ordinatewise to produce the same theorems for $\mathbb{R}^N$.

For $1 \leq i \leq N$, let F_i be a continuous map $F_i : [0, \infty) \to [0, \infty)$ which is strictly increasing, strictly concave and satisfies $F_i(0) = 0$, $F_i(1) = 1$. Then F_i is also subadditive since

$$F_i(s) \geq \frac{s}{s+t} F_i(s+t) \qquad (s, t > 0).$$

Hence we may define an absolute F-norm on $\mathbb{R}^N$ by

$$\|x\| = \sum_{i=1}^{N} F_i(|x_i|) \qquad (x \in \mathbb{R}^N). \tag{2.0.1}$$

Now, for $x \in \mathbb{R}^N$, define

$$\sigma(x) = \inf\{\max(\|y\|, \|z\|) : x = \tfrac{1}{2}(y+z)\}. \tag{2.0.2}$$

We shall need the following properties of σ.

LEMMA 2.1. (a) *If* $x \in \mathbb{R}^N$ *and* $x \geq 0$ *then there exist* $y, z \in \mathbb{R}^N$ *with* $y \geq 0$, $z \geq 0$, $x = \frac{1}{2}(y+z)$ *and* $\|y\| \leq \sigma(x)$, $\|z\| \leq \sigma(x)$.

(b) *For* $x, y \in \mathbb{R}^N$,

$$|\sigma(x) - \sigma(y)| \leq \|x - y\|, \tag{2.1.1}$$

$$\sigma(x) \leq \|x\|. \tag{2.1.2}$$

(c) *If* $x \geq 0$ *and* $\sigma(x) = \|x\| = 1$ *then, for some* k, *we have* $x = e_k$.

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Proof. (a) is an easy consequence of a compactness argument. For (b) (2.1.1), observe that if $x=\frac{1}{2}(z+z')$ then

$$y=\tfrac{1}{2}[(z+y-x)+(z'+y-x)],$$

so that $\sigma(y)\le\sigma(x)+\|y-x\|$ and so (2.1.1) follows. (2.1.2) is an immediate consequence of the definition of σ.

We are grateful to the referee for the following short proof of (c). Suppose $x\ge 0$, $\|x\|=1$, $x_i>0$ and $x_j>0$ where $i\ne j$. We show $\sigma(x)<1$.

Since F_i is concave, it has left and right derivatives at x_i, α_1 and α_2, say, with $0\le\alpha_2\le\alpha_1$. Similiarly F_j has left and right derivatives at x_j, β_1 and β_2 with $0<\beta_2\le\beta_1$. For small $t>0$,

$$\begin{aligned}\|x+t(\beta_1e_i-\alpha_2e_j)\|&<\|x\|,\\ \|x-t(\beta_1e_i-\alpha_2e_j)\|&<\|x\|+t(-\alpha_1\beta_1+\beta_2\alpha_2)\\ &\le\|x\|.\end{aligned}$$

Hence $\sigma(x)<1$.

We conclude that if $\sigma(x)=1$ then $x=e_k$ for some k, $1\le k\le N$.

Now let $$\pi(x)=x_1+\ldots+x_n\qquad(x\in\mathbb{R}^N).$$

THEOREM 2.2. *Suppose $a\in\mathbb{R}^N$, $a\ge 0$ and $\pi(a)=1$. Then there are disjoint Borel subsets $E_1,\ldots,E_N$ of $[0,1)$ with $\lambda(E_i)=a_i$ $(1\le i\le N)$ and a scalar valued dyadic supermartingale θ_n $(0\le n<\infty)$ such that*

$$0\le\theta_n(t)\le 1\qquad(0\le t<1,0\le n<\infty),\tag{2.2.1}$$

$$\lim_{n\to\infty}\theta_n(t)=0\ a.e.\tag{2.2.2}$$

and if

$$u_n=\mathscr{E}\Big(\sum_{i=1}^{N}1_{E_i}e_i\mid\mathscr{B}_n\Big)\qquad(0\le n<\infty)\tag{2.2.3}$$

