The intersection of certain quadrics.

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We investigate in this paper a certain special family of quadric varieties, that is of V^2_{R-1}'s in $[R]$. Now among the more important properties of a quadric in $[R]$ is that it possesses a system or systems of "generators," *i.e.*, the quadric may be taken as the locus of certain families of subspaces, the behaviour of these depending on the parity of R. If R is even, a quadric V^2_{2n-1} in $[2n]$ contains a single family of $[n-1]$'s, so it seems likely that in discussing special families of quadrics in $[2n]$ an important type will be obtained by constraining the quadric to pass through a number of $[n-1]$'s.

The freedom of quadrics in $[2n]$ is $n(2n+3)$, and since the postulation of an $[n-1]$ for quadrics is $\frac{1}{2}n(n+1)$, the freedom of a quadric in $[2n]$ which is to contain k assigned $[n-1]$'s in general position is $n(2n+3) - \frac{1}{2}kn(n+1)$. In order therefore that the freedom should not be negative we must have $k \leqq 4 + 2/(n+1)$, and it follows that (except for the trivial case where $n = 1$) *the maximum number of* $[n-1]$*'s which may be assigned to a quadric is four.* We therefore will discuss here the nature of *the family of quadrics obtained, subject only to the condition of possessing four assigned* $[n-1]$*'s in general position.*

By the reasoning given above, such a family has freedom n, and the quadrics will thus have a common intersection consisting of a V_{n-1} of order 2^{n+1}. It is found that this V_{n-1} (which we shall call the *base* of the family) is highly degenerate, and we shall limit ourselves to an investigation of the component parts of the base and of their relations to one another.

The establishment of the base occupies § 1 to § 4. In § 1 we define $4n$ varieties, of known orders and method of generation, which must belong to the base. We then show in § 2 that these varieties must lie in eight certain $[2n-3]$'s. We calculate the orders of the varieties

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in §3 and show in §4 that the base is composed entirely of the varieties. Then § 5 to § 7 is concerned with the relative positions of the parts of the base. In § 5 we discuss how the eight $[2n-3]$'s in which the base lies cut the different parts of the base and are thus able to discover in § 6 the relative positions of those components of the base lying in the same $[2n-3]$, and in § 7 those of components in different $[2n-3]$'s.

In what follows the geometrical detail is somewhat obscured by the necessity of making n general. It will perhaps make what happens clearer if we describe without proof the configuration of the varieties which is obtained for some of the lower values of n. There is a certain difference, largely superficial, between those cases where n is even and those where n is odd.

The case in which n is 2 is almost trivial. In a $[4]\,S$ we have given four lines a, b, c, d. Then there is a unique line a' crossing b, c, d; a line b' crossing a, c, d, and analogously defined lines c' and d'. *Then the quadrics through* a, b, c, d *cut in a* V_1^8 *which degenerates into the eight lines* a, b, c, d, a', b', c', d'.

The nature of the configuration is better displayed in a less elementary case; let us describe the figures obtained for $n=3$ and $n=4$. If n is 3, we are given in a $[6]\,S$ four generally placed planes a, b, c, d. Now there is just one plane a' which meets a in a line $L(a, a')$ and b, c, d in points $P(b, a')$, $P(c, a')$, $P(d, a')$, and there are analogous planes b', c', d'. The three lines $P(c, b')\,P(d, b')$, $P(d, c')\,P(b, c')$, $P(b, d')\,P(c, d')$ all lie in the $[3]\,A'$, the intersection of the three primes bc, cd, bd (where bc is the [5] containing b and c.) The quadric surface in A' through these three lines is the locus of all the lines meeting b, c, d. Let us call it $\mathcal{V}_1(A')$, and let us define similar quadric surfaces $\mathcal{V}_1(B')$, $\mathcal{V}_1(C')$, $\mathcal{V}_1(D')$. It will be noticed that both a and a' lie in the $[3]\,A$, the intersection of the primes ab, ac, ad; call them $\mathcal{V}_0(A)$ and $\mathcal{V}_2(A)$ respectively and give analogous names to b, c, d, b', c', d'. *Then the base of the quadrics in* S *which are constrained to pass through* a, b, c, d *is a* V_2^{16} *which degenerates into the eight planes* $\mathcal{V}_0(A)$, $\mathcal{V}_0(B)$, $\mathcal{V}_0(C)$, $\mathcal{V}_0(D)$, $\mathcal{V}_2(A)$, $\mathcal{V}_2(B)$, $\mathcal{V}_2(C)$, $\mathcal{V}_2(D)$ *and the four quadric surfaces* $\mathcal{V}_1(A')$, $\mathcal{V}_1(B')$, $\mathcal{V}_1(C')$, $\mathcal{V}_1(D')$. These varieties lie in the eight [3]'s A, B, C, D, A', B', C', D'; the intersections of the planes with one another and with the quadrics is clear from this description; any two of the quadrics cut in two points, $\mathcal{V}_1(A')$ and $\mathcal{V}_1(B')$, *e.g.*, cutting in the points $P(c, d')$ and $P(d, c')$.

If n is 4 we are given in an [8] S four generally placed [3]'s, a, b, c, d which we can call also $\mathcal{V}_0(A)$, $\mathcal{V}_0(B)$, $\mathcal{V}_0(C)$, $\mathcal{V}_0(D)$. We first define four further [3]'s a', b', c', d' which we call also $\mathcal{V}_3(A')$, $\mathcal{V}_3(B')$, $\mathcal{V}_3(C')$, $\mathcal{V}_3(D')$; a' is the unique [3] meeting a in a point $P(a, a')$ and b, c, d in lines $L(b, a')$, $L(c, a')$, $L(d, a')$, with analogous definitions for b', c', d'. Then the three planes $A'_b \equiv P(b, b')\, L(ba')$, $A'_c \equiv P(c, c')\, L(c, a')$, $A'_d \equiv P(d, d')\, L(d, a')$ all lie in the [5] A', the intersection of the three primes bc, bd, cd. Let us define $\mathcal{V}_1(A')$ as the locus of lines in A' meeting these three planes; it is a V_3^3. Define similarly $\mathcal{V}_1(B')$, $\mathcal{V}_1(C')$, $\mathcal{V}_1(D')$. [$\mathcal{V}_1(A')$ does not meet a, but meets b in the plane A'_b and meets c and d analogously; it meets a' in the quadric surface of lines crossing $L(b, a')$, $L(c, a')$, $L(d, a')$, meets b' in the line through $P(b, b')$, crossing $L(c, b')$, $L(d, b')$ and meets c' and d' analogously, and it meets $\mathcal{V}_1(B')$ in two lines, viz., the line through $P(c, c')$ crossing $L(c, a')$ and $L(c, b')$ and the line through $P(d, d')$ crossing $L(d, a')$ and $L(d, b')$, and meets $\mathcal{V}_1(C')$, $\mathcal{V}_1(D')$ analogously.]

Now define the line A_b as that through $P(b, b')$ meeting $L(b, c')$ and $L(b, d')$ and define similar lines A_c, A_d; then these three lines all lie with a in the [5] A, the intersection of the three primes ab, ac, ad. Define $\mathcal{V}_2(A)$ as the locus of planes in A meeting a in a line, A_b, A_c, A_d in points; it is a V_3^3. Define similarly $\mathcal{V}_2(B)$, $\mathcal{V}_2(C)$, $\mathcal{V}_2(D)$. [$\mathcal{V}_2(A)$ meets a in the quadric surface of lines crossing $L(a, b')$, $L(a. c')$, $L(a, d')$, meets b in the line A_b and meets c and d analogously; it does not meet a', but meets b' in the plane $L(a, b')\, P(b, b')$ and meets c' and d' analogously; it meets $\mathcal{V}_2(B)$ in two lines, viz., the line through $P(c, c')$ crossing $L(a, c')$, $L(b, c')$ and the line through $P(d, d')$ crossing $L(a, d')$, $L(b, d')$, and meets $\mathcal{V}_2(C)$ and $\mathcal{V}_2(D)$ analogously. Finally, $\mathcal{V}_1(A')$ does not meet $\mathcal{V}_2(A)$, but $\mathcal{V}_1(A')$ and $\mathcal{V}_2(B)$ meet in a quadric surface q and a line l defined as follows:—take the line B_c through $P(c, c')$ crossing $L(c, a')$ and $L(c, d')$, and the line B_d through $P(d, d')$ crossing $L(a, d')$ and $L(c, d')$; their joining [3] cuts the plane A'_b in a line λ, and the quadric q is the locus of lines meeting B_c, B_d, λ: the join of $P(a, a')$ and the plane A'_b is a [3] meeting B_c, B_d in two points J, K respectively, and the plane $JKP(a, a')$ cuts b in the line l.]

Then the quadrics through a, b, c, d cut in a V_3^{32} which degenerates into the eight [3]'*s*, $\mathcal{V}_0(A)$, $\mathcal{V}_0(B)$, $\mathcal{V}_0(C)$, $\mathcal{V}_0(D)$, $\mathcal{V}_3(A')$, $\mathcal{V}_3(B')$, $\mathcal{V}_3(C')$, $\mathcal{V}_3(D')$ *and the eight* V_3^3*'s* $\mathcal{V}_1(A')$, $\mathcal{V}_1(B')$, $\mathcal{V}_1(C')$, $\mathcal{V}_1(D')$, $\mathcal{V}_2(A)$, $\mathcal{V}_2(B)$, $\mathcal{V}_2(C)$, $\mathcal{V}_2(D)$.

We should perhaps note here that the cases given above are in many respects untypical. For larger values of n the varieties of the base increase both in number and dimension, the low value of n in the above cases makes many of the intersections of the varieties evanescent.

§ 1.

Let us now consider generally four $[n-1]$'s a, b, c, d generally placed in a $[2n]$ S, and the system of quadrics Q which contain a, b, c, d. We use two main tools in discovering the base of the system. (i) We find associated with a, b, c, d certain families of flat spaces every one of which must lie in every quadric Q. (ii) We find certain special quadrics Q in whose special intersection the base must lie.

The Varieties $\mathcal{V}_{2r}(A)$, $\mathcal{V}_{2r}(B)$, $\mathcal{V}_{2r}(C)$, $\mathcal{V}_{2r}(D)$. We shall begin by using the first of these methods. Firstly *consider a* $[2r]$ p *which cuts* a *in an* $[r]$ *and* b, c, d *in* $[r-1]$*s where* $0 \leqq 2r \leqq n-1$. A quadric Q will thus cut p in an $[r]$ and three $[r-1]$s, all generally placed, and since the postulation of these elements for quadrics is $\frac{1}{2}(r+1)(r+2)+3.\frac{1}{2}r(r+1)$ or $2r^2+3r+1$, while the freedom of quadrics in $[2r]$ is $2r^2+3r$, it follows that *any quadric* Q *contains the* $[2r]$ p. For example, if a quadric contains a line and three general points of a plane, then it contains that plane, or if it contains a plane and three general lines of a [4] it contains that [4].

Now in general p is not a fixed $[2r]$. The freedom of p is in fact $n-2r-1$ so that *the locus of* p *is a* V_{n-1}. Let us call it $\mathcal{V}_{2r}(A)$. Since every p lies in every quadric Q it follows that the whole of $\mathcal{V}_{2r}(A)$ lies in every Q. But the base of the family Q is a V_{n-1} and consequently $\mathcal{V}_{2r}(A)$ *is part of this base.* By taking different values of r we obtain a number of such parts; in fact, if n is odd we have $\frac{1}{2}(n+1)$, and if n is even $\frac{1}{2}n$ varieties such as $\mathcal{V}_{2r}(A)$. A noteworthy particular case occurs if $r=0$, $\mathcal{V}_0(A)$ being the locus of points which lie in a, *i.e.*, $\mathcal{V}_0(A)$ is the $[n-1]$ a.

By taking in the same manner those $[2r]$'s which meet b in an $[r]$ and a, c, d in $[r-1]$'s we obtain varieties which we may call $\mathcal{V}_{2r}(B)$, and we may define similar varieties $\mathcal{V}_{2r}(C)$, $\mathcal{V}_{2r}(D)$. All these varieties form part of the base of the family of quadrics Q.

The Varieties $\mathcal{V}_{2r+1}(A')$, $\mathcal{V}_{2r+1}(B')$, $\mathcal{V}_{2r+1}(C')$, $\mathcal{V}_{2r+1}(D')$. We now find a second series of varieties with similar properties. *Consider a* $[2r+1]$ p' *which cuts* a *in an* $[r-1]$ *and* b, c, d *in* $[r]$*'s*, where $0 \leqq 2r+1 \leqq n-1$. A quadric Q cuts p' in an $[r-1]$ and three $[r]$'s;

the postulation of these elements for quadrics is $\frac{1}{2}r(r+1)+3.\frac{1}{2}(r+1)(r+2)$ or $2r^2+5r+3$, while the freedom of quadrics in a $[2r+1]$ is $2r^2+5r+2$, so *every quadric Q must contain the* $[2r+1]\,p'$. For example, if a quadric contains three generally placed lines and a point of a [3], then it contains that [3].

As before, the spaces p' have as their locus a certain variety. The freedom of p' is $n-2r-2$, so *the locus of p' is a V_{n-1}*, which we shall call $\mathcal{V}_{2r+1}(A')$. For the same reasons as before, $\mathcal{V}_{2r+1}(A')$ *is part of the base of the family of quadrics Q*, and by taking different values of r we have a number of different varieties; in fact if n is odd we have $\frac{1}{2}(n-1)$ and if n is even $\frac{1}{2}n$ varieties such as $\mathcal{V}_{2r+1}(A')$. We define $\mathcal{V}_{2r+1}(B')$ as the locus of $[2r+1]$'s meeting b in an $[r-1]$ and a, c, d in $[r]$'s and we define similarly $\mathcal{V}_{2r+1}(C')$ and $\mathcal{V}_{2r+1}(D')$, all belonging to the base. *The total number of varieties we have hitherto defined is* $4n$, whether n be odd or even.

