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PRODUCTS OF POSITIVE REFLECTIONS IN THE ORTHOGONAL GROUP

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Introduction. For G a group, S a subset of G which generates G, the length problem in G with respect to S is to find, for $g \in G$, the least integer r such that g can be written as the product of r elements of S. For G an orthogonal group $O_f(F)$ (here F is a field, and the elements of $O_f(F)$ preserve the quadratic form f) and S the set of reflections in $O_f(F)$ the length problem has been studied by E. Cartan [**2**], J. Dieudonné [**4**, **5**], E. Ellers [**7**], P. Scherk [**8**], and others. In all of these investigations, however, the problem posed by requiring that S be a single conjugacy class of reflections in $O_f(F)$ has been ignored. And it is generally the case that the reflections in $O_f(F)$ fall into several conjugacy classes.

The case $F = \mathbf{R}$, the reals, is the one that will concern us below. Here the reflections are in two conjugacy classes, the elements of which we (naturally) label positive and negative. The group $O_f(\mathbf{R})$ is determined by the type (p, q) of the space $\mathbf{R}^{p+q}$, and we write instead $O_{p,q}(\mathbf{R})$ or, more simply, $O_{p,q}$. Such groups are of physical interest, $O_{3,1}$ being the Lorentz group. The positive reflections in this group are the reflections which preserve the direction of time. It seems natural to ask for a solution to the length problem with respect to such reflections.

The restriction to the reals in this paper seems necessary: The techniques used below are wholly inappropriate in the case of a general field.

1. Preliminaries and notation. Let F be a field, V a vector space over F having a symmetric, bilinear, nondegenerate inner product f. The *orthogonal group*, $O_f(F)$, is the group of linear transformations of V preserving f. If a basis of V be chosen, and J is the nonsingular symmetric matrix representing f, then a matrix A belongs to $O_f(F)$ if, and only if,

$$A^tJA = J.$$

Clearly $\det(A) = \pm 1$. In the case $F = \mathbf{R}$, the reals, $O_f(F)$ is determined up to isomorphism by the signature, (p, q), of f and we write $O_{p,q}(\mathbf{R})$ or $O_{p,q}$ for $O_f(\mathbf{R})$. In the sequel we shall assume that $p, q > 0$. When this is the case $O_{p,q}$ has four connected components, the identity component of which is simple modulo its centre. (See [**5**].) The problem of identifying the elements of these four components will be taken up in Section 3.

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Henceforth we will assume that Char $F \neq 2$.

Reflections. A *reflection* in $O_f(F)$ is an element $R \in O_f(F)$ such that $R^2 = I$ and rank $(I - R) = 1$. Thus there are nonzero vectors $a, b \in V$ such that

$$R = I + ab^tJ.$$

Since $R^2 = I$ we obtain $b^tJa = -2$. As well, since $R^tJR = J$ we have $b = \lambda a$, $\lambda \in F$. Thus

$$R = R_a = I + \lambda aa^tJ$$

with

$$\lambda = -2/a^tJa.$$

The conjugacy classes in $O_f(F)$ into which the R_a fall are labelled by the elements of $F^*/(F^*)^2$; the conjugacy class of R_a is determined by a^tJa mod $(F^*)^2$. In particular, when $F = \mathbf{R}$, the reflections in $O_{p,q}$ fall into two conjugacy classes distinguished by the sign of the length of the vector a determining R_a. We shall say that a reflection R_a is *positive* if $a^tJa > 0$. The positive reflections in $O_{p,q}$ $(p, q > 0)$ generate two of the four components of $O_{p,q}$. We write $G_{p,q}$ for the group so generated.

Let $u \in G_{p,q}$. We denote by $l(u)$ the smallest number of positive reflections whose product is u. (Here we adopt the convention that $l(1) = 0$.) It is the main purpose of this paper to determine $l(u)$ for all $G_{p,q}$ with $p, q > 0$. (The cases $p = 0$ or $q = 0$ are trivial.)

Finally, if u is any linear transformation from V to V, we will denote by $r(u)$ the rank of $1 - u$, and by $E(u)$ the 1-eigenspace of u.

LEMMA 1.1. *Let $u, r_a \in O_f(F)$ with r_a a reflection and* char $(F) \neq 2$. *Then $E(r_au) \supset E(u)$ or $E(r_au) \subset E(u)$ and*

(1) $\dim E(r_au) = \dim E(u) \pm 1.$

Proof. See [**3**], [**5**].

We have immediately:

COROLLARY 1.2. *Let $u, r_a \in O_f(F)$ with r_a a reflection. Then*

$$\dim E(r_au) = \dim E(u) + 1$$

if, and only if, $a \in E(u)^\perp$.

Types in $O_{p,q}$. If u and u' are in the same conjugacy class of $G_{p,q}$ then $l(u) = l(u')$. In [**1**], Bourgoyne and Cushman classified the conjugacy classes of the unitary group $U_{p,q}(\mathbf{C})$, in the following terms: For $u \in U_{p,q}(\mathbf{C})$ acting on $\mathbf{C}^{p+q}$, we have

$$\mathbf{C}^{p+q} = V_1 \perp V_2 \perp \ldots \perp V_k,$$

an orthogonal direct sum, in which each V_i is u-invariant, and irreducible as a u-module. Relative to some basis of V_i ($i = 1, \ldots, k$) the action of u on V_i is that of a Jordan block, or a pair of Jordan blocks. Two elements u and u' are in the same conjugacy class if and only if they have the same decomposition into Jordan blocks.