then

$$u_n(t)\ge\theta_n(t)a\qquad(0\le t<1,0\le n<\infty),\tag{2.2.4}$$

$$\|u_n(t)-\theta_n(t)a\|\le 1\ (0\le t<1,\ 0\le n<\infty).\tag{2.2.5}$$

Proof. To start observe

$$\|a\|=\sum_{i=1}^{N}F_i(a_i)\ge\pi(a)=1.$$

Define $\alpha_0(t)\equiv\alpha_0$ for $0\le t<1$, where $0<\alpha_0\le 1$ and $\|\alpha_0a\|=1$; then let $w_0(t)=\alpha_0a$, $0\le t<1$. We then define inductively sequences $(w_n:n\ge 0)$, $(w_n^*:n\ge 1)$, $(\alpha_n:n\ge 0)$ of

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functions on [0, 1), where

$$w_n \ (n\geq 0) \text{ and } w_n^* \ (n\geq 1) \text{ are } \mathbb{R}^N\text{-valued and } \mathcal{B}_n\text{-measurable,} \tag{2.2.6}$$

$$\alpha_n \ (n\geq 0) \text{ is } \mathbb{R}\text{-valued and } \mathcal{B}_n\text{-measurable,} \tag{2.2.7}$$

$$\begin{aligned} w_n(t)&\geq 0 \qquad (0\leq t<1,\ n\geq 0),\\ w_n^*(t)&\geq 0 \qquad (0\leq t<1,\ n\geq 0),\\ \alpha_n(t)&\geq 0 \qquad (0\leq t<1,\ n\geq 0),\end{aligned} \tag{2.2.8}$$

$$\mathcal{E}(w_{n+1}^* \mid \mathcal{B}_n) = w_n \qquad (n\geq 0), \tag{2.2.9}$$

$$w_n(t) = w_n^*(t) + \alpha_n(t)a \qquad (0\leq t<1,\ n\geq 1), \tag{2.2.10}$$

$$\|w_n(t)\| = 1 \qquad (0\leq t<1,\ n\geq 0), \tag{2.2.11}$$

$$\|w_{n+1}^*(t)\| \leq \sigma(w_n(t)) \qquad (0\leq t<1,\ n\geq 0). \tag{2.2.12}$$

Indeed suppose w_j, w_j^* and α_j have been chosen for $j\leq n$. Then

$$w_n(t) = b_{n,k} \qquad (t\in D_{n,k}),$$

where $\|b_{n,k}\|=1$, and $b_{n,k}\geq 0$. Choose $y_{2k-1}, y_{2k}\geq 0$ so that $\max(\|y_{2k-1}\|, \|y_{2k}\|) = \sigma(b_{n,k})$ and $b_{n,k} = \frac{1}{2}(y_{2k-1}+y_{2k})$ (see Lemma 2.1(a)). Now define

$$w_{n+1}^*(t) = y_k \qquad (t\in D_{n+1,k}).$$

Then (2.2.9) and (2.2.12) are clear. Since

$$\|w_{n+1}^*(t)\| \leq 1 \qquad (0\leq t<1),$$

we can determine α_{n+1} to be $\mathcal{B}_{n+1}$-measurable so that $\alpha_{n+1}\geq 0$ and

$$\|w_{n+1}^*(t) + \alpha_{n+1}(t)a\| = 1 \qquad (0\leq t<1).$$

Now define

$$w_{n+1}(t) = w_{n+1}^*(t) + \alpha_{n+1}(t)a \qquad (0\leq t<1)$$

and clearly (2.2.11) holds.

Observe that

$$\mathcal{E}(w_{n+1} \mid \mathcal{B}_n) = w_n + \mathcal{E}(\alpha_{n+1} \mid \mathcal{B}_n)a$$

and if $m>n$

$$\mathcal{E}(w_n \mid \mathcal{B}_n) = w_n + \left(\sum_{k=n+1}^{m} \mathcal{E}(\alpha_k \mid \mathcal{B}_n)\right)a. \tag{2.2.13}$$

Hence w_n is a submartingale and it is clearly bounded. Thus $\lim_{n\to\infty} w_n(t) = w_\infty(t)$ exists almost everywhere, and $\|w_\infty(t)\|=1$ a.e.