We should notice that if n is odd, $\mathcal{V}_{n-1}(A)$ is the unique $[n-1]$ meeting a in a $[\frac{1}{2}(n-1)]$ and b, c, d in $[\frac{1}{2}(n-3)]$'s and that if n is even, $\mathcal{V}_{n-1}(A')$ is the unique $[n-1]$ meeting a in a $[\frac{1}{2}(n-4)]$, b, c, d in $[\frac{1}{2}(n-2)]$'s. So in either case among the varieties of the base there are contained not only the four original $[n-1]$'s a, b, c, d but also four other $[n-1]$'s, which cross a, b, c, d in a manner depending on the parity of n. For instance in [4] we have the four original lines a, b, c, d and also the four lines each meeting three of a, b, c, d; while in [6] we have the four planes a, b, c, d and four further planes each meeting one of a, b, c, d in a line and the other three in points.

§ 2.

The $[2n-3]$*'s* A, B, C, D, A', B', C', D', *and their relation to* $\mathcal{V}_{2r}(A)$, $\mathcal{V}_{2r+1}(A')$ *etc.* Let us now use our second method of approach to the problem and consider some special members of the family of quadrics Q. Prominent among these are three which consist of prime-pairs. In fact, if we denote the prime joining a to b by (ab) with similar symbols for other joins, we have the quadric consisting of (ab) and (cd), that consisting of (ac) and (bd) and that consisting of (ad) and (bc). These three quadrics cut in a V^8_{2n-3} which obviously degenerates into the eight $[2n-3]$'s

$$A \equiv (ab)\,(ac)\,(ad) \qquad A' \equiv (bc)\,(bd)\,(cd)$$
$$B \equiv (ab)\,(bc)\,(bd) \qquad B' \equiv (ac)\,(ad)\,(cd)$$
$$C \equiv (ac)\,(bc)\,(cd) \qquad C' \equiv (ab)\,(ad)\,(bd)$$
$$D \equiv (ad)\,(bd)\,(cd) \qquad D' \equiv (ab)\,(ac)\,(bc)$$

where, for example, $(ab)(ac)(ad)$ means the $[2n-3]$ common to the three primes (ab), (ac), (ad). It follows therefore that *the base of the family of quadrics Q*, that is, the total intersection of all the quadrics Q, including the three prime-pairs, *must lie in these eight* $[2n-3]$'*s*. There are $n+1$ linearly independent quadrics; if three of these are taken as the prime-pairs, the remaining $n-2$ will cut out the base on the eight $[2n-3]$'s A, B, C, D, A', B', C', D'. They cut A, for example in $n-2$ linearly independent quadrics V^2_{2n-4} of A, whose intersection is a V_{n-1} of order 2^{n-2}, and similarly for the other $[2n-3]$'s making the total base of order 2^{n+1}.

Now we have already discovered $4n$ varieties which must belong to the base; let us consider how they lie with respect to the $[2n-3]$'s.

$\mathcal{V}_{2r}(A)$ is the locus of $[2r]$'s p which cut a in an $[r]$ and b, c, d in $[r-1]$'s. Consequently p lies in the prime (ab), similarly it lies in (ac) and (ad) and hence in A. It follows that *the variety* $\mathcal{V}_{2r}(A)$ *lies in* A and similarly $\mathcal{V}_{2r}(B)$, $\mathcal{V}_{2r}(C)$, $\mathcal{V}_{2r}(D)$ lie respectively in B, C, D. And in the same way $\mathcal{V}_{2r+1}(A')$ is the locus of $[2r+1]$'s p' which cut a in an $[r-1]$ and b, c, d in $[r]$'s. And so p' lies in the primes (bc), (bd), (cd) and hence in A'. Thus *the variety* $\mathcal{V}_{2r+1}(A')$ *lies in* A' and similarly $\mathcal{V}_{2r+1}(B')$, $\mathcal{V}_{2r+1}(C')$, $\mathcal{V}_{2r+1}(D')$ lie respectively in B', C', D'.

Since $\mathcal{V}_{2r}(A)$, for example, lies entirely in A, it should be possible to define it in terms of elements which lie exclusively in A. But in order to do this we must first investigate the traces in A of the fundamental spaces a, b, c, d and of the figure obtained from them by join and section.

The six primes (ab) (ac) (ad) (bc) (bd) (cd) cut in a $[2n-6]$ X. The primes (ab) (ac) (ad) all pass through a, so X cuts a in the $[n-4]$ common to a (bc) (bd) (cd); call it $\mathfrak{a}$. Similarly X cuts b, c, d in $[n-4]$'s $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$. It is clear that all eight of $A, B, C, D, A', B', C', D'$ pass through X.

A contains a and cuts b in the $[n-3]$ $b(ac)(ad) \equiv A_b$ and cuts c and d in similar $[n-3]$'s A_c, A_d, while B, C, D behave analogously. A' cuts a in the $[n-4]$ $\mathfrak{a}$, cuts b in the $[n-2]$ $b(cd) \equiv A'_b$ and cuts c and d in similar $[n-2]$'s A'_c, A'_d, while B', C', D' behave analogously. So a, for example, is cut by B', C', D' in $[n-2]$'s B'_a, C'_a, D'_a, by B, C, D in $[n-3]$'s B_a, C_a, D_a, and by A' in the $[n-4]$ $\mathfrak{a}$. It is obvious from their mode of formation that B'_a, C'_a cut in D_a; C'_a, D'_a cut in B_a; D'_a, B'_a cut in C_a; and that all six pass through $\mathfrak{a}$. In fact, in a, B'_a, C'_a, D'_a are three primes, B_a, C_a, D_a their intersections by pairs, and $\mathfrak{a}$ their total intersection.

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Let us also work out the intersection of any two of A, B, C, D, A', B', C', D'. It will be sufficient to find the section of A with any other $[2n-3]$ and also of A' with any other $[2n-3]$ of the set.

From their definitions it is clear that A and A' cut in the $[2n-6]\,X$. A and B cut in the $[2n-5]\,AB$, the intersection of $(ab)(ac)(ad)(bc)(bd)$, while A and C, A and D cut in similar $[2n-5]$'s AC and AD. A and B' cut in the $[2n-4]\,AB'$, the intersection of $(ab)(ac)(ad)(cd)$, while A and C', A and D' cut in similar $[2n-4]$'s AC' and AD'. AC' and AD' cut in AB, AD' and AB' cut in AC, AB' and AC' cut in AD, while all six go through X. AB', AC', AD' are three primes in A; AB, AC, AD are their intersections by pairs and X is their complete intersection.

In A, X and A_b both belong also to B, and since they cut in $\mathfrak{b}$ their join is a $[2n-5]$ which we can thus identify with AB. Similarly AC and AD are the joins of X to A_c and A_d respectively. And since AB' is the join of AC and AD, AB' is the join of X to A_c and A_d, and similar results are true for AC' and AD'.

Let us now find similarly the intersections of A' with the other $[2n-3]$'s. A' cuts A in the $[2n-6]\,X$, A' and B' in the $[2n-5]$ $A'B'$, the intersection of $(ac)(ad)(bc)(bd)(cd)$, while A' cuts C' and D' similarly in $[2n-5]$'s $A'C'$, $A'D'$. We notice that $A'B'$ is the same $[2n-5]$ as CD, $A'C'$ the same as BD and so on. A' and B cut in the $[2n-4]$ $A'B$, the intersection of $(ab)(bc)(bd)(cd)$, and A' cuts C and D in similar $[2n-4]$'s $A'C$, $A'D$. $A'B$ and $A'C$ cut in $A'D'$, $A'C$ and $A'D$ cut in $A'B'$, $A'D$ and $A'B$ cut in $A'C'$ and all six go through X. In fact, $A'B$, $A'C$, $A'D$ are three primes in A', $A'B'$, $A'C'$, $A'D'$ are their intersections by pairs, and X their complete intersection.

In A', X and A'_b both belong to B, and since they cut in $\mathfrak{b}$ their join is a $[2n-4]$ which we thus identify with $A'B$. Similarly $A'C$, $A'D$ are the joins of X to A'_c and A'_d respectively. $A'B'$ may be defined in A' as the intersection of $A'C$, $A'D$, and similarly for $A'C'$ and $A'D'$.

Returning to the consideration of the varieties $\mathcal{V}_{2r}(A)$, $\mathcal{V}_{2r+1}(A')$ etc., it is now clear that *in A, $\mathcal{V}_{2r}(A)$ may be defined as the locus of $[2r]$'s which cut $\mathfrak{a}$ in an $[r]$ and A_b, A_c, A_d in $[r-1]$'s, and that in A', $\mathcal{V}_{2r+1}(A')$ may be defined as the locus of $[2r+1]$'s which cut $\mathfrak{a}$ in an $[r-1]$ and A'_b, A'_c, A'_d in an $[r]$*, and that similar definitions may be made for the remaining varieties.

§ 3.

This information now permits us to calculate the orders of the varieties $\mathcal{V}_{2r}(A)$, $\mathcal{V}_{2r+1}(A')$. It will be convenient to prove the results in a slightly more general form than is immediately required, so we present the calculation in the form of two Lemmas.

In a $[2n-3]$ Σ *take one* $[n-1]$ α *and three* $[n-3]$*'s*, β, γ, δ, *and consider the set* Γ *of* $[2r]$*'s which meet* α *in an* $[r]$ *and* β, γ, δ *in* $[r-1]$*'s*; their freedom is $n-2r-1$, so they build a V_{n-1}.

Lemma L. The locus of Γ *is a* V_{n-1} *of order* ${}^{n-1}C_{2r}$.

In a $[2n-3]$ Σ' *take one* $[n-4]$ α' *and three* $[n-2]$*'s*, β', γ', δ', *and consider the set* Γ' *of* $[2r+1]$*'s which meet* α' *in an* $[r-1]$ *and* β', γ', δ' *in* $[r]$*'s*; their freedom is $n-2r-2$, so they build a V_{n-1}.

Lemma L'. The locus of Γ' *is a* V_{n-1} *of order* ${}^{n-1}C_{2r+1}$.

The proof of these lemmas is by a simultaneous induction; let us make the temporary hypothesis that both lemmas are true for all values of r if n is replaced by $n-1$. We assume in all this work that the sympol ${}^{p}C_{q}$ with $q > p$ means zero.

Lemma L. In Σ take the section of Γ by a general prime $[2n-4]$ ϖ through γ, δ; ϖ cuts α in an $[n-2]$ α_1, β in an $[n-4]$ β_1. Now $[2r]$'s of Γ either (i) lie entirely in ϖ, or (ii) cut it in $[2r-1]$'s.

(i) $[2r]$'s of Γ in ϖ cut α_1 in an $[r]$, β_1, γ, δ in $[r-1]$'s. Consequently they all lie in the $[2n-5]$ $\alpha_1\beta_1 \equiv \sigma$; σ contains the $[n-2]$ α_1 and the $[n-4]$ β_1 and cuts γ and δ in $[n-4]$'s γ_1 and δ_1. Then the $[2r]$'s we are discussing meet α_1 in an $[r]$, β_1, γ_1, δ_1 in $[r-1]$'s, and therefore afford a case of Lemma L with σ, α_1, β_1, γ_1, δ_1 in place of Σ, α, β, γ, δ, and with n reduced to $n-1$. So they form a V_{n-2} of order ${}^{n-2}C_{2r}$. All the $[2r]$'s of this variety are $[2r]$'s of Γ.

(ii) If a $[2r]$ of Γ cuts ϖ in a $[2r-1]$, this $[2r-1]$ cuts α_1 in an $[r-1]$, β_1 in an $[r-2]$, γ and δ in $[r-1]$'s and therefore lies in the $[2n-5]$ $\gamma\delta \equiv \varsigma$; ς cuts α in an $[n-3]$ α_2, β in an $[n-5]$ β_2. So the $[2r-1]$'s in ς cut α_2, γ, δ in $[r-1]$'s and β_2 in an $[r-2]$ and thus afford a case of Lemma L' with ς, β_2, α_2, γ, δ in place of Σ', α', β', γ', δ', and with n reduced to $n-1$, r reduced to $r-1$. So they form a V_{n-2} of order ${}^{n-2}C_{2r-1}$. But we must now show that the whole of this V_{n-2} lies in the V_{n-1} the locus of Γ. In fact if χ is any $[2r-1]$ of this V_{n-2}, χ and β cut in an $[r-2]$ and thus have as their join an $[r+n-2]$ $\chi\beta$. Then $\chi\beta$ and α will cut in an $[r]$ ρ. ρ and χ cut in the $[r-1]$ common to α and χ, so their join is a $[2r]$ $\chi\rho$. $\chi\rho$ and β both lie in $\chi\beta$, and therefore $\chi\rho$ cuts β in an $[r-1]$ and so $\chi\rho$ is a $[2r]$ of Γ.

So it follows that the V_{n-1} locus of Γ cuts ϖ in two V_{n-2}'s of orders ${}^{n-2}C_{2r}$ and ${}^{n-2}C_{2r-1}$, and therefore the order of V_{n-1} is ${}^{n-2}C_{2r} + {}^{n-2}C_{2r-1}$, that is, is ${}^{n-1}C_{2r}$.

Lemma L'. In Σ' take the section of Γ' by a general prime $[2n-4]\,\varpi'$ through α', β'; ϖ' cuts γ' and δ' in $[n-3]$'s γ'_1, δ'_1. Now $[2r+1]$'s of Γ' either (i) lie entirely in ϖ' or (ii) cut it in $[2r]$'s.