Given this, it is easy to restrict to the subgroup $O_{p,q}$ of $U_{p,q}$ and classify, in similar terms, the conjugacy classes there. We now list and label these actions together with (sometimes) matrix presentations of them, which will be referred to subsequently as *standard*. See [**6**]. The subspace of V on which the action takes place will be called the *carrier space* of the action. In all cases presented, $m \geqq 0$. The matrices A below yield a real irreducible action, and preserve the metric associated with the matrix J which accompanies, i.e., $A^tJA = J$.

Type 1. $\Delta_m(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$; $|\lambda| \neq 1$, $\lambda \notin \mathbf{R}$.
This type can be presented as

$$A = \begin{pmatrix} Q & 0 \\ 0 & (Q^{-1})^t \end{pmatrix}, \ J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where Q is a $(2m + 2) \times (2m + 2)$ real matrix with eigenvalues λ and $\bar{\lambda}$ having only two (complex) eigenvectors.

Type 2. $\Delta_m(\lambda, \lambda^{-1})$; $|\lambda| \neq 1$, $\lambda \in \mathbf{R}$.
This type can be presented as

$$A = \begin{pmatrix} Q & 0 \\ 0 & (Q^{-1})^t \end{pmatrix}, \ J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where Q is an $(m + 1) \times (m + 1)$ Jordan block with eigenvalue λ.

Type 3. $\Delta_m^{\epsilon}(\lambda, \bar{\lambda})$; $|\lambda| = 1$, $\lambda \neq \pm 1$, $\epsilon = \pm$.
The action here is of $(m \times 1) \times (m + 1)$ Jordan λ and $\bar{\lambda}$ blocks on a (complex) space of complex dimension $2m + 2$ and signature

$(m + 1 + \epsilon 1, m + 1 - \epsilon 1)$ if m is even;

$(m + 1, m + 1)$ if m is odd.

This action can be realized on a real space of dimension $2m + 2$. In the special case $m = 0$ this can be presented as

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad J = \epsilon I_2 \quad \lambda = e^{i\theta}.$$

Type 4. $\Delta_{2m+1}^{+}(1) + \Delta_{2m+1}^{-}(1)$.
This can be presented as

$$A = \begin{pmatrix} Q & 0 \\ 0 & (Q^{-1})^t \end{pmatrix} \quad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where Q is a $(2m + 2) \times (2m + 2)$ Jordan block of 1's.

Type 5. $\Delta^{+}_{2m+1}(-1) + \Delta^{-}_{2m+1}(-1)$.
This is similar to the preceding type.

Type 6. $\Delta^{\epsilon}_{2m}(1)$; $\epsilon = \pm$.
The action is that of a $(2m + 1) \times (2m + 1)$ Jordan block of 1's, on a space of signature $(m + 1 + \epsilon 1, m + 1 - \epsilon 1)$.

Type 7. $\Delta^{\epsilon}_{2m}(-1)$; $\epsilon = \pm$.
This is similar to the preceding type.

Every element $u \in O_{p,q}$ induces an orthogonal decomposition

$$V = V_1 \perp \ldots \perp V_k$$

of V into subspaces invariant under u, such that the action of u is one of the actions described in types 1-7. Accordingly, following [**6**], we shall say that u *belongs to the type* $\Delta = \Delta_1 + \ldots + \Delta_k$ associated with the conjugacy class in $O_{p,q}$ of u. If $\Delta_0^+(1)$ appears m times in Δ, $\Delta_0^-(1)$ n times, then the *effective part*, eff (Δ), of Δ is

$$\Delta - n\Delta_0^+(1) - n\Delta_0^-(1).$$

eff (u) is defined similarly.
When r_a is a reflection such that $E(r_a u) \supset E(u)$ we will write

$$u \rightarrow r_a u.$$

If $E(r_a u) \subset E(u)$ we will write

$$u \leftarrow r_a u.$$

2. The main theorem. In this section we state and prove the result concerning lengths of elements in $G_{p,q}$ with respect to positive reflections. We require first

Definition 2.1. $u \in G_{p,q}$ is *exceptional* if either
(i) $E(u)^{\perp}$ is negative semi-definite, or
(ii) $u^2 = 1$ and $E(u)^{\perp}$ is not positive semidefinite. ($u = 1$ is non-exceptional.)

Note. Definition 2.1 could be restated as follows:
$u \in G_{p,q}$ is exceptional if the type of eff (u) is either

$$\text{(i) } \sum_{i=1}^{m} \Delta_0^-(\lambda_i, \bar{\lambda}_i) + n(\Delta_1^+(1) + \Delta_1^-(1)) + p\Delta_0^-(-1) + q\Delta_2^-(1),\ m + n + p + q > 0, \text{ or}$$

(ii) $p\Delta_0^-(-1) + q\Delta_0^+(-1)$, $p > 0$.

LEMMA 2.2. *Let* $u, r_a \in G_{p,q}$ *with* r_a *a reflection and* $f(a, a) = 1$. *Then*

$$\text{trace } (r_a u) = \text{trace } (u) - 2f(a, ua).$$

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Furthermore, to show that $u \to r_a u$ *with* $r_a u$ *having eigenvalues off the unit circle, it suffices to show that* $f(a, ua)$ *can be made arbitrarily large subject to* $f(a, a) = 1$ *and* $a \in E(u)^{\perp}$.