The real-valued submartingale $(\pi \circ w_n : n \geq 0)$ is uniformly bounded and converges to $\pi \circ w_\infty$ a.e. Hence

$$\int_0^1 \pi(w_\infty(t))\, dt = \lim_{n\to\infty} \int_0^1 \pi(w_n(t))\, dt$$
$$= \int_0^1 \pi(w_0(t))\, dt + \sum_{k=1}^{\infty} \int_0^1 \alpha_k(t)\, dt$$

by (2.2.13) since $\pi(a) = 1$. Hence

$$\int_0^1 \sum_{k=1}^{\infty} \alpha_k(t)\, dt < \infty$$

and so (a.e.) $\sum \alpha_k(t) < \infty$. Thus $\alpha_n(t) \to 0$ a.e. and $\|w_{n+1}(t) - w^*_{n+1}(t)\| \to 0$ a.e. Hence $\|w^*_{n+1}(t)\| \to 1$ and $\sigma(w_n(t)) \to 1$ a.e. By Lemma 2.1(b), σ is continuous and so (a.e.)

$$\sigma(w_\infty(t)) = \|w_\infty(t)\| = 1.$$

As $w_\infty(t) \geq 0$, we conclude that

$$w_\infty(t) = \sum_{i=1}^{N} 1_{E_i} e_i \quad \text{a.e.,}$$

where $E_1, \ldots, E_N$ are disjoint Borel sets with $E_1 \cup \ldots \cup E_N = [0, 1)$.

Now define $u_n = \mathscr{E}(w_\infty \mid \mathscr{B}_n)$. Then, since $\{w_n\}$ is uniformly bounded and $w_n \to w_\infty$ a.e.,

$$u_n = \lim_{m\to\infty} \mathscr{E}(w_N \mid \mathscr{B}_n)$$
$$= w_n + \left(\sum_{k=n+1}^{\infty} \mathscr{E}(\alpha_k \mid \mathscr{B}_n) \right) a$$
$$= w_n + \theta_n a,$$

where $\theta_n \geq 0$ is $\mathscr{B}_n$-measurable. Since (w_n) is a submartingale, (θ_n) is a supermartingale. As $u_n - w_n \to 0$ a.e., we have $\theta_n \to 0$ a.e. As $\pi(w_\infty) \leq 1$ a.e., $\pi(u_n) \leq 1$ a.e. and so $\theta_n \leq 1$ a.e. Also $\|u_n - \theta_n a\| = \|w_n\| = 1$. Finally observe

$$u_0 = (\alpha_0 + \theta_0) a$$
$$= \sum_{i=1}^{N} \lambda(E_i) e_i.$$

Hence

$$\pi(u_0) = \sum_{i=1}^{N} \lambda(E_i) = 1$$
$$= \alpha_0 + \theta_0.$$

Thus $\lambda(E_i) = a_i$ $(1 \leq i \leq N)$, and the proof is complete.

In fact we shall not use Theorem 2.2; instead we use its "finite" version.

THEOREM 2.3. *Under the same hypotheses as Theorem 2.2, given $\varepsilon > 0$, there is a finite dyadic martingale $(v_0, v_1, \ldots, v_m)$ with*

$$v_0(t) = a \qquad (0 \le t < 1), \tag{2.3.1}$$

$$\|v_m(t)\| \le 1 + \varepsilon \qquad (0 \le t < 1). \tag{2.3.2}$$

For $1 \le n \le m-1$, there is a positive $\mathcal{B}_n$-measurable function ϕ_n with $\phi_n \le 1$ and

$$\|v_n(t) - \phi_n(t)a\| \le 1 + \varepsilon \qquad (0 \le t < 1). \tag{2.3.3}$$

Proof. Suppose $0 < \delta_0 < \frac{1}{2}$ is chosen so that $\|2\delta_0 a\| < \frac{1}{2}\varepsilon$ and $\|(1-\delta_0)^{-1}\| < 1 + \frac{1}{2}\varepsilon$ whenever $\|x\| < 1$.

