MODELS FOR JOINT ISOMETRIES

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(Received 7 November, 1994)

An N-tuple $\mathcal{T} = (T_1, \ldots, T_N)$ of commuting contractions on a Hilbert space **H** is said to be a joint isometry if $\sum_n \|T_n x\|^2 = \|x\|^2$ for all x in **H**, or, equivalently, if $\sum_n T_n^* T_n = I$. Athavale in [**1**] characterized the joint isometries as subnormal N-tuples whose minimal normal extensions have joint spectra in the unit sphere S^{2N-1}; a geometric perspective of this is given in [**4**]. Subsequently, V. Müller and F.-H. Vasilescu proved that commuting N-tuples which are joint contractions, i.e. $T_1^* T_1 + \ldots + T_N^* T_N \leq I$, can be represented as restrictions of certain weighted shifts direct sum a joint isometry. In this paper we adapt the canonical models of [**3**], and also construct a new canonical model, which completes the previous descriptions by showing joint isometries are indeed restrictions of specific multivariable weighted shifts [**2**].

We use the notation $z = (z_1, \ldots, z_N)$ and $J = (j_1, \ldots, j_N)$ for a multi-index of non-negative integers. We let ϵ_k denote the multi-index J having $j_k = 1$ and $j_l = 0$ otherwise; $(J \pm \epsilon_k)$ has the obvious meaning, but by using $(J - \epsilon_k)$ we imply $j_k \geq 1$. We let $|J| = j_1 + \ldots + j_N$, $J! = j_1! \ldots j_N!$, $z^J = z_1^{j_1} \ldots z_N^{j_N}$, and $\mathcal{T} = T^J = T_1^{j_1} \ldots T_N^{j_N}$.

We let $H^2 = H_H^2(U^N) = \{f = \sum f_J z^J : \sum \|f_J\|_H^2 < \infty\}$ be the standard Hardy space of square-summable **H**-valued analytic functions on the polydisc. Given a bounded net $\{w_{J,k} : k = 1, \ldots, N\}$ such that $w_{J,k} w_{J+\epsilon_k, l} = w_{J,l} w_{J+\epsilon_l, k}$, we define L_k, $k = 1, \ldots, N$ to be the unique bounded linear map on **H** such that

$$L_k(xz^J) = w_{J,k} x z^{J+\epsilon_k},$$

for all x in **H** and all J. We call $\mathcal{L} = (L_1, \ldots, L_N)$ a family of commuting N-variable weighted shifts and it is easy to see that $L_k^*(xz^J) = w_{J-\epsilon_k,k} x z^{J-\epsilon_k}$ if $j_k \geq 1$ and 0 otherwise. Furthermore, we can define a net $\{\beta_J\}$ by $\mathcal{L}^J 1 = \beta_J z^J$ and let

$$H^2(\beta) = H^2_{\beta,\mathbf{H}}(U^N) = \left\{ f = \sum f_J z^J : \sum \|f_J\|_{\mathbf{H}}^2 \beta_J^2 < \infty \right\}$$

be a weighted H^2 space. We then define $\tilde{L}_k$, $k = 1, \ldots, N$ on $H^2(\beta)$ by $\tilde{L}_k f = z_k f$ and have the N-tuples $\mathcal{L}$ and $\tilde{\mathcal{L}}$ unitarily equivalent. These results and the basic theory of N-variable weighted shifts can be found in [**2**].

The construction of our model relies on the following lemma.

Proposition 1. *Let $\mathcal{T} = (T_1, \ldots, T_N)$ be jointly isometric and let $M_J = (|J|!/J!)$ be the J-th multinomial coefficient. Then, for $n = 1, 2, \ldots,$ we have*

$$\sum_{|J|=n} M_J T^{*J} T^J = I;$$

that is, the family $\{M_J^{1/2} T^J : |J| = n\}$ is jointly isometric.