Proof.

$$\begin{aligned} \operatorname{tr}(r_a u) &= \operatorname{tr}(u - 2 - aa^t J_a) \text{ if } f(a, a) = 1 \\ &= \operatorname{tr}(u) - 2a^t Jua \\ &= \operatorname{tr}(u) - 2f(a, ua). \end{aligned}$$

The remaining assertion of the lemma is obvious in view of Lemma 2.1.

We can now state the main theorem. The proof will proceed by a series of lemmas.

THEOREM 2.3. *For* $u \in G_{p,q}$, $p, q > 0$, $l(u) = r(u)$ *unless* u *is exceptional, when* $l(u) = r(u) + 2$.

The following lemmas will establish this fact: If u is nonexceptional, then there is a positive reflection r_a such that

$u \to r_a u$ and

$r_a u$ is nonexceptional.

The first assertion of the theorem will then follow by induction on $r(u)$.

LEMMA 2.4. *If* $u \in G_{p,q}$ *has type* $\Delta = \Delta_m(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$ *with* $|\lambda| \neq 1$ *and* $\lambda \notin \mathbf{R}$, *then* $u \to u'$ *with* u' *nonexceptional.*

Proof. $u' = r_a u$ with a any vector of (say) length 1. Then u' has a single $+1$ eigenvector by Lemma 1.2, and $E(u')^{\perp}$ contains positive vectors since the carrier space of Δ has a subspace of positive type of dimension at least 2. Thus we are done unless $(u')^2 = 1$. In this last case the carrier space of u' admits a basis of ± 1 eigenvectors of u', with the -1 eigenvectors spanning a space of dimension at least 3. But since $r(r_a) = 1$, and $u = r_a u'$,

$$\dim E(-u) \geqq 2$$

and this is impossible since u has no -1 eigenvectors.

LEMMA 2.5. *If* u *contains a type* $\Delta_n(\lambda, \lambda^{-1})$ *with* $\lambda \in \mathbf{R}$, $|\lambda| \neq 1$, *then* $u \to u'$ *with* u' *nonexceptional.*

Proof. If $m > 0$ we may proceed as in the last lemma, choosing a vector a in the carrier space of $\Delta_m(\lambda, \lambda^{-1})$ so that $u' = r_a u$ and $f(a, a) = 1$. If $m = 0$ we let

$$\Delta_0(\lambda, \lambda^{-1}) \to \Delta_0^{\epsilon}(1) + \Delta_0^{-\epsilon}(-1),\ \epsilon = \pm.$$

If the type of eff (u) is $\Delta_0(\lambda, \lambda^{-1})$ then, as will be seen in the next section,

$\lambda > 0$. In this case we must choose a in the carrier space of $\Delta_0(\lambda, \lambda^{-1})$. We may present the 2-dimensional problem as follows:

$$A = \text{eff}\,(u) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$a = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{with } 2\xi\eta = 1.$$

Then

$$R_a = I - 2\begin{pmatrix} \xi \\ \eta \end{pmatrix}(\xi \quad \eta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -2\begin{pmatrix} 0 & \xi^2 \\ \eta^2 & 0 \end{pmatrix}$$

and

$$R_aA = -2\begin{pmatrix} 0 & \xi^2\lambda^{-1} \\ \eta^2\lambda & 0 \end{pmatrix}$$

and elementary computation shows that the $+1$ eigenvector of R_aA is of negative type. Thus $\epsilon = -1$, and since $\Delta_0^-(1) + \Delta_0^+(-1)$ corresponds to a positive reflection, we are done. Otherwise, $E(u')^\perp$ is negative semidefinite, or else $(u')^2 = 1$. In either case, u must contain one of the types

$$\Delta_1 = \Delta_0(\lambda, \lambda^{-1}) + \Delta_0^\epsilon(-1),\ \epsilon = \pm;$$
$$\Delta_2 = \Delta_0(\lambda, \lambda^{-1}) + \Delta_0^-(\lambda, \bar{\lambda}),\ |\lambda| = 1, \lambda \neq \pm 1;$$
$$\Delta_3 = \Delta_0(\lambda, \lambda^{-1}) + \Delta_1^+(1) + \Delta_1^-(1), \text{ or};$$
$$\Delta_4 = \Delta_0(\lambda, \lambda^{-1}) + \Delta_2^-(1).$$

In the first case we can present eff (u) and a as

$$A_1 = \begin{pmatrix} \lambda & & \\ & \lambda^{-1} & \\ & & -1 \end{pmatrix} \quad J_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \epsilon 1 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \text{ with } 2\xi\eta + \epsilon\zeta^2 = 1.$$

We choose a_1 so that

$$a_1{}^tJ_1A_1a_1 = \xi\eta(\lambda + \lambda^{-1}) - \epsilon\zeta^2 = \frac{\lambda + \lambda^{-1}}{2} - \epsilon\zeta^2\left(\frac{\lambda + \lambda^{-1}}{2} + 1\right)$$

is arbitrarily large, which is possible since

$$\frac{\lambda + \lambda^{-1}}{2} + 1 \neq 0.$$

This shows, by Lemma 2.2, that we can choose r_a so that $u \to u'$ with u'

having eigenvalues off the unit circle. Hence u' is nonexceptional, and we are done.