Let u_n, θ_n be chosen as in Theorem 2.2 and select m so that

$$\int_0^1 \theta_m(t)\, dt = \delta \le \delta_0.$$

Define

$$v_m = (1-\delta)^{-1}(u_m - \theta_m a)$$

and

$$v_n = \mathcal{E}(v_m \mid \mathcal{B}_n) \qquad (0 \le n \le m).$$

Then $\|v_m\| \le 1 + \varepsilon$ and

$$\begin{aligned} v_n &= (1-\delta)^{-1}(u_n - \mathcal{E}(\theta_m \mid \mathcal{B}_n)a) \\ &= (1-\delta)^{-1}(u_n - \theta_n a) + (1-\delta)^{-1}(\theta_n - \mathcal{E}(\theta_m \mid \mathcal{B}_n))a. \end{aligned}$$

Define

$$\phi_n = \theta_n - \mathcal{E}(\theta_m \mid \mathcal{B}_n) \qquad (0 \le n \le m).$$

Then $0 \le \phi_n \le \theta_n \le 1$ and

$$v_n - \phi_n a = (1-\delta)^{-1}(u_n - \theta_n a) + \delta(1-\delta)^{-1}\phi_n a$$

and so

$$\|v_n - \phi_n a\| \le 1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = 1 + \varepsilon.$$

3. Main results. We now turn to the general infinite-dimensional problem.

LEMMA 3.1. *Suppose X is an F-space with a strictly concave F-norm. Suppose $x_0 \ne 0$ and that $x_0 \in \operatorname{co}\{x : \|x\| \le \delta\}$. Then there is a finite dyadic martingale u_n $(0 \le n \le m)$ with $u_0(t) \equiv x_0$, and*

$$\|u_m(t)\| \le 2\delta \qquad (0 \le t < 1), \tag{3.1.1}$$

$$\|u_n(t)\| \le \|x_0\| + 2\delta \qquad (0 \le t < 1, 0 \le n \le m). \tag{3.1.2}$$

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Proof. There exist $y_1, \ldots, y_N \in X$ with $y_i \neq 0$ $(1 \leq i \leq N)$, $\|y_i\| \leq \delta$ and $x_0 = a_1 y_1 + \ldots + a_N y_N$, where $a_i \geq 0$ and $a_1 + a_2 + \ldots + a_N = 1$.

For $0 \leq t < \infty$, define

$$F_i(t) = \|ty_i\| / \|y_i\|.$$

Then F_i is strictly concave. Define the absolute norm on $\mathbb{R}^N$ by

$$\|b\| = \sum_{i=1}^{N} F_i(|b_i|).$$

Now, by Theorem 2.3, there is a finite $\mathbb{R}^N$-valued dyadic martingale $(v_n : 0 \leq n \leq m)$ with (taking $\varepsilon = 1$)

$$v_0(t) \equiv a = (a_1, \ldots, a_N) \qquad (0 \leq t < 1),$$
$$\|v_m(t)\| \leq 2 \qquad (0 \leq t < 1)$$

and

$$\|v_n(t) - \phi_n(t)a\| \leq 2 \qquad (0 \leq t < 1, 0 \leq n < m),$$

where $0 \leq \phi_n(t) \leq 1$. Define $T : \mathbb{R}^N \to X$ by

$$Tb = \sum_{i=1}^{N} b_i y_i.$$

Then

$$\begin{aligned} \|Tb\| &\leq \sum_{i=1}^{N} \|b_i y_i\| \\ &\leq \sum_{i=1}^{N} \|y_i\| \, F_i(|b_i|) \\ &\leq \delta \, \|b\|. \end{aligned}$$

Now let $u_n = Tv_n$. Then $u_0(t) \equiv x_0$ and $\|u_m(t)\| \leq 2\delta$. Also

$$\begin{aligned} \|u_n(t)\| &\leq \|\phi_n(t) x_0\| + 2\delta \\ &\leq \|x_0\| + 2\delta. \end{aligned}$$