Proof. We note that for $|J| = n+1$, we have the generalized Pascal triangle equation $M_J = \sum_{k=1}^{N} M_{(J-\varepsilon_k)}$. If we assume the proposition holds for all $|J| = n$, we have

$$\begin{aligned}\sum_{|J|=n+1} M_J T^{*J} T^J &= \sum_{|J|=n+1} \sum_{k=1}^{N} M_{(J-\varepsilon_k)} T^{*J} T^J \\ &= \sum_{k=1}^{N} T_k^*\Big(\sum_{|J|=n+1} M_{J-\varepsilon_k} T^{*(J-\varepsilon_k)} T^{J-\varepsilon_k}\Big) T_k \\ &= \sum_{k=1}^{N} T_k^*\Big(\sum_{J=n} M_J T^{*J} T^J\Big) T_k \\ &= \sum T_k^* I T_k = I.\end{aligned}$$

To construct our model we let

$$w_{J,k} = \left(\frac{j_k+1}{|J|+1}\right)^{1/2}\sqrt{2}, \qquad k = 1, \dots, N$$

and let $\mathscr{L} = (L_1 \dots, L_N)$ be the corresponding family of weighted shifts on H^2.

THEOREM 3. *Let $\mathscr{T} = (T_1, \dots, T_N)$ be jointly isometric. There exists a closed subspace $M \subset H^2$ invariant under $L_1, \dots, L_N$ and a unitary $W: \mathbf{H} \to M^\perp \subset H^2$ such that*

$$WT_kW^* = L_k^*|_{M^\perp}, \qquad k = 1, \dots, N.$$

Thus, the restriction $\mathscr{L}^*|_{M^\perp}$ forms a canonical model for T.

Proof. For x in $\mathbf{H}$, let

$$Wx = \sum_J (\sqrt{2})^{-(|J|+1)} M_J^{1/2}(T^J x) z^J.$$

Then

$$\begin{aligned}\|Wx\|_{H^2}^2 &= \sum_J 2^{-(|J|+1)} M_J \|T^J x\|^2 \\ &= \sum_{n=0}^{\infty} 2^{-(n+1)}\Big(\sum_{|J|=n} M^J \|T^J x\|^2\Big) \\ &= \sum_{n=0}^{\infty} 2^{-(n+1)} \|x\|^2 \\ &= \|x\|^2\end{aligned}$$

so W is isometric. Further,

$$\begin{aligned}L_k^* Wx &= \sum_J (\sqrt{2})^{-(|J|+1)} M_J^{1/2} L_k^*(T^J x) z^J \\ &= \sum_{j_k \geq 1} (\sqrt{2})^{-(|J|+1)} (|J|!/J!)^{1/2} (\sqrt{2}\, j_k/|J|)^{1/2} (T^J x) z^{J-\varepsilon_k} \\ &= \sum_J (\sqrt{2})^{-(|J|+1)} (|J|!/J!)^{1/2} (T^{J+\varepsilon_k} x) z^J \\ &= WT_k x.\end{aligned}$$

https://doi.org/10.1017/S0017089500031426 Published online by Cambridge University Press

Hence, $L_k^*W = WT_k$, so the range of W, which is closed since W is isometric, is invariant under $\{L_k^*\}$. Hence, W maps $\mathbf{H}$ unitarily onto $M^\perp$ for some M invariant under the weighted shifts $\{L_k\}$ and the theorem follows.

We note that more generally, we can define

$$w'_{J,k} = (\alpha_{|J|}/\alpha_{(|J|+1)})\left(\frac{j_k+1}{|J|+1}\right)^{1/2}$$

for any $\{\alpha_n\}$ such that $\sum_{n=1}^{\infty} \alpha_n^2 = 1$ and have $W'x = \sum \alpha_{|J|} M_J^{1/2}(T^J x)z^J$ define a unitary equivalence between $\mathcal{T}$ and $\mathcal{L}'^*/M^\perp$ where $\mathcal{L}'$ is the N-tuple of weighted shifts corresponding to $\{w'_{J,k}\}$.

Also, for $\beta_J = \sqrt{2}^{|J|}(J!/|J|!)^{1/2}$, the weighted shifts $\tilde{L}_k$ on H^2_β are unitarily equivalent to the L_k above. Hence, $\tilde{\mathcal{L}}^* = (\tilde{L}_1^*, \ldots, \tilde{L}_N^*)$ restricted to $\tilde{M}^\perp \subset H^2(\beta)$ can alternatively be used for the canonical model.