In the second case we can take

$$A_2 = \begin{pmatrix} \lambda & & & \\ & \lambda^{-1} & & 0 \\ & 0 & \cos\theta & -\sin\theta \\ & & \sin\theta & \cos\theta \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$a_2 = \begin{pmatrix} \xi \\ \eta \\ \zeta \\ \delta \end{pmatrix} \quad \text{with } 2\xi\eta - \zeta^2 - \delta^2 = 1.$$

Then

$$a_2{}^t J_2 A_2 a_2 = \xi\eta(\lambda + \lambda^{-1}) - (\zeta^2 + \delta^2)\cos\theta$$
$$= (\zeta^2 + \delta^2)\left(\frac{\lambda + \lambda^{-1}}{2} - \cos\theta\right) + \frac{\lambda + \lambda^{-1}}{2}.$$

Since $\cos(\theta) \neq \pm 1$, this can be made arbitrarily large by choice of ζ and δ and we are again done by Lemma 2.2.

In the third case we note that the contribution to the invariant (see Section 3) of Δ_3 is $+1$ if $\lambda > 0$, -1 if $\lambda < 0$. Hence, in order that $u \in G_{p,q}$ it is necessary either that $\lambda > 0$, whence

$$\Delta \rightarrow \Delta_0^-(1) + \Delta_0^+(-1) + \Delta_1^+(1) + \Delta_1^-(1)$$

(which is nonexceptional) or that u contains types different from $\Delta_1^+(1) + \Delta_1^-(1)$. Accordingly, we proceed to type 4, which may be presented as

$$A_4 = \begin{pmatrix} \lambda & & & & \\ & \lambda^{-1} & & & \\ & & 1 & & \\ & & 1 & 1 & \\ & & 0 & 1 & 1 \end{pmatrix} \quad J_4 = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & \frac{1}{2} & 1 \\ & & \frac{1}{2} & -1 & 0 \\ & & 1 & 0 & 0 \end{pmatrix}$$

$$a_4 = \begin{pmatrix} \xi \\ \eta \\ 0 \\ \zeta \\ \delta \end{pmatrix} \text{with } 2\xi\eta - \zeta^2 = 1.$$

(Note that $a_4 \in E(u)^{\perp}$.) We have

$$a_4{}^t J_4 A_4 a_4 = \xi\eta(\lambda + \lambda^{-1}) - \zeta^2 = \zeta^2\left(\frac{\lambda + \lambda^{-1}}{2} - 1\right) + \frac{\lambda + \lambda^{-1}}{2}$$

and since $\lambda \neq 1$, this can be made arbitrarily large. Again by Lemma 2.2 we are done.

LEMMA 2.6. *If $u \in G_{p,q}$ contains any of the types*

(i) $\Delta_m{}^\epsilon(\lambda, \bar{\lambda})$, $|\lambda| = 1$, $\lambda \neq \pm 1$, $m > 0$

(ii) $\Delta_{2k+1}^+(-1) + \Delta_{2k+1}^-(-1)$, $k \geqq 0$

(iii) $\Delta_{2k}^\epsilon(-1)$, $k \geqq 1$, $\epsilon = \pm$

(iv) $\Delta_{2k+1}^+(1) + \Delta_{2k+1}^-(1)$, $k \geqq 1$

(v) $\Delta_{2k}{}^+(1)$, $k \geqq 1$

(vi) $\Delta_{2k}{}^-(1)$, $k \geqq 2$

then $u \to u'$ with u' nonexceptional.

Proof. (i) The signature of the carrier space of $\Delta_m(\lambda, \bar{\lambda})$ is $(m + 2, m)$ or $(m, m + 2)$ if m is even, and $(m + 1, m + 1)$ if m is odd. Since $m > 0$, the claim follows as in the proof of Lemma 2.4.

(ii) For $k \geqq 1$ the claim follows as before (noting that the dimension of the -1 eigenspace corresponding to $\Delta_{2k+1}^+(-1) + \Delta_{2k+1}^-(-1) = \Delta$ is only 2). For $k = 0$ (using r_a so that a lies in the carrier space of Δ) we have

$$(2) \quad \Delta = \Delta_1{}^+(-1) + \Delta_1{}^-(-1) \to \Delta_0{}^\epsilon(1) + \Delta'$$

or

$$(3) \quad \Delta = \Delta_1{}^+(-1) + \Delta_1{}^-(-1) \to \Delta_2{}^\epsilon(1) + \Delta_0{}^{-\epsilon}(-1) = \Delta''.$$

Here the restrictions on the right hand side are imposed by the fact that $\dim E(r_a u) = \dim E(u) + 1$; the choice of $-\epsilon$ in $\Delta_0{}^{-\epsilon}(-1)$ is forced by considerations in Section 3.

Now (3) is impossible since $E(-\Delta)$ contains a totally isotropic subspace of dimension 2 while $E(-\Delta'')$ contains none. In the former case (2) it suffices, by the signature argument, that

$$\Delta' \neq r\Delta_0{}^+(-1) + s\Delta_0{}^-(-1).$$

But, letting $u' = r_a u$, we must choose the reflection vector a so that $a \notin E^\perp(-\Delta)$, since $E^\perp(-\Delta)$ is totally isotropic. Hence, by Corollary 1.2,

$$\dim E(-\Delta') < \dim E(-\Delta).$$

This case therefore cannot arise.