THEOREM 3.2. *Suppose X is an F-space with trivial dual, and that $x_0 \in X$. Then there is a dyadic martingale $(u_n : n \geq 0)$ with $u_0(t) \equiv x_0$ and*

$$\max_{0 \leq t < 1} \|u_n(t)\| \to 0 \quad \textit{as} \quad n \to \infty. \tag{3.2.1}$$

Proof. As explained in the introduction we may suppose that the F-norm on X is strictly concave (passing to an equivalent F-norm does not affect (3.2.1)). The hypotheses guarantee that the convex hull of any neighborhood of zero is X. The construction is inductive, based on Lemma 3.1. To start the construction we may find a finite martingale

$(u_n : 0 \le n \le N_1)$ so that $u_0(t) = x_0$, $\|u_{N_1}(t)\| \le \frac{1}{2}\|x_0\|$ and $\|u_n(t)\| \le 2\|x_0\|$ $(1 \le n \le N_1)$, by applying Lemma 3.1 with $\delta = \frac{1}{4}\|x_0\|$ if $x_0 \ne 0$ (the case $x_0 = 0$ is trivial).

Suppose now we have defined $(u_n : 1 \le n \le N_k)$ so that

$$\|u_{N_j}(t)\| \le (\tfrac{1}{2})^j \|x_0\| \qquad (1 \le j \le k), \tag{3.2.2}$$

$$\|u_n(t)\| \le 2(\tfrac{1}{2})^j \|x_0\| \qquad (N_j < n < N_{j+1},\ 1 \le j \le k-1). \tag{3.2.3}$$

We shall show how to extend to a finite dyadic martingale $(u_n : 1 \le n \le N_{k+1})$ so that (3.2.2) and (3.2.3) hold for $j \le k+1$ and $j \le k$ respectively.

We have

$$u_{N_k}(t) = y_l \qquad (t \in D_{N_k,l}).$$

For each y_l, there is a finite martingale $(v_n^l : 0 \le n \le M)$ with

$$\begin{aligned} v_0^l(t) &= y_l \qquad (0 \le t \le 1), \\ \|v_M^l(t)\| &\le (\tfrac{1}{2})^{k+1}\|x_0\| \qquad (0 \le t \le 1), \\ \|v_n^l(t)\| &\le \|y_l\| + (\tfrac{1}{2})^{k+1}\|x_0\| \\ &\le (\tfrac{1}{2})^{k-1}\|x_0\| \qquad (0 \le t \le 1, 0 \le n \le M). \end{aligned}$$

Here M may be taken independent of l by simply extending the martingale where necessary by adding further terms equal to the last term of the sequence.

Now let $N_{k+1} = N_k + M$ and define

$$u_{N_k+i} = v_1^l(2^{N_k}t - l + 1) \qquad (t \in D_{N_k,l}).$$

It is now easy to verify that conditions (3.2.2) and (3.2.3) hold where applicable. Continuing in this way we clearly have (3.2.1) for the (infinite) martingale (u_n).

The step from Theorem 3.2 to our main result is a very simple one if X is a quasi-Banach space or more generally is exponentially galbed (see Turpin [**8**]). In such space there is a natural correspondence between curves with uniform zero derivative and dyadic martingales converging uniformly to 0. In a general F-space a little more sublety is required in the proof of the main theorem.

THEOREM 3.3. *Suppose X is an F-space with trivial dual and that $x_0, x_1 \in X$. Then there is a curve $g:[0,1] \to X$ with $g(0) = x_0$, $g(1) = x_1$ and*

$$\lim_{|t-s| \to 0} \frac{g(t) - g(s)}{t - s} = 0 \quad \textit{uniformly for} \quad 0 \le s, t \le 1. \tag{3.3.1}$$

In particular $g'(t) = 0$ for $0 \le t \le 1$.