(i) $[2r+1]$'s of Γ' in ϖ' cut α' in an $[r-1]$, β', γ'_1, δ'_1 in $[r]$'s. Consequently they all lie in the $[2n-5]\,\gamma'_1\,\delta'_1 \equiv \sigma'$; σ' cuts α' in an $[n-5]\,\alpha'_1$, cuts β' in an $[n-3]\,\beta'_1$ and contains the $[n-3]$'s γ'_1, δ'_1. Then the $[2r+1]$'s we are discussing meet α'_1 in an $[r-1]$, β'_1, γ'_1, δ'_1 in $[r]$'s and therefore afford a case of Lemma L' with σ', α'_1, β'_1, γ'_1, δ'_1 in place of Σ', α', β', γ', δ' and with n reduced to $n-1$. So these $[2r+1]$'s form a V_{n-2} of order ${}^{n-2}C_{2r+1}$. All the $[2r+1]$'s of this variety are $[2r+1]$'s of Γ'.

(ii) If a $[2r+1]$ of Γ' cuts ϖ' in a $[2r]$, this $[2r]$ cuts α' in an $[r-1]$, β' in an $[r]$, γ'_1 and δ'_1 in $[r-1]$'s and therefore lies in the $[2n-5]\,\alpha'\,\beta' \equiv \varsigma'$; ς' cuts γ'_1 and δ'_1 in $[n-4]$'s γ'_2, δ'_2. So the $[2r]$'s in ς' cut α', γ'_2, δ'_2 in $[r-1]$'s and β' in an $[r]$, and thus afford a case of Lemma L with ς', β', α', γ'_2, δ'_2 in place of Σ, α, β, γ, δ, and with n reduced to $n-1$. So they form a V_{n-2} of order ${}^{n-2}C_{2r}$. We must now show that the whole of this V_{n-2} lies in the V_{n-1} locus of Γ'. In fact, if χ' is any $[2r]$ of this V_{n-2}, χ' and δ' cut in an $[r-1]$ and thus have as their join an $[r+n-1]\,\chi'\,\delta'$. Then $\chi'\,\delta'$ and γ' will cut in an $[r]\,\rho'$. ρ' and χ' cut in the $[r-1]$ common to γ' and χ', so their join is a $[2r+1]\,\rho'\,\chi'$. $\rho'\,\chi'$ and δ' both lie in $\chi'\,\delta'$ and therefore $\chi'\,\rho'$ cuts δ' in an $[r]$, and so $\chi'\,\rho'$ is a $[2r+1]$ of Γ'.

So the V_{n-1} locus of Γ' cuts ϖ' in two V_{n-2}'s of orders ${}^{n-2}C_{2r+1}$ and ${}^{n-2}C_{2r}$, and therefore the order of V_{n-1} is ${}^{n-2}C_{2r+1} + {}^{n-2}C_{2r}$, that its, ${}^{n-1}C_{2r+1}$.

If therefore we can prove the two lemmas for any particular value of n, it will follow that they are true for all higher values. But in the particular case when n is 3, the lemmas are easily verified. As regards Lemma L, α is a plane and β, γ, δ three points in the $[3]\,\Sigma$. There are two possible values of r, viz., $r=0$, $r=1$. In the first case Γ is the set of points lying in α; their locus is thus the $V_2^1\,\alpha$ of order ${}^2C_0$; and in the second case, Γ is the set of planes meeting α in a line and passing through β, γ, δ; their locus is the $V_2^1\,\beta\gamma\delta$ of order ${}^2C_2$. And for Lemma L', β', γ', δ' are three lines in the $[3]\,\Sigma'$. The only value of r is 0, when Γ' consists of the set of lines meeting β', γ', δ' in points; so the locus of Γ' is a V_2^2 whose order is ${}^2C_1$.

The two Lemmas are thus proved generally.

We have proved these Lemmas in order to find the orders of $\mathcal{V}_{2r}(A)$ and $\mathcal{V}_{2r+1}(A')$. Now $\mathcal{V}_{2r}(A)$ is a case of the locus in Lemma L, where A, a, A_b, A_c, A_d take the place of Σ, α, β, γ, δ. *So* $\mathcal{V}_{2r}(A)$ *is a* V_{n-1} *of order* ${}^{n-1}C_{2r}$. Similarly $\mathcal{V}_{2r+1}(A')$ is a case of the locus in Lemma L', where A', $\mathfrak{a}$, A'_b, A'_c, A'_d take the place of Σ', α', β', γ', δ'. *So* $\mathcal{V}_{2r+1}(A')$ *is a* V_{n-1} *of order* ${}^{n-1}C_{2r+1}$. There are of course similar results for $\mathcal{V}_{2r}(B)$, etc.

Now let us recall that (i) the base of the system of quadrics lies entirely in $A, B, C, D, A', B', C', D'$, (ii) the total order of the base in any of these $[2n-3]$'s is 2^{n-2}, (iii) the varieties $\mathcal{V}_{2r}(A)$, $\mathcal{V}_{2r+1}(A')$, etc., belong to the base.

But by what we have just proved, the sum of the orders of all the varieties $\mathcal{V}_{2r}(A)$ (*i.e.* of $\mathcal{V}_0(A)$, $\mathcal{V}_2(A)$, $\mathcal{V}_4(A)$, etc.) is $\Sigma_r {}^{n-1}C_{2r}$, $0 \leqq 2r \leqq n-1$. But by elementary algebra this sum equals 2^{n-2}, which is just the order we require for the total order of the base in A.

And similarly, the sum of the orders of all the varieties $\mathcal{V}_{2r+1}(A')$ (*i.e.*, of $\mathcal{V}_1(A')$, $\mathcal{V}_3(A')$, $\mathcal{V}_5(A')$, etc.) is $\Sigma_r {}^{n-1}C_{2r+1}$, $0 < 2r+1 \leqq n-1$. But this is of course again equal to 2^{n-2} which is the number required for the total order of the base in A'.

Before concluding that the base consists of these varieties there is a doubt which must be resolved. We have hitherto tacitly assumed that the $n-2$ linearly independent quadrics in A, say, cut in a V_{n-1}. If this is so, then the varieties $\mathcal{V}_{2r}(A)$ will make up the entire base in A, since their total order is 2^{n-2}. But might not these quadrics cut in a V_n, or some variety of higher dimension, in the same way as three quadrics in [3] may cut in a twisted cubic curve? We shall show that they cannot, by considering certain sections of the figure which we shall develop in the next paragraph.

§ 4.

The spaces X, Y, Z. We have already defined the $[2n-6]$ $X \equiv (ab)(ac)(ad)(bc)(bd)(cd)$. It contains the $[n-4]$'s $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$. Beginning with these we can build up in X a configuration similar to that in the original space S, with $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ taking the place of a, b, c, d. For example, the three $[2n-7]$'s $(\mathfrak{a}, \mathfrak{b})$ $(\mathfrak{a}\mathfrak{c})(\mathfrak{a}\mathfrak{d})$ cut in the $[2n-9]$ $\mathfrak{A}$, the three $[2n-7]$'s $(\mathfrak{b}\mathfrak{c})(\mathfrak{b}\mathfrak{d})(\mathfrak{c}\mathfrak{d})$ cut in the $[2n-9]$ $\mathfrak{A}'$, while all six $[2n-7]$'s cut in the $[2n-12]$ $\mathfrak{X}$. In $\mathfrak{A}$, the locus of $[2r]$'s meeting $\mathfrak{a}$ in an $[r]$ and $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in $[r-1]$'s is a V_{n-4} of order ${}^{n-4}C_{2r}$ which we may call $\mathcal{V}_{2r}(\mathfrak{A})$, and in $\mathfrak{A}'$, the locus of $[2r+1]$'s meeting $\mathfrak{a}$ in an $[r-1]$ and $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in $[r]$'s is a V_{n-4} of order

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${}^{n-4}C_{2r+1}$ which we may call $\mathfrak{V}_{2r+1}(\mathfrak{A}')$. In fact, the properties of the configuration in X are exactly analogous to those of the general configuration in S with n reduced to $n-3$. Furthermore $\mathfrak{X}$ cuts $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in $[n-7]$'s $\mathfrak{a}^*$, $\mathfrak{b}^*$, $\mathfrak{c}^*$, $\mathfrak{d}^*$ and from these can be obtained a similar configuration in $\mathfrak{X}$ with n reduced to $n-6$; in $\mathfrak{X}$ we have a $[2n-18]$ $\mathfrak{X}^*$ in which is another similar configuration, and the process may be continued until the dimensions of the spaces concerned become zero, giving a "nest" of configurations each lying inside all that precede it.

We introduce also two rather similar, but asymetrically placed, spaces Y and Z in which exist configurations analogous to that in S but with n reduced to $n-2$ and $n-1$ respectively.

Define Y as the $[2n-4]$ $(ac)(ad)(bc)(bd)$. Then it cuts a in $a(bc)(bd)$, *i.e.*, in B_a, and similarly cuts b in A_b, c in D_c and d in C_d. Call these $[n-3]$'s a_i, b_i, c_i, d_i. From them we may build up in Y a configuration similar to that in S with n reduced to $n-2$. It is important to be able to recognise the positions which certain elements in this configuration hold in the original configuration in S. Let us identify A_i, B , C'_i, D'_i.

The $[2n-5]$ (a_i, b_i) is the join of B_a and A_b; it is consequently the $[2n-5]$ AB, and so A_i lies in AB. Now AB cuts a in B_a, b in A_b, c in $\mathfrak{c}$, d in $\mathfrak{d}$. The $[2n-6]$ $\mathfrak{c}B_a$ joining $\mathfrak{c}$ to B_a lies in $a_i c_i$ for B_a is a_i and $\mathfrak{c}$ lies in D_c which is c_i; so the $[2n-6]$ $\mathfrak{c}B_a$ is the $[2n-6]$ $(a_i b_i)(a_i c_i)$. For similar reasons the $[2n-6]$ $\mathfrak{d}B_a$ is the $[2n-6]$ $(a_i b_i)(a_i d_i)$. Consequently the $[2n-7]$ $A_i \equiv (a_i b_i)(a_i c_i)(a_i d_i)$ is the intersection of the $[2n-6]$'s $\mathfrak{c}B_a$ and $\mathfrak{d}B_a$ in the $[2n-5]$ AB, and similarly B_i is the intersection of $\mathfrak{c}A_b$ and $\mathfrak{d}A_b$. The $[2n-6]$ $\mathfrak{d}B_a$ lies in $(a_i d_i)$ and the $[2n-6]$ $\mathfrak{d}A_b$ lies in $(b_i d_i)$, so these $[2n-6]$'s are $(a_i b_i)(a_i d_i)$ and $(a_i b_i)(b_i d_i)$, and their intersection is $(a_i b_i)(a_i d_i)(b_i d_i)$, *i.e.*, is C'_i, and similarly D'_i is the intersection of $\mathfrak{c}B_a$ and $\mathfrak{c}A_b$.

The locus of $[2r]$'s meeting a_i in an $[r]$ and b_i, c_i, d_i in $[r-1]$'s is a V_{n-3} of order ${}^{n-3}C_{2r}$, called $\mathfrak{V}_{2r}(A_i)$, and the locus of $[2r+1]$'s meeting a_i in an $[r-1]$ and b_i, c_i, d_i in $[r]$'s is a V_{n-3} of order ${}^{n-3}C_{2r+1}$, called $\mathfrak{V}_{2r+1}(A'_i)$.

It is of course possible to define Y_i as the $[2n-8]$ $(a_i c_i)(a_i d_i)$ $(b_i c_i)(b_i d_i)$; it cuts a_i, b_i, c_i, d_i in the $[n-5]$'s a_{ii}, b_{ii}, c_{ii}, d_{ii} and a configuration may be obtained from these in Y_i similar to the configuration in S, but with n replaced by $n-4$; and the process may be continued until the dimensions of the spaces concerned vanish, giving a nest of configurations each lying in all that precede.

In the same kind of way let us define Z as the $[2n-2]$ $(ab)(cd)$. Then Z cuts a in $a(cd)$, *i.e.* in B'_a, and similarly cuts b, c, d in A'_b, D'_c, C'_d. Call these $[n-2]$'s a_1, b_1, c_1, d_1. The configuration obtained in Z from these is similar to that in S with n replaced by $n-1$. The locus of $[2r]$'s meeting a_1 in an $[r]$ and b_1, c_1, d_1 in $[r-1]$'s is a V_{n-2} of order ${}^{n-2}C_{2r}$, called $\mathcal{V}_{2r}(A_1)$, and we can define in a similar way $\mathcal{V}_{2r+1}(A'_1)$.

Let us identify in S A_1 and B'_1. We shall show that A_1 is the join of B'_a and $\mathfrak{b}$. Since B'_a is a_1, B'_a lies in $(a_1 b_1)$, $(a_1 c_1)$, $(a_1 d_1)$ and thus in A_1. $\mathfrak{b}$ is $b(ac)(ad)(cd)$, and b_1 is $b(cd)$ so that $\mathfrak{b}$ lies in b_1 and thus in $(a_1 b_1)$. Now $(a_1 c_1)$ is $(ac)(ab)(cd)$ for each of (ac), (ab), (cd) contains both a_1 and c_1, and $(a_1 c_1)$ and $(ac)(ab)(cd)$ are both $[2n-3]$'s. Hence $\mathfrak{b}$ lies in $(a_1 c_1)$ and for similar reasons $\mathfrak{b}$ lies in $(a_1 d_1)$. Consequently $\mathfrak{b}$ lies in A_1. Now the join of B'_a and b is a $[2n-5]$ and consequently coincides with A_1.

We shall show similarly that B'_1 is the join of A_c and A_d. B'_1 is $(a_1 c_1)(a_1 d_1)(c_1 d_1)$, and c_1 is D'_c and therefore contains A_c, so that A_c lies in $(a_1 c_1)$ and in $(c_1 d_1)$. Now $(a_1 d_1)$ is $(ad)(ab)(cd)$, so A_c lies in $(a_1 d_1)$. Therefore B'_1 contains A_c and similarly contains A_d, and since the join of A_c and A_d is a $[2n-5]$ it coincides with B'_1.