(iii) If a is any vector of unit length in the carrier space of $\Delta_{2k}{}^\epsilon(-1)$ then $u \to r_a u$. Choose a so that

$$a \notin E^\perp(-\Delta_{2k}{}^\epsilon(-1)).$$

Then if

$$\Delta_{2k}^\epsilon(-1) \to \Delta'$$

then $E(-\Delta') = (0)$ by Corollary 1.2 and

$$\dim E(-\Delta_{2k}^\epsilon(-1)) = 1.$$

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Thus, in order to show that $r_a u$ is nonexceptional, we need only consider the case when $E^{\perp}(\Delta')$ is negative semidefinite. Since the carrier space of $\Delta_{2k}^{\epsilon}(-1)$ has signature $(k, k + 1)$ or $(k + 1, k)$, it suffices to consider the case $k = 1$. For $\Delta = \Delta_2^+(-1)$ we have

$$\Delta' = \Delta_2^+(1) \text{ or}$$

$$\Delta' = \Delta_0^{\epsilon}(1) + \Delta'', \epsilon = \pm$$

and in both cases $E(\Delta')^{\perp}$ contains positive vectors. For $\Delta = \Delta_2^-(-1)$ we have the matrix presentation

$$A = \begin{pmatrix} -1 & & \\ 1 & -1 & \\ 0 & 1 & -1 \end{pmatrix} \quad J = \begin{pmatrix} 0 & -\frac{1}{2} & 1 \\ -\frac{1}{2} & -1 & \\ 1 & & \end{pmatrix}.$$

With coordinates ξ, η, ζ for the reflection vector a we find that

$$a^tJAa = -\xi^2/2 + \xi\eta + \eta^2 - 2\xi\zeta$$

subject to

$$-\xi\eta - \eta^2 + 2\xi\zeta = 1.$$

Hence

$$a^tJAa = -\xi^2/2 - 1$$

and a^tJAa can be made arbitrarily large and so (Lemma 2.2) we have, by proper choice of a,

$$\Delta \rightarrow \Delta_0^-(1) + \Delta_0(\lambda, \lambda^{-1}); \lambda \in \mathbf{R}, |\lambda| \neq 1.$$

This is of the required form.

Before proceeding we require the following

LEMMATA. *If V is a space of signature (p, q) and W is a subspace of V of dimension $< p$, then $W^{\perp}$ is not negative semidefinite.*

Proof. $\dim (W^{\perp}) = p + q - \dim (W) > q$. Let R be the radical of $W^{\perp}$. Then, assuming that $W^{\perp}$ is negative semidefinite, $W^{\perp} = R \oplus T$ where T is negative definite, and $R \subset T^{\perp}$. But

$$\dim (R) \leqq \min (p, q - \dim T)$$

and in particular

$$\dim (R) \leqq q - \dim (T)$$

so that

$$\dim (W) = \dim (T) + \dim (R) \leqq q$$

a contradiction.

We now return to the proof of the lemma.

(iv) The result is clear, as in Lemma 2.4, unless $k = 1$. We have

$$\Delta_3^+(1) + \Delta_3^-(1) \to \Delta'$$

with $\dim E(\Delta') = 3$ and, since the carrier space of Δ' has signature $(4, 4)$, the possibility that $E(\Delta')^\perp$ is negative semidefinite is excluded by the lemmata. The possibility

$$\Delta' = p\Delta_0^+(1) + (3-p)\Delta_0^-(1) + q\Delta_0^+(-1) \\ + (5-q)\Delta_0^-(-1)$$

is excluded by

$$\dim E(-\Delta') \leqq 1$$

in view of Lemma 1.1.

(v) The result follows from the lemmata unless $k = 1$. We have

$$\Delta_2^+(1) \to \Delta'$$

and, since $E(\Delta')$ contains an isotropic vector and has dimension 2, and $\omega(\Delta_2^+(1)) = 1$ (see Section 3)

$$\Delta' = \Delta_0^+(1) + \Delta_0^-(1) + \Delta_0^+(-1)$$

and u' is nonexceptional unless the types contained in u are $\Delta_2^+(1)$, $\Delta_0^\epsilon(-1)$, $\Delta_0^\epsilon(\epsilon = \pm)$ with $\Delta_0^-(-1)$ present. We consider then

$$\Delta_2^+(1) + \Delta_0^-(-1) \to \Delta'$$

which we claim we can do with Δ' having eigenvalues off the unit circle. We can represent $\Delta_2^+(1) + \Delta_0^-(-1)$ and the reflection vector a of r_a in the form

$$A = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ & & & -1 \end{pmatrix} \quad J = \begin{pmatrix} 0 & -\frac{1}{2} & -1 & \\ -\frac{1}{2} & 1 & 0 & \\ -1 & 0 & 0 & \\ & & & -1 \end{pmatrix} \quad a = \begin{pmatrix} \xi \\ \eta \\ \zeta \\ \delta \end{pmatrix}.$$

We require $a \in E(u)^\perp$ and this yields $\xi = 0$. For $a^tJa = 1$ we require $\eta^2 = 1 + \delta^2$. Finally,

$$a^tJAa = \eta^2 + \delta^2 = 1 + 2\delta^2$$

and this can be made arbitrarily large. The claim follows.

(vi) The claim follows from the lemmata unless $k = 2$. We then have $\Delta_4^-(1) \to \Delta'$, $\dim E(\Delta') = 2$, and $E(\Delta')$ contains isotropic vectors. If

$$\Delta' \supset \Delta_0^+(1) + \Delta_0^-(1)$$

then $E(\Delta')^\perp$ is not negative semidefinite (since $E(\Delta')$ has signature $(1, 2)$) and $(u')^2 \neq 1$ (since $\dim E(-\Delta')^\perp \leqq 1$ in the carrier space of

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$\Delta_4^-(1)$). The only other possibility is

$$\Delta' = \Delta_1^+(1) + \Delta_1^-(1) + \Delta_0^-(-1)$$

(the term $\Delta_0^-(-1)$ being dictated by the determinant, its sign by Section 3), when $E(\Delta')$ is totally isotropic. We seek to avoid this case by choice of $a = (1 - u)x$, $x \in E(r_a u)\backslash E(u)$. It suffices to find such a vector x which is nonisotropic. Otherwise,

$$f((1 - u)x, (1 - u)x) > 0 \Rightarrow f(x, x) = 0$$

or

$$2f(x, x) > f(x, (u + u^{-1})x) \Rightarrow f(x, x) = 0$$

or

$$2f(x, x) \leqq f(x, (u + u^{-1})x) \text{ whenever } f(x, x) \neq 0.$$

Continuity then gives

$$2f(x, x) \leqq f(x, (u + u^{-1})x) \text{ for all } x.$$

But this implies $f(a, a) \leqq 0$ for all choices of x, and this is false since $E(\Delta')^\perp$ is not negative semidefinite.