In this context, we note [2] that if $w = (w_1, \ldots, w_n) \in \mathbb{C}^N$ is a bounded point evaluation on $H^2(\beta)$, then for $f = \sum f_J z^J \in H^2(\beta)$, $f(w) = \sum f_J w^J \in \mathbb{C}$ is well defined. We can then consider $f = f(z)$ to be a function analytic at w in $\mathbb{C}^N$. Further, w is a bounded point evaluation on $H^2(\beta)$ if and only if $\sum_J |w_1|^{2j_1} \ldots |w_N|^{2j_N}/\beta_J^2 = \sum |w|^{2J}/\beta_J^2 < \infty$. In our particular case, we have

$$\begin{aligned}
\sum_J |w|^{2J}/\beta_J^2 &= \sum (|w|^{2J}\, |J|!)/(2^{|J|}J!) \\
&= \sum_{n=0}^{\infty} 2^{-n}\left(\sum_{|J|=n} M_J\, |w|^{2J}\right) \\
&= \sum_{n=0}^{\infty} \frac{(|w_1|^2 + \ldots + |w_N|^2)}{2^n} \\
&< \infty
\end{aligned}$$

if $\sum_{j=1}^{N} |w_j|^2 = \|w\|^2 < 2$. Thus, we have the following theorem.

THEOREM 4. *$H^2(\beta)$ consists of functions analytic on the polydisc of radius $\sqrt{2}$.*

Similarly, choosing $\alpha_n = \left(\frac{1}{\sqrt{M}}\right)^n$, $n = 2, 3, \ldots$, where $M > 1$ is fixed and α_1 is chosen such that $\sum \alpha_n^2 = 1$, yields $\{w'_{J,k}\}$ and corresponding $\{\beta'_J\}$ with $\beta'_J = \sqrt{M}^{|J|}(J!/|J|!)^{1/2}$. We then have that $H^2(\beta')$ consists of functions analytic on the polydisc of radius $\sqrt{M}$. If we let $\alpha_n = (\sqrt{n})^{-n}$, we can have $H^2(\beta')$ consisting of entire functions.

A model was given in [3] for an N-tuple of commuting contractions $\mathcal{S} = (S_1, \ldots, S_N)$ such that $\sum S_k^* S_k < I$ and $\sum B_J M_J^{1/2} S^J x$ converges for all X in $\mathbf{H}$. If $\mathcal{J}$ is a joint isometry, then provided $0 < r < 1$, $r\mathcal{J} = (rT_1, \ldots, rT_N)$ satisfies these conditions since

$$\sum_{k=1}^{N} (rT_k)^*(rT_k) = r^2 I < I,$$

and

$$\sum_J \|M_J^{1/2}(rT)^J x\|^2 = \sum_{n=0}^{\infty} r^n \sum_{|J|=n} M_J \|T^J x\|^2$$
$$= \sum_{n=0}^{\infty} r^n \|x\|^2$$
$$= (1-r)^{-1} \|x\|^2.$$

Thus, from [**3**], we deduce the following result.

THEOREM 5. *Let $(T_1, \ldots, T_N)$ be jointly isometric in* $\mathbf{H}$. *Then provided* $0 < r < 1$,

$$Wx = \sum_J (1-r^2)^{1/2} M_J^{1/2} (T^J x) z^J$$

is a unitary map from $\mathbf{H}$ *onto* $M^{\perp} \subset H^2$, *where M is invariant under* $(L_1'', \ldots, L_N'')$ *such that* $W(rT_k)W^* = L_k''^*|_{M^{\perp}}$. *Here* $\mathcal{L}'' = (L_1'', \ldots, L_N'')$ *is the family of weighted shifts corresponding to the net* $w_{J,k}'' = [(j_k+1)/(|J|+1)]^{1/2}$. *Thus*, $(\mathcal{L}'')^*$ *is a canonical model for* (rJ). *By taking* $\lim_{r \to 1}$ *we have a model for T.*

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