We can define Z_1 as $(a_1 b_1)(c_1 d_1)$. Z_1 cuts a_1, b_1, c_1, d_1 in $[n-3]$'s a_2, b_2, c_2, d_2 and a configuration may be obtained from these in Z_1 similar to that in S with n replaced by $n-2$. And this process may be continued until the dimensions of the spaces concerned vanish.

Let us now recall the position we had reached with the base of the family of quadrics Q. We wish to ascertain that the $n-2$ linearly independent quadrics in A do not cut in more than a V_{n-1}. To do this cut the configuration in A by the prime $[2n-4]$ AB'. There is no member of the family in A containing this prime, so the family Q cuts AB' in $n-2$ linearly independent quadrics. AB' cuts a in the $[n-2]$ B'_a, cuts A_b in the $[n-4]$ $\mathfrak{b}$, and contains the $[n-3]$'s A_c and A_d, and the $n-2$ quadrics pass through these. Now if the base in A were a V_n, say, these $n-2$ quadrics in AB' would have to cut in a V_{n-1}. But one of these independent quadrics in AB' may be taken as the prime-pair consisting of the join of $\mathfrak{b}B'_a$ and the join of $A_c A_d$, *i.e.* as A_1 and B'_1, and thus the V_{n-1} must break up and lie in A_1 and B'_1. The configurations in A_1 and B'_1 are like those in A and B' with n reduced by one, so we now see that if the base in A consists of more than a V_{n-1}, then the base in either A_1 or B'_1 or both consists of more than a V_{n-2}. By a sufficient continuation we can show similarly that there is a [3] A_{n-3} or B'_{n-3} where the base consists of more than a V_2. But this is obviously not the case. A similar proof applies to B', cutting it by the prime AB'.

We arrive therefore at the theorem:—

The base of the family of quadrics through a, b, c, d consists of the $4n$ *varieties* $\mathcal{V}_{2r}(A)$, $\mathcal{V}_{2r}(B)$, $\mathcal{V}_{2r}(C)$, $\mathcal{V}_{2r}(D)$, $\mathcal{V}_{2r+1}(A')$, $\mathcal{V}_{2r+1}(B')$, $\mathcal{V}_{2r+1}(C')$, $\mathcal{V}_{2r+1}(D')$.

Thus to take a specific case, the base of the family of quadrics passing through four [5]'s a, b, c, d in a [12] S breaks up into twenty-four V_5's, three lying in each of eight [9]'s A, B, C, D, A', B', C', D'. A contains the [5] a and three [3]'s A_b, A_c, A_d; A' contains the [2] α and three [4]'s A'_b, A'_c, A'_d. Then the varieties are

in A	in A'.
$\mathcal{V}_0(A)$, a V_5^1, locus of points in a, (*i.e.* it is a),	$\mathcal{V}_1(A')$, a V_5^5, locus of lines meeting A'_b, A'_c, A'_d in points,
$\mathcal{V}_2(A)$, a V_5^{10}, locus of planes meeting a in a line, A_b, A_c, A_d in points,	$\mathcal{V}_3(A')$, a V_5^{10}, locus of [3]'s meeting α in a point, A'_b, A'_c, A'_d in lines,
$\mathcal{V}_4(A)$, a V_5^5, locus of [4]'s meeting a in a plane, A_b, A_c, A_d in lines,	$\mathcal{V}_5(A')$, a V_5^1, the [5] meeting α in a line, A'_b, A'_c, A'_d in planes.

There are similar configurations in the remaining [9]'s, B, C, D, B', C', D'.

§ 5.

Our task is now to establish the manner in which these varieties are mutually related. But since they lie in the eight $[2n-3]$'s A, B, C, D, A', B', C', D' we had better find first how these varieties cut the $[2n-3]$'s. There are six kinds of intersection possible; let us take a typical specimen of each type:—the intersection of $\mathcal{V}_{2r}(A)$ with A', with B, with B'; the intersection of $\mathcal{V}_{2r+1}(A')$ with A, with B', with B.

The intersection of $\mathcal{V}_{2r}(A)$ *with* A'. $\mathcal{V}_{2r}(A)$ lies in A, and A' cuts A in X, so the intersection is the intersection in A of $\mathcal{V}_{2r}(A)$ with X. If ϖ is a $[2r]$ of $\mathcal{V}_{2r}(A)$, ϖ meets A_b, A_c, A_d in $[r-1]$'s, and thus meets $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in $[r-2]$'s. So ϖ meets X in at least a $[2r-3]\,\lambda$, but it might meet it in a $[2r-2]\,\mu$, or a $[2r-1]\,\nu$, or might lie completely in X.

(i) *If ϖ meets X in a* $[2r-3]\,\lambda$, then λ meets $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in $[r-2]$'s. ϖ meets a in an $[r]\,\sigma$ and λ and σ lie in ϖ and so cut in an $[r-3]$.

So λ cuts $\mathfrak{a}$ in an $[r-3]$ and $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in $[r-2]$'s and is thus a $[2r-3]$ of $\mathcal{V}_{2r-3}(\mathfrak{A}')$.

We should now show that all of $\mathcal{V}_{2r-3}(\mathfrak{A}')$ lies in $\mathcal{V}_{2r}(A)$. Take a general $[2r-3]\, l$ of $\mathcal{V}_{2r-3}(\mathfrak{A}')$; it cuts a in an $[r-3]$, so their join al is an $[n+r-1]$. So al cuts A_b in an $[r-1]$ meeting l in an $[r-2]$, and it cuts similarly A_c, A_d. These $[r-1]$'s and l have as their join a $[2r]\, p$, meeting A_b, A_c, A_d in $[r-1]$'s, and since p and a lie in al, p cuts a in an $[r]$, and is therefore a $[2r]$ of $\mathcal{V}_{2r}(A)$. Consequently through a general $[2r-3]$ of $\mathcal{V}_{2r-3}(\mathfrak{A}')$ goes a $[2r]$ of $\mathcal{V}_{2r}(A)$.

(ii) *If ϖ cuts X in a* $[2r-2]\,\mu$, μ must meet $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in at least $[r-2]$'s; μ and σ lie in ϖ and so cut in an $[r-2]$, so μ cuts $\mathfrak{a}$ in an $[r-2]$. And, moreover, μ must meet one of $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in an $[r-1]$, for otherwise the $[2n-4]\, X\varpi$ would cut the $[n-3]\, A_b$ in the $[n-4]\, \mathfrak{b}$ and in additional points and would thus contain it; it would also contain similarly A_c, A_d and so A_b, A_c, A_d would all lie in $X\varpi$ which is impossible. So μ is a $[2r-2]$ of $\mathcal{V}_{2r-2}(\mathfrak{B})$, $\mathcal{V}_{2r-2}(\mathfrak{C})$ or $\mathcal{V}_{2r-2}(\mathfrak{D})$.

Now if m is taken as a general $[2r-2]$ of $\mathcal{V}_{2r-2}(\mathfrak{B})$ say, it meets $\mathfrak{a}$ in an $[r-2]$. So m and a meet in an $[r-2]$ and their join is an $[n+r-1]\, am$. Then am cuts the $[n-3]\, A_c$ in an $[r-1]$ through the $[r-2]$ common to $\mathfrak{c}$ and m, and similarly cuts A_d in an $[r-1]$ through the $[r-2]$ common to $\mathfrak{d}$ and m. By joining these two $[r-1]$'s to m, we determine a $[2r]\, p$, meeting A_b, A_c, A_d in $[r-1]$'s. And since p and a lie in am, p cuts a in an $[r]$, and is thus a $[2r]$ of $\mathcal{V}_{2r}(A)$. Consequently through a general $[2r-2]$ of $\mathcal{V}_{2r-2}(\mathfrak{B})$ there goes a $[2r]$ of $\mathcal{V}_{2r}(A)$, and similarly for general $[2r-2]$'s of $\mathcal{V}_{2r-2}(\mathfrak{C})$, $\mathcal{V}_{2r-2}(\mathfrak{D})$.

(iii) *If ϖ cuts X in a* $[2r-1]\,\nu$, then ν meets $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in at least $[r-2]$'s; ν and σ lie in ϖ, and thus cut in an $[r-1]$, so that ν meets $\mathfrak{a}$ in an $[r-1]$. Then ν must meet two of $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in $[r-1]$'s, for if say it meet neither $\mathfrak{c}$ nor $\mathfrak{d}$ in $[r-1]$'s then ϖ meets A_c, A_d outside of $\mathfrak{c}$, $\mathfrak{d}$, and so the $[2n-5]\, X\varpi$ contains both A_c and A_d, which is impossible. So ν is a $[2r-1]$ of $\mathcal{V}_{2r-1}(\mathfrak{B}')$, $\mathcal{V}_{2r-1}(\mathfrak{C}')$ or $\mathcal{V}_{2r-1}(\mathfrak{D}')$.

And now if n is a general $[2r-1]$ of say $\mathcal{V}_{2r-1}(\mathfrak{B}')$, then n cuts a in an $[r-1]$, so na is an $[n+r-1]$ and cuts the $[n-3]\, A_b$ in an $[r-1]$ through the $[r-2]$ common to n and $\mathfrak{b}$. This $[r-1]$ joined to n gives a $[2r]\, p$. And p and a lie in na and thus cut in an $[r]$. Consequently through a general $[2r-1]$ of $\mathcal{V}_{2r-1}(\mathfrak{B}')$ goes a $[2r]$ of $\mathcal{V}_{2r}(A)$, and similarly for general $[2r-1]$'s of $\mathcal{V}_{2r-1}(C')$, $\mathcal{V}_{2r-1}(D')$.

(iv) Lastly, *if a $[2r]$ of $\mathcal{V}_{2r}(A)$ lies wholly in X* it is a $[2r]$ of $\mathcal{V}_{2r}(\mathfrak{A})$, and any $[2r]$ of $\mathcal{V}_{2r}(\mathfrak{A})$ is a $[2r]$ of $\mathcal{V}_{2r}(A)$.

Consequently X cuts $\mathcal{V}_{2r}(A)$ *in the eight varieties.*

(*i*) $\mathcal{V}_{2r-3}(\mathfrak{A}')$,
(*ii*) $\mathcal{V}_{2r-2}(\mathfrak{B})$, $\mathcal{V}_{2r-2}(\mathfrak{C})$, $\mathcal{V}_{2r-2}(\mathfrak{D})$,
(*iii*) $\mathcal{V}_{2r-1}(\mathfrak{B}')$, $\mathcal{V}_{2r-1}(\mathfrak{C}')$, $\mathcal{V}_{2r-1}(\mathfrak{D}')$,
(*iv*) $\mathcal{V}_{2r}(\mathfrak{A})$.

The order of the section of $\mathcal{V}_{2r}(A)$ *by X is* ${}^{n-4}C_{2r-3} + 3.{}^{n-4}C_{2r-2} + 3.{}^{n-4}C_{2r-1} + {}^{n-4}C_{2r}$, *which equals* ${}^{n-1}C_{2r}$, *the order of* $\mathcal{V}_{2r}(A)$. This follows immediately from the fact that the order of $\mathcal{V}_k(\mathfrak{A})$ or of $\mathcal{V}_k(\mathfrak{A}')$ is ${}^{n-4}C_k$.

We must recollect that the values of the suffixes in this table must lie between 0 and $n-4$, both inclusive. Thus for example, if $n=10$, $\mathcal{V}_4(A)$ and $\mathcal{V}_6(A)$ will each cut X in eight varieties, as given in the table, but $\mathcal{V}_2(A)$ cuts X in the seven varieties $\mathcal{V}_0(\mathfrak{B})$, $\mathcal{V}_0(\mathfrak{C})$, $\mathcal{V}_0(\mathfrak{D})$, $\mathcal{V}_1(\mathfrak{B}')$, $\mathcal{V}_1(\mathfrak{C}')$, $\mathcal{V}_1(\mathfrak{D}')$, $\mathcal{V}_2(\mathfrak{A})$, as no variety $\mathcal{V}_{-1}(\mathfrak{A}')$ exists, while $\mathcal{V}_8(A)$ cuts X in the four varieties $\mathcal{V}_5(\mathfrak{A}')$, $\mathcal{V}_6(\mathfrak{B})$, $\mathcal{V}_6(\mathfrak{C})$, $\mathcal{V}_6(\mathfrak{D})$, as the varieties $\mathcal{V}_7(\mathfrak{B}')$, $\mathcal{V}_7(\mathfrak{C}')$, $\mathcal{V}_7(\mathfrak{D}')$, $\mathcal{V}_8(\mathfrak{A})$ do not exist in this case.

The Intersection of $\mathcal{V}_{2r+1}(A')$ *with A.* In the same way as in the last case, this is the intersection in A' of $\mathcal{V}_{2r+1}(A')$ with X. By an argument of somewhat similar type to that given above (though it is not exactly analogous) we find the following result

The Intersection of X with $\mathcal{V}_{2r+1}(A')$ *consists of the eight varieties*

(*i*) $\mathcal{V}_{2r-2}(\mathfrak{A})$,
(*ii*) $\mathcal{V}_{2r-1}(\mathfrak{B}')$, $\mathcal{V}_{2r-1}(\mathfrak{C}')$, $\mathcal{V}_{2r-1}(\mathfrak{D}')$,
(*iii*) $\mathcal{V}_{2r}(\mathfrak{B})$, $\mathcal{V}_{2r}(\mathfrak{C})$, $\mathcal{V}_{2r}(\mathfrak{D})$,
(*iv*) $\mathcal{V}_{2r+1}(\mathfrak{A}')$.