This completes the proof of the lemma.

The remaining cases are now all of low dimension.

LEMMA 2.7. *If $u \in G_{p,q}$, with $\Delta_0^+(-1)$ belonging to the type of u, as well as any of*

(i) $\Delta_0^\epsilon(\lambda, \bar{\lambda})$, $|\lambda| = 1$, $\lambda \neq \pm 1$, $\epsilon = \pm$
(ii) $\Delta_1^+(1) + \Delta_1^-(1)$
(iii) $\Delta_2^-(1)$

then $u \rightarrow u'$ with u' nonexceptional.

Proof. If Δ, the type of u, contains $\Delta_0^+(-1) + \Delta_0^+(\lambda, \bar{\lambda}) = \Delta_1$ then we have, by choosing the reflection vector a in the carrier space of $\Delta_0^+(\lambda, \bar{\lambda})$,

$$\Delta_1 \rightarrow \Delta_0^+(1) + 2\Delta_0^+(-1)$$

and we are done unless the remaining types of u are $\Delta_0^\epsilon(\pm 1)$ with $\Delta_0^-(-1)$ occurring. In this case we take $\Delta = \Delta_0^-(-1) + \Delta_0^+(\lambda, \bar{\lambda})$ and show that $\Delta \rightarrow \Delta'$ with Δ' having eigenvalues off the unit circle. We have matrices and reflection vector given by

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad J = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$a = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \text{ with } \xi^2 + \eta^2 - \zeta^2 = 1.$$

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Also, $\cos\theta \neq \pm 1$. Now

$$\begin{aligned} a^tJAa &= \cos\theta(\xi^2+\eta^2)+\zeta^2 \\ &= (1+\cos\theta)(\xi^2+\eta^2)-1 \end{aligned}$$

which can be made arbitrarily large, proving the last assertion, and hence the claim of the lemma.

The case $u \supset \Delta = \Delta_0^+(-1) + \Delta_0^-(\lambda, \bar{\lambda})$ is the same as above with $-J$ in place of J.

If $u \supset \Delta = \Delta_0^+(-1) + \Delta_2^-(1)$, we can proceed as in the proof of (v), Lemma 2.6, using $-J$ in place of J, to show that $\Delta \to \Delta'$ with Δ' non-exceptional.

Finally, suppose that $u \supset \Delta = \Delta_0^+(-1) + \Delta_1^+(1) + \Delta_1^-(1)$. We claim that

$$\Delta \to \Delta' = \Delta_2^+(1) + \Delta_0^+(1) + \Delta_0^-(1)$$

via $u' = r_a u$, with a a unit vector in the carrier space of Δ. Again, we let A be a matrix representing Δ, preserving a symmetric form represented by a matrix J. We can take

$$A = \begin{pmatrix} 1 & 0 & & & \\ 1 & 1 & & & \\ & & 1 & -1 & \\ & & 0 & 1 & \\ & & & & -1 \end{pmatrix} \quad J = \begin{pmatrix} 0 & I_2 & \\ I_2 & 0 & \\ & & 1 \end{pmatrix} \quad x = \begin{pmatrix} b \\ c \\ d \\ e \\ f \end{pmatrix}.$$

Here $a = (1-A)x$. The condition $a^tJa = 1$ is $f^2 = 1$. We choose x not orthogonal to $E(u)$. Now $E(u') = E(u) \oplus (x)$ is not totally isotropic, and in fact has radical of dimension 1. Hence

$$\Delta' \not\supset \Delta_1^+(1) + \Delta_1^-(1) + \Delta_0^+(1)$$

and so

$$\Delta' = \Delta_2^{\epsilon}(1) + \Delta_0^+(1) + \Delta_0^{-\epsilon}(1), \ \epsilon = \pm$$

and $\omega(\Delta) = 1$ forces (see Section 3) $\epsilon = +$. The conclusion follows.

LEMMA 2.8. *If $u \in G_{p,q}$ and u contains the type $\Delta_0^+(\lambda, \bar{\lambda})$ with $|\lambda| = 1$, $\lambda \neq \pm 1$ and u contains as well one of the types*

(i) $\Delta_0^-(-1)$

(ii) $\Delta_1^+(1) + \Delta_1^-(1)$

(iii) $\Delta_2^-(1)$

then $u \to u'$ with u' non-exceptional.

Proof. We have

$$\Delta_0^+(\lambda, \bar{\lambda}) \to \Delta_0^+(1) + \Delta_0^+(-1)$$

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and the result follows from Lemma 2.7 unless we have the case (i) above, which was dealt with in the proof of Lemma 2.7.