Proof. It suffices to suppose $x_0 = 0$. Then there is a dyadic martingale $(u_n : n \ge 0)$ with

$$u_0(t) = x_1 \qquad (0 \le t < 1),$$

$$\max_{0 \le t < 1} \|u_n(t)\| = \varepsilon_n \to 0.$$

Choose $N_0=0$. Since each u_n has finite range it is possible to choose a strictly increasing sequence of positive integers $(N_k : k \geq 1)$ so that

$$\|2^{N_j-N_k}(u_k(t)-u_{k-1}(t))\| \leq 2^{j-k}\varepsilon_j \tag{3.3.2}$$

for $0 \leq j \leq k-1$, $0 \leq t < 1$. Each $t \in [0, 1)$ has a unique binary expansion

$$t = \sum_{j=1}^{\infty} \tau_j 2^{-j},$$

where each τ_j is zero or one and $\tau_j = 0$ infinitely often. Now define

$$v_k(t) = u_k\left(\sum_{j=1}^{k} \tau_{N_j} 2^{-j}\right).$$

(Recall that u_k is constant on the interval $\sum_{j=1}^{k} \tau_{N_j} 2^{-j} \leq t < \sum_{j=1}^{k} \tau_{N_j} 2^{-j} + 2^{-k}$.) Then we observe that v_k is a $\mathscr{B}_{N_k}$-martingale, with

$$\max_{0 \leq t < 1} \|v_k(t)\| = \varepsilon_k,$$

$$\mathscr{E}(v_k \mid \mathscr{B}_0) = \int_0^1 v_k(t)\, dt = \int_0^1 u_k(t)\, dt = x_1.$$

In fact we observe that

$$\mathscr{E}(v_k \mid \mathscr{B}_{N_k-1}) = v_{k-1}. \tag{3.3.3}$$

For $k \geq 1$ and $0 \leq t \leq 1$, we define

$$g_k(t) = \int_0^t v_k(s)\, ds$$

(the integrand is simple). Then each g_k is continuous and from (3.3.3) we have

$$g_k(t) = g_{k-1}(t) \quad \text{if} \quad 2^{N_k-1} t \in \mathbb{Z}.$$

Now suppose that $0 < t < 1$ and that $2l \leq 2^{N_k} t < 2l+1$, where l is an integer. Then

$$\begin{aligned} g_k(t) - g_{k-1}(t) &= \int_{2l/2^{N_k}}^{t} (v_k(s) - v_{k-1}(s))\, ds \\ &= (t - 2l(2^{-N_k}))(v_k(t) - v_{k-1}(t)). \end{aligned} \tag{3.3.4}$$

Equally, if $2l+1 \leq 2^{N_k} t < 2l+2$,

$$g_k(t) - g_{k-1}(t) = ((2l+2)2^{-N_k} - t)(v_k(t) - v_{k-1}(t)). \tag{3.3.5}$$

Combining these results, we have

$$\begin{aligned} \|g_k(t) - g_{k-1}(t)\| &\leq \max_{0 \leq t < 1} \|2^{-N_k}(v_k(t) - v_{k-1}(t))\| \\ &= \max_{0 \leq t < 1} \|2^{-N_k}(u_k(t) - u_{k-1}(t))\| \\ &\leq 2^{-k}\varepsilon_0. \end{aligned}$$

Hence (g_k) converges uniformly to a continuous function g on $[0, 1]$, and $g(0)=0$, $g(1)=x_1$.

Now suppose $0 \le s < t \le 1$. Then there is a least integer n so that for some integer l we have $2^n s \le l < l+1 \le 2^n t$. Clearly $2^{n-1}t - 2^{n-1}s < 2$ and $2^n t - 2^n s \ge 1$. Hence $2^{-n} \le t-s < 4 \, . \, 2^{-n}$ and $n \ge \log_2 1/(t-s)$.