The order of the section is ${}^{n-4}C_{2r-2}+3.{}^{n-4}C_{2r-1}+3.{}^{n-4}C_{2r}+{}^{n-4}C_{2r+1}$, *which is equal to* ${}^{n-1}C_{2r+1}$, *the order of* $\mathcal{V}_{2r+1}(A')$.

The suffixes in this table run from 0 to $n-4$, both inclusive.

These two tables may be interpreted slightly differently. They imply that $\mathcal{V}_{2r}(\mathfrak{A})$ *lies in* $\mathcal{V}_{2r}(A)$, $\mathcal{V}_{2r+1}(B')$, $\mathcal{V}_{2r+1}(C')$, $\mathcal{V}_{2r+1}(D')$, $\mathcal{V}_{2r+2}(B)$, $\mathcal{V}_{2r+2}(C)$, $\mathcal{V}_{2r+2}(D)$, $\mathcal{V}_{2r+3}(A')$, *and that* $\mathcal{V}_{2r+1}(\mathfrak{A}')$ *lies in* $\mathcal{V}_{2r+1}(A')$, $\mathcal{V}_{2r+2}(B)$, $\mathcal{V}_{2r+2}(C)$, $\mathcal{V}_{2r+2}(D)$, $\mathcal{V}_{2r+3}(B')$, $\mathcal{V}_{2r+3}(C')$, $\mathcal{V}_{2r+3}(D')$, $\mathcal{V}_{2r+4}(A)$. We must of course omit from this table any varieties which do not exist for the particular value of r under discussion.

The Intersection of $\mathcal{V}_{2r}(A)$ *with B.* Since $\mathcal{V}_{2r}(A)$ lies in A, the intersection of $\mathcal{V}_{2r}(A)$ with B is the intersection in A of $\mathcal{V}_{2r}(A)$ with AB. Let us recall that A contains a, and cuts b, c, d in A_b, A_c, A_d, and that AB cuts a in B_a, contains A_b and cuts A_c, A_d in $\mathfrak{c}$, $\mathfrak{d}$ respectively. Then a $[2r]$ ϖ of $\mathcal{V}_{2r}(A)$ meets B_a in an $[r-2]$, A_b in an

$[r-1]$, and $\mathfrak{c}$, $\mathfrak{d}$ in $[r-2]$'s, so it cuts AB in at least a $[2r-2]\,\lambda$, but might cut it in a $[2r-1]\,\mu$, or might lie entirely in it.

(i) *If ϖ cuts AB in a* $[2r-2]\,\lambda$, then λ cuts A_b in an $[r-1]$ and $\mathfrak{c}$ in an $[r-2]$ and thus lies in the $[2n-6]\,\mathfrak{c}A_b$; similarly it lies in the $[2n-6]\,\mathfrak{d}A_b$. Consequently it lies in their intersection, a $[2n-7]$ which we have agreed to call B_i. Now B_i cuts B_a in B_{ai} (the analogue in B_i of B_a in B), contains b_i and cuts $\mathfrak{c}$ and $\mathfrak{d}$ in B_{ci} and B_{di}, so the locus of λ is $\mathcal{V}_{2r-2}(B_i)$. Since these $[2r-2]$'s are those $[2r-2]$'s of $\mathcal{V}_{2r-2}(B)$ which lie in A we can denote their locus also by the longer but more symmetrical notation $\mathcal{V}_{2r-2}(B, A)$. Each $[2r-2]$ of the locus cuts X in a $[2r-3]$ which meets $\mathfrak{a}$ in an $[r-3]$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ in $[r-2]$'s, *i.e.* in a $[2r-3]$ of $\mathcal{V}_{2r-3}(\mathfrak{A}')$, or it lies entirely in X; in fact, $\mathcal{V}_{2r-2}(B, A)$ cuts X in $\mathcal{V}_{2r-3}(\mathfrak{A}') + \mathcal{V}_{2r-2}(\mathfrak{B})$.

Now let l be any $[2r-2]$ of $\mathcal{V}_{2r-2}(B, A)$ cutting X in a $[2r-3]\,\sigma$ of $\mathcal{V}_{2r-3}(\mathfrak{A}')$. Then l cuts a in an $[r-2]$ so they lie in an $[n+r-1]\,al$, al and A_c lie in A and thus cut in an $[r-1]\,Q$. Similarly al and A_d cut in an $[r-1]\,R$, while we know that l cuts A_b in an $[r-1]\,P$. Each of P, Q, R cuts σ in an $[r-2]$ and so they determine with σ a $[2r]\,p$. Since p and a lie in al they cut in an $[r]$, and consequently p is a $[2r]$ of $\mathcal{V}_{2r}(A)$, and so through every $[2r-2]$ of $\mathcal{V}_{2r}(B, A)$ goes a $[2r]$ of $\mathcal{V}_{2r}(A)$; (since we have already shown that there is one if p lies entirely in X).

(ii) *Suppose that a* $[2r]\,\varpi$ of $\mathcal{V}_{2r}(A)$ *cuts AB in a* $[2r-1]\,\mu$ and in no more. Then μ cuts B_a in an $[r-1]$, A_b in an $[r-1]$, $\mathfrak{c}$ and $\mathfrak{d}$ in $[r-2]$'s at least. Now ϖ meets A_c in an $[r-1]$, but if any of this $[r-1]$ lies outside of $\mathfrak{c}$, then ϖ lies wholly in the $[2n-4]$ XA_bA_c, *i.e.* in AD', for it cuts XA_b in μ already. So if ϖ meets both A_c and A_d in $[r-1]$'s outside of $\mathfrak{c}$ and $\mathfrak{d}$, then ϖ lies in both AD' and in AC' and thus in AB which is contrary to our present hypothesis. Consequently ϖ meets either $\mathfrak{c}$ or $\mathfrak{d}$ in an $[r-1]$; we will at present discuss the first possibility. It implies that μ meets B_a, A_b, $\mathfrak{c}$ in $[r-1]$'s and $\mathfrak{d}$ in an $[r-2]$. So μ lies in the two $[2n-6]$'s $\mathfrak{c}A_b$, $\mathfrak{c}B_a$, and therefore in their intersection, a $[2n-7]$ which we have called D'_i. Now D'_i cuts B_a in D'_{ai}, A_b in D'_{bi}, $\mathfrak{c}$ in D'_{ci} and $\mathfrak{d}$ in $\mathfrak{d}_i$, so μ is a $[2r-1]$ of $\mathcal{V}_{2r-1}(D'_i)$. Since such $[2r-1]$'s are those $[2r-1]$'s of $\mathcal{V}_{2r-1}(D')$ which lie in C', we can give the locus the name of $\mathcal{V}_{2r-1}(D', C')$. A $[2r-1]$ of the locus cuts X in a $[2r-2]$ meeting $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{d}$ in $[r-2]$'s, $\mathfrak{c}$ in an $[r-1]$, or it lies in X; in fact $\mathcal{V}_{2r-1}(D', C')$ cuts X in $\mathcal{V}_{2r-2}(\mathfrak{C}) + \mathcal{V}_{2r-1}(\mathfrak{D}')$.

Now let m be any $[2r-1]$ of $\mathcal{V}_{2r-1}(D', C')$ cutting X in a

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$[2r-2]\sigma$ of $\mathcal{V}_{2r-2}(\mathfrak{C})$. Then since m cuts a in an $[r-1]$ their join is an $[n+r-1]\,am$. But am and A_d lie in A, and so cut in an $[r-1]\,Q$, while we know that A_b and m cut in an $[r-1]\,P$; P and Q meet σ in $[r-2]$'s. So the join of σ, P and Q is a $[2r]\,p$ meeting A_b, A_c, A_d in $[r-1]$'s, and p and a cut in an $[r]$ for they both lie in am. So p is a $[2r]$ of $\mathcal{V}_{2r}(A)$, and through any $[2r-1]$ of $\mathcal{V}_{2r-1}(D', C')$ goes a $[2r]$ of $\mathcal{V}_{2r}(A)$.

We have similarly a second V_{n-3}, $\mathcal{V}_{2r-1}(C', D')$.

(iii) *If a* $[2r]\,\varpi$ *of* $\mathcal{V}_{2r}(A)$ *lies entirely in* AB, it cuts $\mathfrak{a}$, $\mathfrak{c}$, $\mathfrak{d}$ in $[r-1]$'s and $\mathfrak{b}$ in an $[r-2]$, and so cuts X in a $[2r-1]$ of $\mathcal{V}_{2r-1}(\mathfrak{B}')$. Since ϖ cuts B_a in an $[r]$ and $\mathfrak{c}$ in an $[r-1]$ it lies in $\mathfrak{c}B_a$; it lies similarly in $\mathfrak{d}B_a$ and thus in their intersection, A_i, and ϖ is in fact a $[2r]$ of $\mathcal{V}_{2r}(A_i)$. This locus, consisting as it does of those $[2r]$'s of $\mathcal{V}_{2r}(A)$ which lie in B, can be called $\mathcal{V}_{2r}(A, B)$. It cuts X in $\mathcal{V}_{2r-1}(\mathfrak{B}') + \mathcal{V}_{2r}(\mathfrak{A})$. It is clear that any $[2r]\,p$ of $\mathcal{V}_{2r}(A, B)$ is a $[2r]$ of $\mathcal{V}_{2r}(A)$.

So we find that AB *cuts* $\mathcal{V}_{2r}(A)$ *in the four varieties*

(*i*) $\mathcal{V}_{2r-2}(B, A)$, *i.e.* $\mathcal{V}_{2r-2}(B_i)$,

(*ii*) $\mathcal{V}_{2r-1}(C', D')$, $\mathcal{V}_{2r-1}(D', C')$, *i.e.* $\mathcal{V}_{2r-1}(C'_i)$, $\mathcal{V}_{2r-1}(D'_i)$,

(*iii*) $\mathcal{V}_{2r}(A, B)$, *i.e.* $\mathcal{V}_{2r}(A_i)$.

The total order of the section of $\mathcal{V}_{2r}(A)$ *by* AB *is* ${}^{n-3}C_{2r-2} + 2.{}^{n-3}C_{2r-1} + {}^{n-3}C_{2r}$, *which equals* ${}^{n-1}C_{2r}$, *the order of* $\mathcal{V}_{2r}(A)$.

The intersection of $\mathcal{V}_{2r+1}(C')$ *with* D'. It will be found convenient to study this intersection, rather than, say, the intersection of $\mathcal{V}_{2r+1}(A')$ with B'. It is the intersection in C' of $\mathcal{V}_{2r+1}(C')$ with $C'D'$. By applying the same general principles as in the previous section, though with some differences in method, we may arrive at the following conclusion.

The intersection of $\mathcal{V}_{2r+1}(C')$ *with* $C'D'$ *consists of the four varieties.*

(*i*) $\mathcal{V}_{2r-1}(D', C')$, *i.e.* $\mathcal{V}_{2r-1}(D'_i)$,

(*ii*) $\mathcal{V}_{2r}(B, A)$, $\mathcal{V}_{2r}(A, B)$, *i.e.* $\mathcal{V}_{2r}(B_i)$, $\mathcal{V}_{2r}(A_i)$,

(*iii*) $\mathcal{V}_{2r+1}(C', D')$, *i.e.* $\mathcal{V}_{2r+1}(C'_i)$.

The total order of the section is ${}^{n-3}C_{2r-1} + 2.{}^{n-3}C_{2r} + {}^{n-3}C_{2r+1}$, *which equals* ${}^{n-1}C_{2r+1}$, *the order of* $\mathcal{V}_{2r+1}(C')$.

In both these cases the suffixes lie between 0 and $n-3$ inclusive.

These two tables imply that $\mathcal{V}_{2r}(A; B)$ *lies in* $\mathcal{V}_{2r}(A)$, $\mathcal{V}_{2r+1}(C')$, $\mathcal{V}_{2r+1}(D')$, $\mathcal{V}_{2r+2}(B)$, *and that* $\mathcal{V}_{2r+1}(A', B')$ *lies in* $\mathcal{V}_{2r+1}(A')$, $\mathcal{V}_{2r+2}(C)$, $\mathcal{V}_{2r+2}(D)$, $\mathcal{V}_{2r+3}(B')$, these varieties being supposed to exist.

The intersection of $\mho_{2r}(A)$ *with* B'. This is the intersection in A of $\mho_{2r}(A)$ with AB', which cuts a in B'_a, cuts A_b in $\mathfrak{b}$, and contains A_c and A_d. A $[2r]\,\varpi$ of $\mho_{2r}(A)$ cuts a in an $[r]$, A_b, A_c, A_d in $[r-1]$'s and so cuts AB' in at least a $[2r-1]\,\lambda$, but might lie wholly in AB'.

(i) *In the former case* λ cuts A_c and A_d in $[r-1]$'s, and therefore lies in their join, which is the $[2n-5]$ we have called B'_1. So λ is a $[2r-1]$ of $\mho_{2r-1}(B'_1)$. Since the $[2r-1]$'s of this locus are those $[2r-1]$'s of $\mho_{2r-1}(B')$ which lie in A, the locus may be denoted by $\mho_{2r-1}(B', A)$.

Now let l be any $[2r-1]$ of $\mho_{2r-1}(B', A)$. Since it meets a in an $[r-1]$, al is an $[n+r-1]$ which cuts A_b in an $[r-1]\,P$ going through the $[r-2]$ common to l and $\mathfrak{b}$. So lP is a $[2r]$ of $\mho_{2r}(A)$.

(ii) *If the* $[2r]\,\varpi$ *lies wholly in* AB' it cuts B'_a in an $[r]$ and $\mathfrak{b}$ in an $[r-1]$ and consequently lies in their join, the $[2n-5]$ we have called A_1. In fact, ϖ is a $[2r]$ of $\mho_{2r}(A_1)$. The $[2r]$'s of this locus being those $[2r]$'s of $\mho_{2r}(A)$ which lie in B', the locus may also be called $\mho_{2r}(A, B')$. Any $[2r]\,p$ of this locus is clearly a $[2r]$ of $\mho_{2r}(A)$.