LEMMA 2.9. *If* $u \in G_{p,q}$ *contains* $\Delta = \Delta_1^+(-1) + \Delta_1^-(-1)$ *then* $u \to u'$ *with* u' *nonexceptional.*

Proof. For any reflection vector a chosen in the carrier space of Δ we have $\Delta \to \Delta'$. If $\Delta' \supset \Delta_0^\epsilon(1)$ then we are done unless

$$\Delta' = \Delta_0^\epsilon(1) + \Delta_0^\epsilon(-1) + 2\Delta_0^{-\epsilon}(-1).$$

This can be avoided by choosing the reflection vector outside of $E(-u)^\perp$. Otherwise we have

$$\Delta' = \Delta_2^\epsilon(1) + \Delta_0^{-\epsilon}(-1),\ \epsilon = \pm$$

and again u' is nonexceptional.

Proof. (of the theorem). By Lemmas 2.4-2.9 we have $u \to u'$ with u' nonexceptional unless (by Lemmas 2.4, 2.5, 2.6, 2.9) the types contained in u are $\Delta_0^\epsilon(\lambda, \bar{\lambda})$, $|\lambda| = 1$, $\lambda \neq \pm 1$; $\Delta_0^\epsilon(-1)$; $\Delta_1^+(1) + \Delta_1^-(1)$; $\Delta_2^-(1)$; $\Delta_0^\epsilon(1)$. The types $\Delta_0^\epsilon(1)$ can be ignored. By Lemma 2.7, if $\Delta_0^+(-1)$ belongs to the type of u, as well as one of the types above (other than $\Delta_0^\epsilon(-1)$), then $u \to u'$ with u' nonexceptional. Hence either u is exceptional, or we have $u^2 = 1$ with $\Delta_0^-(-1)$ not in the type of u. But in this case clearly $l(u) = r(u)$ since the type of eff (u) is $k\Delta_0^+(-1)$. Hence we can remove $\Delta_0^+(-1)$ from the list above. By Lemma 2.8 we can remove $\Delta_0^+(\lambda, \bar{\lambda})$ from the shortened list which now is

$$\Delta_0^-(\lambda, \bar{\lambda}),\ |\lambda| = 1,\ \lambda \neq \pm 1;$$

$$\Delta_0^-(-1);\ \Delta_1^+(1) + \Delta_1^-(1);\ \Delta_2^-(1);\ \Delta_0^\epsilon(1).$$

However, if these are the types in u, then u is exceptional.

It only remains to show that if u is exceptional, then $l(u) = r(u) + 2$. If $E(u)^\perp$ is negative semidefinite then (Corollary 1.2) for any choice of a positive reflection we have

$$u \leftarrow u'$$

and since $\dim E(u') = \dim E(u) - 1$, we have

$$l(u) \geqq r(u) + 2.$$

On the other hand, we can choose a positive reflection r_a so that tr $(r_a u)$ is arbitrarily large (this is easily checked) so that $r_a u$ has eigenvalues off the unit circle. The result follows in this case.

If $u^2 = 1$ with $E(u)^\perp$ not positive definite, and $u \to u'$ then also $(u')^2 = 1$ with $E(u')^\perp$ not positive definite. (This follows from the observation that $\dim E(-u') = \dim E(-u) - 1$ and that the reflection vector a is a positive vector in $E(-u)$.) The result now follows by

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induction, and the observation that if

$$u \to u' \to \ldots \to v$$

then we must arrive at a transformation v for which $E(v)^{\perp}$ is negative definite.

This completes the proof.

3. The invariant. In another paper [**6**], with D. Ž. Djokovic, concerning the length problem with respect to reflections in $U_{p,q}(\mathbf{C})$, it was necessary to introduce a construction called the *invariant*. This was defined as follows: For $u \in U_{p,q}(\mathbf{C})$, if $\det(1 - u) \neq 0$, then the invariant, $\omega(u)$, is given by

$$\omega(u) = (-1)^q \det(1 - u)/|\det(1 - u)|.$$

In the case $\det(1 - u) = 0$, let $d = \dim E(u)$. Then there are d positive reflections $r_1, \ldots, r_d$ so that $\hat{u} = r_1 \ldots r_d u$ satisfies $\det(1 - \hat{u}) \neq 0$ and we define $\omega(u) = \omega(\hat{u})$. $\omega(u)$ is well-defined and $\omega(u) = \pm 1, \pm i$ when $\det(u) = \pm 1$. $\omega(u)$ is called the invariant in [**6**] because if

$$u \to u' \text{ or } u \leftarrow u'$$

then $\omega(u) = \omega(u')$. (There is in $U_{p,q}(\mathbf{C})$ the further possibility that if $u' = r_a u$ then $E(u') = E(u)$. In this case $\omega(u) = \pm i\omega(u')$.)

The mapping $u \to \omega(u)$ is not a homomorphism in $U_{p,q}(\mathbf{C})$. However, the construction is "inherited" by $O_{p,q}(\mathbf{R})$, with the same properties. Since now $\omega(u)$ is real we have $\omega(u) = \pm 1$. Furthermore, the mapping $u \to \omega(u)$ is a homomorphism. Also, as we shall see, together with the mapping $u \to \det(u)$ the four connected components of $O_{p,q}(\mathbf{R})$ $(p, q > 0)$ are distinguished.

Notation. We label the identity component of $O_{p,q}$ $(p, q > 0)$ by $A_{p,q}$; $G_{p,q} \backslash A_{p,q}$ by $B_{p,q}$; the part generated by negative reflections and having determinant -1 we label by $C_{p,q}$, and; $SO_{p,q} \backslash A_{p,q}$ by $D_{p,q}$.