Now suppose $N_{k-1} \le n < N_k$, where $1 \le k < \infty$. Suppose l_1 is the least integer not less than $2^n s$ and l_2 is the greatest integer not greater than $2^n t$. Then

$$2^n(g_{k-1}(t) - g_{k-1}(l_2 2^{-n})) = (2^n t - l_2) v_{k-1}(l_2 2^{-n}),$$
$$2^n(g_{k-1}(l_1 2^{-n}) - g_{k-1}(s)) = (l_1 - 2^n s) v_{k-1}(l_1 2^{-n}),$$
$$2^n(g_{k-1}(i2^{-n}) - g_{k-1}((i-1)2^{-n})) = v_{k-1}((i-1)2^{-n}).$$

Hence

$$\|2^n(g_{k-1}(l_2 2^{-n}) - g_{k-1}(l_1 2^{-n}))\| \le (l_2 - l_1)\varepsilon_{k-1}$$

and

$$\|2^n(g_{k-1}(t) - g_{k-1}(s))\| \le (l_2 - l_1 + 2)\varepsilon_{k-1}.$$

However $l_2 - l_1 \le 2^n(t-s) < 4$ so that $l_2 - l_1 + 2 \le 5$. Hence

$$\|2^n(g_{k-1}(t) - g_{k-1}(s))\| \le 5\varepsilon_{k-1}. \tag{3.3.6}$$

Now

$$2^n(g_k(t) - g_{k-1}(t)) = 2^{n-N_k}\rho(v_k(t) - v_{k-1}(t)),$$

where $0 \le \rho \le 1$, by (3.3.4) and (3.3.5). Hence

$$\|2^n(g_k(t) - g_{k-1}(t))\| \le \varepsilon_k + \varepsilon_{k-1}. \tag{3.3.7}$$

A similar inequality holds for s.

If $r > k$

$$2^n(g_r(t) - g_{r-1}(t)) = 2^{n-N_r}\rho(v_r(t) - v_{r-1}(t)),$$

where $0 \le \rho \le 1$, and so

$$\begin{aligned}\|2^n(g_r(t) - g_{r-1}(t))\| &\le \|2^{N_k - N_r}(v_r(t) - v_{r-1}(t))\| \\ &\le \max_{0 \le t < 1} \|2^{N_k - N_r}(u_r(t) - u_{r-1}(t))\| \\ &\le 2^{k-r}\varepsilon_k\end{aligned}$$

by (3.3.2). Hence

$$\|2^n(g(t) - g_k(t))\| \le \Big(\sum_{r>k} 2^{k-r}\Big)\varepsilon_k = \varepsilon_k. \tag{3.3.8}$$

A similar inequality holds for s.

Combining (3.3.6), (3.3.7) and (3.3.8) and the similar results for s we obtain

$$\|2^n(g(t)-g(s))\| \leq 7\varepsilon_{k-1}+4\varepsilon_k$$

and hence

$$\left\|\frac{g(t)-g(s)}{t-s}\right\| \leq 7\varepsilon_{k-1}+4\varepsilon_k,$$

where $N_k > \log_2 1/(t-s)$. Hence g has the properties specified in the theorem.

Every F-space X has a unique maximal linear subspace with trivial dual; this subspace is closed. Let us call this maximal subspace the *core* of X. If core $(X)=\{0\}$, it does not necessarily follow that X has a separating dual; for a detailed investigation of related ideas see Ribe [**3**]. We conclude with a simple corollary.

COROLLARY 3.4. *Suppose X is an F-space and $x \in X$. In order that there exists a curve $g:[0,1] \to X$ with $g(0)=0$, $g(1)=x$ and $g'(t)=0$ for $0 \leq t \leq 1$ it is necessary and sufficient that $x \in \operatorname{core}(X)$.*

Proof. If $x \in \operatorname{core}(X)$ the existence of g is given by Theorem 3.3. Suppose conversely such a g exists and let Y be the closed linear span of $\{g(t): 0 \leq t \leq 1\}$. Suppose ϕ is a continuous linear functional on Y. Then $(\phi \circ g)'(t)=0$ $(0 \leq t \leq 1)$ and hence by the Mean Value Theorem $\phi(g(t))=0$ $(0 \leq t \leq 1)$. Thus $\phi=0$ and so $Y \subset \operatorname{core}(X)$; in particular $x \in \operatorname{core}(X)$.

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