So $\mho_{2r}(A)$ *cuts* AB' *in the two varieties.*

(*i*) $\mho_{2r-1}(B', A)$, *i.e.* $\mho_{2r-1}(B'_1)$,

(*ii*) $\mho_{2r}(A, B')$, *i.e.* $\mho_{2r}(A_1)$.

The orders of these varieties are ${}^{n-2}C_{2r-1}$ *and* ${}^{n-2}C_{2r}$, *while the order of* $\mho_{2r}(A)$ *is their sum*, ${}^{n-1}C_{2r}$.

Since AD' contains AB it would be well to see how the parts of the base in AD' contain those in AB. AD' cuts $\mho_{2r}(A)$ in $\mho_{2r-1}(D', A) + \mho_{2r}(A, D')$. Now a $[2r-1]$ of $\mho_{2r-1}(D', A)$ either lies in AB or cuts it in a $[2r-2]$. AB is the intersection of AD' and AC', so if the $[2r-1]$ lies in AB it is a $[2r-1]$ of $\mho_{2r-1}(D', C')$. Otherwise the $[2r-1]$ cuts d in an $[r-2]$, a, b, c in $[r-1]$'s, so its section by AC' cuts b in an $[r-1]$, a, c, d in $[r-2]$'s and consequently the $[2r-2]$ is a $[2r-2]$ of $\mho_{2r-2}(B, A)$. Similarly a $[2r]$ of $\mho_{2r}(A, D')$ either lies in AB or cuts it in a $[2r-1]$. If it lies in AB it belongs to $\mho_{2r}(A, B)$. Otherwise it cuts a in an $[r]$, b, c, d in $[r-1]$'s, so its section in AC' cuts a, b, d in $[r-1]$'s, c in an $[r-2]$, so the $[2r-1]$ is a $[2r-1]$ of $\mho_{2r-1}(C', D')$. But we know that AB cuts $\mho_{2r}(A)$ in $\mho_{2r-2}(B, A) + \mho_{2r-1}(C', D') + \mho_{2r-1}(D', C') + \mho_{2r}(A, B)$, so the entire section is accounted for.

The intersection of $\mho_{2r+1}(B')$ *with* A. We shall find it more convenient again to work out this intersection than the corresponding

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intersection say of $\mathcal{V}_{2r+1}(A')$ with B. It is the intersection in B' of $\mathcal{V}_{2r+1}(B')$ with AB'. By an application of the same type of argument as that used above we may reach the following conclusion.

The intersection of $V_{2r+1}(B')$ *with* AB' *consists of the two varieties.*

(*i*) $\mathcal{V}_{2r}(A, B')$, *i.e.* $\mathcal{V}_{2r}(A_1)$,

(*ii*) $\mathcal{V}_{2r+1}(B', A)$, *i.e.* $\mathcal{V}_{2r+1}(B'_1)$.

The sum of the orders of these varieties is ${}^{n-2}C_{2r} + {}^{n-2}C_{2r+1}$, *which is* ${}^{n-1}C_{2r+1}$, *the order of* $\mathcal{V}_{2r+1}(B')$.

We already have worked out the intersections of these varieties with say $B'C'$, remembering that $B'C'$ is the same space as AD.

These two tables may be interpreted as saying that $\mathcal{V}_{2r}(A, B')$ *lies in* $\mathcal{V}_{2r}(A)$ *and in* $\mathcal{V}_{2r+1}(B')$, *and that* $\mathcal{V}_{2r+1}(A', B)$ *lies in* $\mathcal{V}_{2r+1}(A')$ *and in* $\mathcal{V}_{2r+2}(B)$.

The suffixes in these tables lie between 0 and $n-2$, both inclusive.

§ 6.

The intersections of the varieties of the base in the same $[2n-3]$ A *or* A'. We shall now discuss how two of the base varieties in A, such for instance as $\mathcal{V}_{2p}(A)$ and $\mathcal{V}_{2q}(A)$, cut one another.

(i) *Consider a* $[2s]$ λ *of* $\mathcal{V}_{2s}(A)$ *which lies entirely in a* $[2r]$ μ *of* $\mathcal{V}_{2r}(A)$. *We shall show that there is a* $[2s+2]$ ν *of* $\mathcal{V}_{2s+2}(A)$ *through* λ *lying entirely in* μ. In μ project from λ into a skew $[2r-2s-1]$; then the sections by μ of a, A_b, A_c, A_d project into four $[r-s-1]$'s, since a meets λ in an $[s]$ and A_b, A_c, A_d meet λ in $[s-1]$'s. Now in a $[2r-2s-1]$ there is a finite number of lines meeting four $[r-s-1]$'s; in fact, it is easy to show that there are $r-s$ of them. Any one joined to λ gives a $[2s+2]$ ν, which cuts a in an $[s+1]$, and A_b, A_c, A_d in $[s]s$, so ν is a $[2s+2]$ of $\mathcal{V}_{2s+2}(A)$.

As an obvious corollary, *there is a* $[2q]$ ν' *of* $\mathcal{V}_{2q}(A)$ *which contains* λ *and lies in* μ, where q has any value from s to r.

(ii) *We now show that if two generating elements of two base varieties in* A *intersect, then their section is an element in a third base variety in* A. More precisely, consider $\mathcal{V}_{2p}(A)$ and $\mathcal{V}_{2q}(A)$ with $p > q$; let α be any $[2p]$ of $\mathcal{V}_{2p}(A)$ and β any $[2q]$ of $\mathcal{V}_{2q}(A)$, and let α and β cut in an $[s]$ γ. Then let the intersection of γ with a be a $[k]$ $a\gamma$, where if γ is skew to a, $k=-1$. We shall show that γ is a $[2k]$ of $\mathcal{V}_{2k}(A)$.

The join R of α and β is a $[2p+2q-s]$ which cuts a in a $[p+q-k]$ Ra. Now γ must cut a and A_b in skew spaces (since they are themselves skew) so the intersection γA_b of γ and A_b must be an $[s-k-1-x]$, where $x \geqq 0$. Then R cuts A_b in a $[p+q-s+k-1+x]$ RA_b; so, since Ra and RA_b must be skew in R,

$$2p+2q-s \geqq (p+q-k)+(p+q-s+k-1+x)+1,$$

that is, $x \leqq 0$. Consequently $x = 0$, and so γA_b is an $[s-k-1]$, RA^b is a $[p+q-s+k-1]$, and similarly γA_c, γA_d are $[s-k-1]$'s and RA_c, RA_d are $[p+q-s+k-1]$'s.

Now RA_b and RA_c must be skew in R; *i.e.* $2p+2q-2s+2k-2+1 \leqq 2p+2q-s$, *i.e.* $-s+2k-1 \leqq 0$, and γA_b and γA_c must be skew in γ, *i.e.* $2s-2k-2+1 \leqq s$, *i.e.* $s-2k-1 \leqq 0$. These results imply that $s = 2k-1+y$, where $y = 0, 1, 2$. We now show that $y = 1$.

Substituting the value of s, γ is a $[2k-1+y]$, γa is a $[k]$, γA_b, γA_c, γA_d are $[k-2+y]$'s, while R is a $[2p+2q-2k+1-y]$, Ra is a $[p+q-k]$, RA_b, RA_c, RA_d are $[p+q-k-y]$'s. Consequently the freedom f of γ in A is

$$\begin{aligned} f &= (2n-2k-2-y)(2k+y)-(n-k-1-y)(k+1)-3(n-k-1)(k-1+y) \\ &= (n-k)(2-y)-ky-y^2+2y-2, \end{aligned}$$

while the freedom F of R in A is

$$\begin{aligned} F &= (2n-2p-2q-2k-4+y)(2p+2q-2k+2-y)-(n-p-q+k-3+y) \\ &\qquad (p+q-k+1)-3(n-p-q+k-1)(p+q-k-y+1) \\ &= (n-p-q+k)y+(p+q-k)(y-2)-y^2+2y-2. \end{aligned}$$

And so, if $y = 2$, $f = -2(k+1)$, and consequently there are no spaces such as γ existing in A, while if $y = 0$, $F = -2(p+q-k+1)$, and since it is clear that $p+q \geqq k$, there are no spaces such as R existing in A. Therefore $y = 1$.

It follows that γ is a $[2k]$ meeting a in a $[k]$, A_b, A_c, A_d in $[k-1]$'s, that is, γ is a $[2k]$ of $\mathcal{V}_{2k}(A)$.

Now from (i) through γ goes a $[2q]$ of $\mathcal{V}_{2q}(A)$ which lies entirely in a. *Therefore the intersection of* $\mathcal{V}_{2q}(A)$ *and* $\mathcal{V}_{2p}(A)$ *consists of those* $[2q]$*'s of* $\mathcal{V}_{2q}(A)$ *which lie entirely in* $[2p]$*'s of* $\mathcal{V}_{2p}(A)$.

The freedom of a $[2q]$ of $\mathcal{V}_{2q}(A)$ in a given $[2p]$ of $\mathcal{V}_{2p}(A)$ is $p-q$, while the freedom of $[2p]$'s of $\mathcal{V}_{2p}(A)$ is $n-2p-1$. So the freedom of the $[2q]$'s of the intersection of $\mathcal{V}_{2q}(A)$ and $\mathcal{V}_{2p}(A)$ is $n-1-p-q$, and *consequently the intersection is a variety of dimension* $n-1-(p-q)$. Let us denote the variety by $\mathcal{V}_{2p,\,2q}(A)$. We shall at present denote its order by the symbol $\{n;\, 2p,\, 2q\}$; we shall later find an explicit expression for this order.

(iii) Consider the three varieties $\mathcal{V}_{2p,2q}(A)$, $\mathcal{V}_{2r,2q}(A)$, $\mathcal{V}_{2p,2q}(A)$, with $p > r > q$. *Then the intersection of* $\mathcal{V}_{2p,2r}(A)$ *and* $\mathcal{V}_{2r,2q}(A)$ *is* $\mathcal{V}_{2p,2q}(A)$. For if H is any point of the intersection there goes through H a $[2q]\,\beta$ of $\mathcal{V}_{2q}(A)$ lying in a $[2r]\,\delta$ of $\mathcal{V}_{2r}(A)$, and also a $[2r]\,\delta'$ of $\mathcal{V}_{2r}(A)$ lying in a $[2p]\,\alpha$ of $\mathcal{V}_{2p}(A)$. So since β cuts δ' it follows by (ii) that there is in δ' a $[2q]\,\beta'$ of $\mathcal{V}_{2q}(A)$ through H, and thus β' lies in α and H lies in $\mathcal{V}_{2p,2q}(A)$. And conversely, if β is any $[2q]$ of $\mathcal{V}_{2p,2q}(A)$, lying therefore in a $[2p]\,\alpha$ of $\mathcal{V}_{2p}(A)$, then through β goes a $[2r]\,\delta$ of $\mathcal{V}_{2r}(A)$ such that δ lies in α, by (i); that is, β lies in $\mathcal{V}_{2p,2r}(A)$ and in $\mathcal{V}_{2r,2q}(A)$.

If we write out the varieties in A in the following way,

$$\begin{array}{ccccccccc}
\mathcal{V}_0(A) & & \mathcal{V}_2(A) & & \mathcal{V}_4(A) & & \mathcal{V}_6(A) & & \ldots\ldots \\
& \mathcal{V}_{0,2}(A) & & \mathcal{V}_{2,4}(A) & & \mathcal{V}_{4,6}(A) & & \ldots\ldots & \\
& & \mathcal{V}_{0,4}(A) & & \mathcal{V}_{2,6}(A) & & \ldots\ldots & & \\
& & & \mathcal{V}_{0,6}(A) & & \ldots\ldots & & & \\
& & & & \ldots\ldots, & & & &
\end{array}$$

it follows simply that any variety includes all those which may be reached by descending diagonally in any manner, and that the intersection of any two varieties is that lying diagonally below both, *e.g.* $\mathcal{V}_2(A)$ cuts $\mathcal{V}_{4,6}(A)$ in $\mathcal{V}_{2,6}(A)$, while $\mathcal{V}_2(A)$ contains $\mathcal{V}_{0,6}(A)$.

(iv) Let us now carry out similar work for the varieties in B' (rather that A'). We recall that B' is a $[2n-3]$ containing the $[n-4]\,\mathfrak{b}$, and the $[n-2]$'s B'_a, B'_c, B'_d and that the $[2r+1]$'s of $\mathcal{V}_{2r+1}(B')$ cut $\mathfrak{b}$ in an $[r-1]$, B'_a, B'_c, B'_d in $[r]$'s. In the same general manner as in (i) we show that *if a* $[2s+1]\,\lambda$ *of* $\mathcal{V}_{2s+1}(B')$ *lies entirely in a* $[2r+1]\,\mu$ *of* $\mathcal{V}_{2r+1}(B')$, *then there is through* λ *a* $[2q+1]$ *of* $\mathcal{V}_{2q+1}(B')$ *lying in* μ, *where* $s < q < r$.

(v) Consider now $\mathcal{V}_{2p+1}(B')$ and $\mathcal{V}_{2q+1}(B')$. Suppose $p > q$ and let α be a $[2p+1]$ of $\mathcal{V}_{2p+1}(B')$ and β a $[2q+1]$ of $\mathcal{V}_{2q+1}(B')$, such that α and β cut in an $[s]\,\gamma$, and let γ cut $\mathfrak{b}$ in a $[k-1]\,\mathfrak{b}\gamma$. *Then we shall show that* γ *is a* $[2k+1]$ *of* $\mathcal{V}_{2k+1}(B')$.