LEMMA 3.1. *The mapping* $u \to \omega(u)$ *of* $O_{p,q}$ *to* ± 1 *is a homomorphism in which* $G_{p,q} \to +1$ *and* $C_{p,q}$, $D_{p,q} \to -1$. *Thus together with the mapping* $u \to \det(u)$, *all of the connected components of* $O_{p,q}$ *have been distinguished.*

Proof. If r_a is a positive reflection then, if $u' = r_a u$,

$$u \to u' \text{ or } u \leftarrow u'$$

and so $\omega(u) = \omega(u')$ and it follows that $\omega(u) = \omega(v)$ for any $u, v \in G_{p,q}$ since $G_{p,q}$ is generated by positive reflections. Since, [**6**], $\omega(u) = \omega(\text{eff } u)$, and the type eff (r_a) is $\Delta_0^+(-1)$ it is easy to check that $\omega(r_a) = 1$. Now let r_b be a negative reflection; i.e., $f(b, b) = -1$. Each element of $O_{p,q} \backslash G_{p,q}$ is in the coset $r_b G_{p,q}$. Again since $G_{p,q}$ is generated by positive reflections it follows that $\omega(u) = \omega(v)$ for any element u, v of $O_{p,q}\backslash G_{p,q}$. In particular, $\omega(u) = \omega(r_b)$, and since the type of eff (r_b) is $\Delta_0^-(-1)$ we find that $\omega(r_b) = -1$. This completes the proof.

If the type, Δ, of $u \in O_{p,q}$ decomposes into irreducible types $\Delta = \Delta_1 + \ldots + \Delta_k$ then

$$\omega(\Delta) = \omega(u) = \omega(\Delta_1) \times \ldots \times \omega(\Delta_k).$$

Thus $\omega(u)$ can be computed from a knowledge of the irreducible types contained in u. The computation of $\omega(\Delta)$ for Δ irreducible is the subject of the next lemma.

LEMMA 3.2. *If Δ is an irreducible type, then $\omega(\Delta)$ is as given in the list below:*

$\omega(\Delta)$	Δ	
1	$\Delta_m(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$	$\lvert\lambda\rvert \neq 1, \lambda \notin \mathbf{R}$
1 if $\lambda > 0$	$\Delta_m(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert \neq 1, \lambda \in \mathbf{R}$
$(-1)^{m+1}$ if $\lambda < 0$		
1	$\Delta_m(\lambda, \bar{\lambda})$	$\lvert\lambda\rvert = 1, \lambda \neq \pm 1$
1	$\Delta_{2m+1}^+(-1) + \Delta_{2m+1}^-(-1)$	
1	$\Delta_{2m+1}^+(-1) + \Delta_{2m+1}^-(-1)$	
1	$\Delta_{2m}{}^\epsilon(1)$	$\epsilon = \pm$
$(-1)^m$ *if* $\epsilon = +$	$\Delta_{2m}{}^\epsilon(-1)$	$\epsilon = \pm$
$(-1)^{m+1}$ *if* $\epsilon = -$		

Proof. We remark first that $\Delta_{2m}{}^\epsilon(1)$, for example, acts on a space of signature $(m + 1, m)$ if ϵ is $+$, and a space of signature $(m, m + 1)$ if ϵ is $-$. The computation of $\omega(\Delta)$ is straightforward when Δ has no $+1$ eigenvalues. For example,

$$\omega(\Delta_m(\lambda, \lambda^{-1})) = (-1)^{m+1} \frac{(1 - \lambda)^{m+1}(1 - \lambda^{-1})^{m+1}}{\lvert(1 - \lambda)^{m+1}(1 - \lambda^{-1})^{m+1}\rvert}.$$

If $\lambda < 0$ this is just $(-1)^{m+1}$, as claimed. If $\lambda > 0$ then exactly one of $(1 - \lambda)$, $(1 - \lambda^{-1})$ is negative, and

$$\omega(\Delta_m(\lambda, \lambda^{-1})) = 1.$$

If Δ is one of the types $\Delta_{2m+1}^+(1) + \Delta_{2m+1}^-(1)$ or $\Delta_{2m}{}^\epsilon(1)$ then we can

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represent these using Jordan blocks of the form

$$\begin{pmatrix} 1 & & & & 0 \\ \alpha & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & \alpha & 1 \end{pmatrix}$$

with $0 \neq \alpha$ small. In this way we see that these blocks are in the connected component of the identity, and so $\omega(\Delta) = 1$ in these cases.

We have immediately:

COROLLARY 3.3. *An element $u \in O_{p,q}$ belongs to $G_{p,q}$ if and only if the types of u having negative eigenvalues act on a space of type (r, s) with s odd.*

Note. The connected components of $O_f(F)$, F a field with Char $F \neq 2$ are usually distinguished by the *spinor norm*: If $u \in O_f(F)$ is a product of reflections $u = r_{a_1} \dots r_{a_k}$ then the mapping

$$\phi : u \rightarrow f(a_1, a_1) \dots f(a_k, a_k) \bmod (F^*)^2$$

of $O_f(F)$ into $F^*/(F^*)^2$ is a homomorphism of $O_f(F)$ which, together with $u \rightarrow \det(u)$, distinguishes the components of $O_f(F)$. When $F = \mathbf{R}$, $\phi(u) = \omega(u)$. For a general field F we have

$$\phi(u) = \det\left(\frac{1+u}{2}\right) \bmod (F^*)^2$$

provided that $\det(1 + u) \neq 0$.

Added in proof. D. Ž. Djoković has recently proved this result using other methods, generalizing it to the case where the form f is possibly degenerate.

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