The work is similar to that given in (ii). We consider R, the join of α and β. Since γ cuts $\mathfrak{b}$ and B'_a in skew spaces $\gamma B'_a$ is an $[s-k-x]$ where $x \geqq 0$. Writing as before the condition that R cuts $\mathfrak{b}$ and B'_a in skew spaces, we have $x \leqq 2$; hence $x = 0$, 1 or 2. Similarly $\gamma B'_c$, $\gamma B'_d$ are $[s-k-x']$, $[s-k-x'']$, $x', x'' = 0, 1, 2$. Then if we make RB'_a and RB'_c skew in R we have $-s+2k+x+x'-1 \leqq 0$, and if we make $\gamma B'_a$ and $\gamma B'_c$ skew in γ we have $s-2k-x-x'+1 \leqq 0$; therefore $x+x' = s+2k+1$, and similarly $x'+x'' = s+2k+1$,

$x'' + x = s + 2k + 1$, whence $x = x' = x'' = \frac{1}{2}(s - 2k + 1)$. Eliminating s by this relation, γ is a $[2k - 1 + 2x]$, $\gamma\mathfrak{b}$ is a $[k - 1]$, $\gamma B'_a$, $\gamma B'_c$, $\gamma B'_d$ are $[k - 1 + x]$'s, while R is a $[2p + 2q - 2k + 3 - 2x]$, $R\mathfrak{b}$ is a $[p + q - k - 1]$, RB'_a, RB'_c, RB'_d are $[p + q - k + 1 - x]$'s, where $x = 0, 1, 2$. Now in the same way as in (ii) we show that $x = 1$, so that γ is a $[2k + 1]$ of $\mathcal{V}_{2k+1}(B')$. *Therefore the intersection of* $\mathcal{V}_{2q+1}(B')$ *and* $\mathcal{V}_{2p+1}(B')$ *consists of those* $[2q + 1]$'s *of* $\mathcal{V}_{2q+1}(B')$ *which lie entirely in* $[2p + 1]$'s *of* $\mathcal{V}_{2p+1}(B')$.

The calculation of dimension proceeds as before. *The intersection is a variety of dimension* $n - 1 - (p - q)$. Denote it by $\mathcal{V}_{2p+1, 2q+1}(B')$, and denote its order by $\{n;\ 2p + 1,\ 2q + 1\}$.

(vi) By an exactly similar argument to that in (iii) we may show that if $p > r > q$, *the intersection of* $\mathcal{V}_{2p+1, 2r+1}(B')$ *and* $\mathcal{V}_{2r+1, 2q+1}(B')$ *is* $\mathcal{V}_{2p+1, 2q+1}(B')$. We can set up a similar scheme showing intersections as that given there.

(vii) *The orders of* $\mathcal{V}_{2p, 2q}(A)$, $\mathcal{V}_{2p+1, 2q+1}(B')$. Let us now find the values of $\{n;\ 2p,\ 2q\}$, $\{n;\ 2p + 1,\ 2q + 1\}$. Let us agree that $\{n;\ 2p,\ 2p\}$ and $\{n;\ 2p + 1,\ 2p + 1\}$ shall denote the orders of $\mathcal{V}_{2p}(A)$ and $\mathcal{V}_{2p+1}(B')$, and that $\{n;\ 2p,\ 2q\}$ shall be considered zero if either $\mathcal{V}_{2p}(A)$ or $\mathcal{V}_{2q}(A)$ does not exist (if, *e.g.* $2p$ were negative, or greater than $n - 1$), with a similar proviso for $\{n;\ 2p + 1,\ 2q + 1\}$.

Then $\{n;\ 2p,\ 2q\} = {}^{n-p+q-1}C_{p-q} \times {}^{n-2p+2q-1}C_{2q}$,

$\{n;\ 2p + 1,\ 2q + 1\} = {}^{n-p+q-1}C_{p-q} \times {}^{n-2p+2q-1}C_{2q+1}$.

It is easy to verify that these results are true for all values of p and q if n is 3; the only symbols with non-zero values are found by immediate inspection to be $\{3;\ 2,\ 2\} = 1$, $\{3;\ 0,\ 0\} = 1$, $\{3;\ 2,\ 0\} = 1$, $\{3;\ 1,\ 1\} = 2$, and these results are in agreement with the enunciation. We shall proceed to prove two difference formulae, and by means of these shall prove the general results by induction.

Consider $\mathcal{V}_{2p, 2q}(A)$. Let us find its order by finding the order of its section by the prime AB'. As we saw in § 4, AB' cuts $\mathcal{V}_{2p}(A)$ in $\mathcal{V}_{2p-1}(B'_1) + \mathcal{V}_{2p}(A_1)$ and cuts $\mathcal{V}_{2q}(A)$ in $\mathcal{V}_{2q-1}(B'_1) + \mathcal{V}_{2q}(A_1)$, and consequently AB' cuts $\mathcal{V}_{2p, 2q}(A)$ in the intersection of $\mathcal{V}_{2p-1}(B'_1) + \mathcal{V}_{2p}(A_1)$ with $\mathcal{V}_{2q-1}(B'_1) + \mathcal{V}_{2q}(A_1)$. It is clear that the intersection is degenerate, and consists of

1⁰ the intersection of $\mathcal{V}_{2p}(A_1)$ and $\mathcal{V}_{2q}(A_1)$
2⁰ the intersection of $\mathcal{V}_{2p-1}(B'_1)$ and $\mathcal{V}_{2q-1}(B'_1)$
3⁰ the intersection of $\mathcal{V}_{2p}(A_1)$ and $\mathcal{V}_{2q-1}(B'_1)$
4⁰ the intersection of $\mathcal{V}_{2p-1}(B'_1)$ and $\mathcal{V}$ (A_1).

Now 1^0 is the $V_{n-2-(p-q)}$ $\mathcal{V}_{2p,2q}(A_1)$, which is of order $\{n-1;\ 2p,\ 2q\}$ and 2^0 is the $V_{n-2-(p-q)}$ $\mathcal{V}_{2p-1,2q-1}(B'_1)$, which is of order $\{n-1;\ 2p-1,\ 2q-1\}$. As regards 3^0, the intersections of $\mathcal{V}_{2p}(A_1)$ and $\mathcal{V}_{2q-1}(B'_1)$ must lie in $A_1B'_1$. But $A_1B'_1$ cuts $\mathcal{V}_{2p}(A_1)$ in $\mathcal{V}_{2p}(A_2) + \mathcal{V}_{2p-1}(B'_2)$, and cuts $\mathcal{V}_{2q-1}(B'_1)$ in $\mathcal{V}_{2q-1}(B'_2) + \mathcal{V}_{2q-2}(A_2)$. So the intersection 3^0 breaks up into four parts:—$3{\cdot}1^0$ the intersection of $\mathcal{V}_{2p}(A_2)$ and $\mathcal{V}_{2q-1}(B'_2)$; these varieties lie respectively in $\mathcal{V}_{2p}(A_1)$ and $\mathcal{V}_{2q}(A_1)$, so any intersection they have lies in $\mathcal{V}_{2p,2q}(A_1)$ *i.e.* in 1^0 : $3{\cdot}2^0$, $\mathcal{V}_{2p}(A_2)$ and $\mathcal{V}_{2q-2}(A_2)$ lie in $\mathcal{V}_{2p}(A_1)$ and $\mathcal{V}_{2q-2}(A_1)$ respectively, so their section lies in $\mathcal{V}_{2p,2q-2}(A_1)$ which lies in $\mathcal{V}_{2p,2q}(A_1)$: $3{\cdot}3^0$, the section of $\mathcal{V}_{2p-1}(B'_2)$ and $\mathcal{V}_{2q-1}(B'_2)$ lies in $\mathcal{V}_{2p-1,2q-1}(B'_1)$, for similar reasons: $3{\cdot}4^0$ the section of $\mathcal{V}_{2p-1}(B'_2)$ and $\mathcal{V}_{2q-2}(A_2)$ lies in $\mathcal{V}_{2p-1,2q-1}(B'_1)$. Considering 4^0 in the same way, it is the section of $\mathcal{V}_{2p-1}(B'_2) + \mathcal{V}_{2p-2}(A_2)$ with $\mathcal{V}_{2q}(A_2) + \mathcal{V}_{2q-1}(B'_2)$, and thus breaks into four parts:— $4{\cdot}1^0$ the section of $\mathcal{V}_{2p-1}(B'_2)$ and $\mathcal{V}_{2q}(A_2)$ lies in $\mathcal{V}_{2p,2q}(A_1)$: $4{\cdot}2^0$ the section of $\mathcal{V}_{2p-1}(B'_2)$ and $\mathcal{V}_{2q-1}(B'_2)$ is case $3{\cdot}3^0$: $4{\cdot}3^0$: the section of $\mathcal{V}_{2p-2}(A_2)$ and $\mathcal{V}_{2q-1}(B'_2)$ lies in $\mathcal{V}_{2p-1,2q-1}(B'_1)$ for the same reasons as before. So all these cases give no new section. But the final case, $4{\cdot}4^0$, the section of $\mathcal{V}_{2p-2}(A_2)$ and $\mathcal{V}_{2q}(A_2)$ is $\mathcal{V}_{2p-2,2q}(A_2)$ a variety of dimension $n-2-(p-q)$ and order $\{n-2;\ 2p-2,\ 2q\}$.

The conclusion at which we arrive is therefore that the $V_{n-1-(p-q)}$ $\mathcal{V}_{2p,2q}(A)$ *is cut by the prime* AB' *in the three* $\mathcal{V}_{n-2-(p-q)}$*'s* $\mathcal{V}_{2p,2q}(A_1)$, $\mathcal{V}_{2p-1,2q-2}(B'_1)$, $\mathcal{V}_{2p-2,2q}(A_2)$. *Therefore*

$$\{n;\ 2p,\ 2q\} = \{n-1;\ 2p,\ 2q\} + \{n-1;\ 2p-1,\ 2q-1\} + \{n-2;\ 2p-2,\ 2q\}.$$

Once this relation is established, the induction from the known case $n=3$ to higher values is immediate, for assuming the enunciated results for values of n less than n, the right hand side of this equation becomes

$$^{n-p+q-2}C_{p-q}.\ ^{n-2p+2q-2}C_{2q} + {}^{n-p+q-2}C_{p-q}.\ ^{n-2p+2q-2}C_{2q-1} + {}^{n-p+q-2}C_{p-q-1}.\ ^{n-2p+2q-1}C_{2q},$$

which by elementary algebra equals $^{n-p-q-1}C_{p-q}.\ ^{n-2p+2q-1}C_{2q}$, the enunciated value for $\{n;\ 2p,\ 2q\}$.

(viii) *In the same way the section of* $\mathcal{V}_{2p+1,2q+1}(B')$ *by the prime* AB' *consists of* $\mathcal{V}_{2p+1,2q+1}(B'_1)$, $\mathcal{V}_{2p,2q}(A_1)$, $\mathcal{V}_{2p-1,2q+1}(B'_2)$. *Therefore*

$$\{n;\ 2p+1,\ 2q+1\}$$
$$= \{n-1;\ 2p+1,\ 2q+1\} + \{n-1;\ 2p,\ 2q\} + \{n-2;\ 2p-1,\ 2q+1\}$$

and from this we verify the induction exactly as above.

§ 7.

In this paragraph we shall write $[\mathcal{V}_{2p}(A), \mathcal{V}_{2q}(B)]$ to stand for the intersection of $\mathcal{V}_{2p}(A)$ with $\mathcal{V}_{2q}(B)$. We shall discard the convention that in $\mathcal{V}_{2p,2q}(A)$, *e.g.* p is greater than q, and write $\mathcal{V}_{2p,2q}(A) \equiv \mathcal{V}_{2q,2p}(A)$.

The intersections which occur when a variety in one of the $[2n-3]$'s $A, B, C, D, A', B', C', D'$ cuts a variety in a different $[2n-3]$ are of four types. We name a typical specimen of each kind;

$$[\mathcal{V}_{2p}(A), \mathcal{V}_{2q+1}(B')], \qquad [\mathcal{V}_{2p}(A), \mathcal{V}_{2q}(B)],$$
$$[\mathcal{V}_{2p+1}(C'), \mathcal{V}_{2q+1}(D')], \qquad [\mathcal{V}_{2p}(A), \mathcal{V}_{2q+1}(A')].$$

The intersection $[\mathcal{V}_{2p}(A), \mathcal{V}_{2q+1}(B')]$. Since $\mathcal{V}_{2p}(A)$ lies in A and $\mathcal{V}_{2q+1}(B')$ lies in B', $[\mathcal{V}_{2p}(A), \mathcal{V}_{2q+1}(B')]$ lies in AB'. But AB' cuts $\mathcal{V}_{2p}(A)$ in $\mathcal{V}_{2p}(A_1) + \mathcal{V}_{2p-1}(B'_1)$ and cuts $\mathcal{V}_{2q+1}(B')$ in $\mathcal{V}_{2q+1}(B'_1) + \mathcal{V}_{2q}(A_1)$. Consequently $[\mathcal{V}_{2p}(A), \mathcal{V}_{2q+1}(B')]$ consists of $\mathcal{V}_{2p,2q}(A_1) + \mathcal{V}_{2p-1,2q+1}(B'_1) + [\mathcal{V}_{2p}(A_1), \mathcal{V}_{2q+1}(B'_1)] + [\mathcal{V}_{2p-1}(B'_1), \mathcal{V}_{2q}(A_1)]$.

Now $[\mathcal{V}_{2p}(A_1), \mathcal{V}_{2q+1}(B'_1)]$ lies in $A_1 B'_1$ and therefore by an argument similar to the above consists of $\mathcal{V}_{2p,2q}(A_2)$, which lies in $\mathcal{V}_{2p,2q}(A_1)$, $\mathcal{V}_{2p-1,2q+1}(B'_2)$ which lies in $\mathcal{V}_{2p-1,2q+1}(B'_1)$, $[\mathcal{V}_{2p-1}(B'_2), \mathcal{V}_{2q}(A_2)]$ which lies in $\mathcal{V}_{2p-1,2q+1}(B'_1)$, and $[\mathcal{V}_{2p}(A_2), \mathcal{V}_{2q+1}(B'_2)]$.

Similarly $[\mathcal{V}_{2p-1}(B'_1), \mathcal{V}_{2q}(A_1)]$ consists of four parts, $\mathcal{V}_{2p-1,2q-1}(B'_2)$ in $\mathcal{V}_{2p,2q}(A_1)$, $\mathcal{V}_{2p-2,2q}(A_2)$ lying in $\mathcal{V}_{2p-1,2q+1}(B'_1)$, $[\mathcal{V}_{2p-1}(B'_2), \mathcal{V}_{2q}(A_2)]$ in $\mathcal{V}_{2p-1,2q+1}(B'_1)$, and $[\mathcal{V}_{2p-2}(A_2), \mathcal{V}_{2q-1}(B'_2)]$.

Consequently, apart from $\mathcal{V}_{2p,2q}(A_1)$ and $\mathcal{V}_{2p-1,2q+1}(B'_1)$ we have only $[\mathcal{V}_{2p}(A_2), \mathcal{V}_{2q+1}(B'_2)]$ and $[\mathcal{V}_{2p-2}(A_2), \mathcal{V}_{2q-1}(B'_2)]$. But by a repetition of the argument already used, the only parts of these not in $\mathcal{V}_{2p,2q}(A_1)$ or $\mathcal{V}_{2p-1,2q+1}(B'_1)$ are $[\mathcal{V}_{2p}(A_4), \mathcal{V}_{2q+1}(B'_4)]$ and $[\mathcal{V}_{2p-4}(A_4), \mathcal{V}_{2q-3}(B'_4)]$, and again the only "new" parts of these are $[\mathcal{V}_{2p}(A_6), \mathcal{V}_{2q+1}(B'_6)]$ and $[\mathcal{V}_{2p-6}(A_6), \mathcal{V}_{2q-5}(B'_6)]$. Continuing in this way we arrive at the varieties $[\mathcal{V}_{2p}(A_{2x}), \mathcal{V}_{2q+1}(B'_{2x})]$ and $[\mathcal{V}_{2p-2x}(A_{2x}), \mathcal{V}_{2q-2x+1}(B'_{2x})]$. But if $2x > n-2p-1$, $\mathcal{V}_{2p}(A_{2x})$ no longer exists, so at a certain stage $[\mathcal{V}_{2p}(A_{2x}), \mathcal{V}_{2q+1}(B'_{2x})]$ disappears, and a similar argument applies to the other intersection.

So $\mathcal{V}_{2p}(A)$ *and* $\mathcal{V}_{2q+1}(B')$ *cut in*

1^0 $\mathcal{V}_{2p,2q}(A_1)$, *a variety of dimension* $n-2-|p-q|$ *and of order* $\{n-1\,;\,2p, 2q\}$,

2^0 $\mathcal{V}_{2p-1,2q+1}(B'_1)$, *a variety of dimension* $n-2-|p-q-1|$ *and of order* $\{n-1\,;\,2p-1, 2q+1\}$.

We may observe that these may be expressed more symmetrically as $\mathcal{V}_{2p,2q}(A, B')$ and $\mathcal{V}_{2p-1,2q+1}(B', A)$. $\mathcal{V}_{2p,2q}(A, B')$ may be described as the locus of those $[2q]$'s of $\mathcal{V}_{2q}(A)$ which (i) lie in B', (ii) lie in a $[2p]$ of $\mathcal{V}_{2p}(A)$ which lies in B'.

For the sake of clarity let us see how this works out in a particular case; consider for example the intersections when $n = 10$ of $\mathcal{V}_{2p}(A)$ with $\mathcal{V}_5(B')$.

$p = 0$	$\mathcal{V}_{4,0}(A_1)$	a V_6^{15}
$p = 1$	$\mathcal{V}_{4,2}(A_1) + \mathcal{V}_{5,1}(B'_1)$	a $V_7^{105} + V_6^{60}$
$p = 2$	$\mathcal{V}_4\ (A_1) + \mathcal{V}_{5,3}(B'_1)$	a $V_8^{70} + V_7^{140}$
$p = 3$	$\mathcal{V}_{4,6}(A_1) + \mathcal{V}_5\ (B'_1)$	a $V_7^{105} + V_8^{56}$
$p = 4$	$\mathcal{V}_{4,8}(A_1) + \mathcal{V}_{5,7}(B'_1)$	a $V_6^{15} + V_7^{42}$.

The intersections $[\mathcal{V}_{2p}(A), \mathcal{V}_{2q}(B)]$, $[\mathcal{V}_{2p+1}(C'), \mathcal{V}_{2q+1}(D')]$. Consider first $[\mathcal{V}_{2p}(A), \mathcal{V}_{2q}(B)]$. It must lie in AB, and since AB cuts $\mathcal{V}_{2p}(A)$ in $\mathcal{V}_{2p-2}(B_i) + \mathcal{V}_{2p-1}(C'_i) + \mathcal{V}_{2p-1}(D'_i) + \mathcal{V}_{2p}(A_i)$ and cuts $\mathcal{V}_{2q}(B)$ in $\mathcal{V}_{2q-2}(A_i) + \mathcal{V}_{2q-1}(C'_i) + \mathcal{V}_{2q-1}(D'_i) + \mathcal{V}_{2q}(B_i)$, the intersection breaks up into sixteen parts. Four of these are 1^0 $\mathcal{V}_{2p,2q-2}(A_i)$, 2^0 $\mathcal{V}_{2p-1,2q-1}(C'_i)$, 3^0 $\mathcal{V}_{2p-1,2q-1}(D'_i)$, 4^0 $\mathcal{V}_{2p-2,2q}(B_i)$. The twelve remaining ones break up again, and we consider each part seperately. For the sake of brevity we will omit the details; we may conclude that all the parts lie in the varieties 1^0 to 4^0 above, with the possible exception of $[\mathcal{V}_{2p}(A_{ii}), \mathcal{V}_{2q}(B_{ii})]$ and $[\mathcal{V}_{2p-4}(A_{ii}), \mathcal{V}_{2q-4}(B_{ii})]$. But these again degenerate, into parts all lying in 1^0 to 4^0 with the possible exception of $[\mathcal{V}_{2p}(A_{iv}), \mathcal{V}_{2q}(B_{iv})]$ and $[\mathcal{V}_{2p-8}(A_{iv}), \mathcal{V}_{2q-8}(B_{iv})]$. And continuing the argument far enough, we reach varieties which disappear.

The intersection of $\mathcal{V}_{2p}(A)$ *with* $\mathcal{V}_{2q}(B)$ *consists of the four varieties* 1^0 $\mathcal{V}_{2p,2q-2}(A_i)$, *of dimension* $n - 3 - |p - q + 1|$, *and order* $\{n - 2;\ 2p, 2q - 2\}$

2^0, 3^0 $\mathcal{V}_{2p-1,2q-1}(C'_i)$, $\mathcal{V}_{2p-1,2q-1}(D'_i)$, *of dimension* $n - 3 - |p - q|$, *and order* $\{n - 2;\ 2p - 1, 2q - 1\}$

4^0 $\mathcal{V}_{2p-2,2q}(B_i)$ *of dimension* $n - 3 - |p - q - 1|$, *and order* $\{n - 2;\ 2p - 2, 2q\}$. *We may write these as*

$\mathcal{V}_{2p,2q-2}(A, B)$, $\mathcal{V}_{2p-1,2q-1}(C', D')$, $\mathcal{V}_{2p-1,2q-1}(D', C')$, $\mathcal{V}_{2p-2,2q}(B, A)$.

The investigation of $[\mathcal{V}_{2p+1}(C'), \mathcal{V}_{2q+1}(D')]$ follows similar lines.

The intersection of $\mathcal{V}_{2p+1}(C')$ *with* $\mathcal{V}_{2q+1}(D')$ *consists of the four varieties* 1^0 $\mathcal{V}_{2p+1,2q-1}(C'_i)$, *of dimension* $n - 3 - |p - q + 1|$, *and*

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order $\{n-2;\ 2p+1,\ 2q-1\}$

2^0, 3^0 $\mathcal{V}_{2p,2q}(A_i)$, $\mathcal{V}_{2p,2q}(B_i)$, *of dimension* $n-3-|p-q|$, *and order* $\{n-2;\ 2p,\ 2q\}$

4^0 $\mathcal{V}_{2p-1,2q+1}(D'_i)$, *of dimension* $n-3-|p-q-1|$ *and order* $\{n-2;\ 2p-1,\ 2q+1\}$. We may write these also as $\mathcal{V}_{2p+1,2q-1}(C', D')$, $\mathcal{V}_{2p,2q}(A, B)$, $\mathcal{V}_{2p,2q}(B, A)$, $\mathcal{V}_{2p-1,2q+1}(D', C')$.

The intersection $[\mathcal{V}_{2p}(A), \mathcal{V}_{2q+1}(A')]$. This intersection must lie in AA', *i.e.* in X, and consequently it consists of the intersections of the sections by X of $\mathcal{V}_{2p}(A)$ and $\mathcal{V}_{2q+1}(A')$, *i.e.* it is the intersection $[\mathcal{V}_{2p}(\mathfrak{A}) + \mathcal{V}_{2p-1}(\mathfrak{B}') + \mathcal{V}_{2p-1}(\mathfrak{C}') + \mathcal{V}_{2p-1}(\mathfrak{D}') + \mathcal{V}_{2p-2}(\mathfrak{B}) + \mathcal{V}_{2p-2}(\mathfrak{C}) + \mathcal{V}_{2p-2}(\mathfrak{D}) + \mathcal{V}_{2p-3}(\mathfrak{A}'),\ \mathcal{V}_{2q+1}(\mathfrak{A}') + \mathcal{V}_{2q}(\mathfrak{B}) + \mathcal{V}_{2q}(\mathfrak{C}) + \mathcal{V}_{2q}(\mathfrak{D}) + \mathcal{V}_{2q-1}(\mathfrak{B}') + \mathcal{V}_{2q-1}(\mathfrak{C}') + \mathcal{V}_{2q-1}(\mathfrak{D}') + \mathcal{V}_{2q-2}(\mathfrak{A})]$. It therefore breaks up into sixty four separate portions. Eight of these are, 1^0 $\mathcal{V}_{2p,2q-2}(\mathfrak{A})$, 2^0 $\mathcal{V}_{2p-1,2q-1}(\mathfrak{B}')$, 3^0 $\mathcal{V}_{2p-1,2q-1}(\mathfrak{C}')$, 4^0 $\mathcal{V}_{2p-1,2q-1}(\mathfrak{D}')$, 5^0 $\mathcal{V}_{2p-2,2q}(\mathfrak{B})$, 6^0 $\mathcal{V}_{2p-2,2q}(\mathfrak{C})$, 7^0 $\mathcal{V}_{2p-2,2q}(\mathfrak{D})$, 8^0 $\mathcal{V}_{2p-3,2q+1}(\mathfrak{A}')$. If we consider each of the remainder in turn it is possible to show that all lie in 1^0 to 8^0 or break into parts which lie in 1^0 to 8^0, just as in the previous cases. For example $[\mathcal{V}_{2p}(\mathfrak{A}), \mathcal{V}_{2q-}\ (\mathfrak{B}')]$ is $\mathcal{V}_{2p,2q-2}(\mathfrak{A}, \mathfrak{B}') + \mathcal{V}_{2p-1,2q-1}(\mathfrak{B}', \mathfrak{A})$, which lie in 1^0 and 2^0, and $[\mathcal{V}_{2p}(\mathfrak{A}), \mathcal{V}_{q}(\mathfrak{B})]$ is $\mathcal{V}_{2p,2q-2}(\mathfrak{A}, \mathfrak{B}) + \mathcal{V}_{2p-2,2q}(\mathfrak{B}, \mathfrak{A}) + \mathcal{V}_{2p-1,2q-1}(\mathfrak{C}', \mathfrak{D}') + \mathcal{V}_{2p-1,2q-1}(\mathfrak{D}', \mathfrak{C}')$ which lie respectively in 1^0, 5^0, 3^0, 4^0.

The intersection of $\mathcal{V}_{2p}(A)$ *and* $\mathcal{V}_{2q+1}(A')$ *consists of the eight varieties* 1^0 $\mathcal{V}_{2p,2q-2}(\mathfrak{A})$, *of dimension* $n-4-|p-q+1|$ *and order* $\{n-3;\ 2p,\ 2q-2\}$

2^0, 3^0, 4^0, $\mathcal{V}_{2p-1,2q-1}(\mathfrak{B}')$, $\mathcal{V}_{2p-1,2q-1}(\mathfrak{C}')$, $\mathcal{V}_{2p-1,2q-1}(\mathfrak{D}')$, *of dimension* $n-4-|p-q|$ *and order* $\{n-3;\ 2p-1,\ 2q-1\}$

5^0, 6^0, 7^0 $\mathcal{V}_{2p-2,2q}(\mathfrak{B})$, $\mathcal{V}_{2p-2,2q}(\mathfrak{C})$, $\mathcal{V}_{2p-2,2q}(\mathfrak{D})$, *of dimension* $n-4-|p-q-1|$ *and order* $\{n-3;\ 2p-2,\ 2q\}$

8^0 $\mathcal{V}_{2p-3,2q+1}(\mathfrak{A}')$, *of dimension* $n-4-|p-q-2|$ *and order* $\{n-3;\ 2p-3,\ 2q+1\}$.

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