ON THE RATE OF CONVERGENCE OF MOMENTS IN THE CENTRAL LIMIT THEOREM

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Abstract

An early extension of Lindeberg's central limit theorem was Bernstein's (1939) discovery of necessary and sufficient conditions for the convergence of moments in the central limit theorem. Von Bahr (1965) made a study of some asymptotic expansions in the central limit theorem, and obtained rates of convergence for moments. However, his results do not in general imply that the moments converge. Some better rates have been obtained by Bhattacharya and Rao for moments between the second and third. In this paper we give improved rates of convergence for absolute moments between the third and fourth.

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1. Introduction and summary

The moments of a distribution are among its most useful and accessible characteristics. Hence it is not surprising that one of the earliest extensions of Lindeberg's central limit theorem was Bernstein's (1939) discovery of a necessary and sufficient condition for the convergence of absolute moments in the central limit theorem. Bernstein's theorem may be stated as follows: Suppose that for each $n \geqslant 1$, $X_{n1}, X_{n2}, ..., X_{nk_n}$ are independent random variables with zero means and finite variances satisfying $\sum_j EX_{nj}^2 = 1$, and that the X_{nj} are asymptotically negligible in the sense that $\max_{j \leqslant k_n} P(|X_{nj}| > \varepsilon) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$. Then

$$S_n = \sum_j X_{nj} \xrightarrow{\;\mathscr{D}\;} N(0, 1)$$

and the pth absolute moment of S_n ($p > 2$) converges to that of the standard normal distribution if and only if

$$L_n = \sum_j E|X_{nj}|^p \to 0.$$

Brown (1969, 1970) gave a characteristic function proof of Bernstein's result. Von Bahr (1965) made a study of some asymptotic expansions in the central limit

250

theorem and their application to the rate of convergence of moments. Bhattacharya and Rao (1976) have improved on some of von Bahr's results in the case $2 < p \leqslant 3$. In this connection see also Michel (1976), who tackles the problem via non-uniform estimates of the rate of convergence in the central limit theorem. Our aim in this paper is to obtain simplified rates of convergence of moments, especially in the case $3 < p \leqslant 4$.

Let μ_p $(p > 2)$ denote the pth absolute moment of a standard normal variable. If p is an even integer it is well known that the rate of convergence of $E|S_n|^p$ to μ_p depends on whether earlier moments match those of the standard normal distribution. For example, if $S_n = \sum_1^n X_j/\sqrt{n}$ where the X_j are i.i.d. (independent and identically distributed) with finite sixth moments and with third and fourth moments equal to m_3 and m_4 respectively, then

$$ES_n^6 = 15 + \{10m_3^2 + 15(m_4 - 3)\}/n + O(n^{-2}).$$

If $m_4 = \mu_4 = 3$ and $m_3 = 0$ then the rate of convergence is $O(n^{-2})$; if $10m_3^2 + 15(m_4 - 3) \neq 0$ then the rate is $O(n^{-1})$. When $p > 3$ and the X_j are not i.i.d., it is more difficult to see how to frame the results so as to obtain a good rate of convergence of $E|S_n|^p$ to μ_p. We will confine our attention to the range $2 < p \leqslant 4$. Our results are stated in Section 2 and the proofs given in Section 3.

2. Rates of convergence

Von Bahr's most general rates of convergence are given in his Theorem 4, and in the case $2 < p \leqslant 4$ they can be summarized as follows: If $k_n = n$ then for all $q \leqslant p$,

$$|E|S_n|^q - \mu_q| = \begin{cases} O(n^{(p-2)/2} L_n^2 + L_n^{(q+1)/(p-2)}), & \text{if } 2 < p < 3, \\ O(n^{(p-2)/2} L_n^2 + n^{(q+1)(p-3)/p} L_n^{3(q+1)/p}), & \text{if } 3 \leqslant p \leqslant 4. \end{cases}$$

(Actually von Bahr's results apply in the case when $X_{nj} = X_j(\sum_1^n EX_j^2)^{-\frac{1}{2}}$, where $\{X_n, n \geqslant 1\}$ is a sequence of independent random variables. Also, his result in the case $2 < p < 3$ is rather more complicated than that given here.) In general this does not imply that $E|S_n|^q \to \mu_q$, although Bernstein's work tells us that this *is* the case for all $q \leqslant p$, provided only that $L_n \to 0$ and that the random variables in each row are asymptotically negligible. In the cases (a) $0 < q \leqslant p = 3$, and (b) $0 < q \leqslant 2$, $2 < p \leqslant 3$, Bhattacharya and Rao (1976), Theorem 18.1, have shown that $|E|S_n|^q - \mu_q| \leqslant Cd_{n,p} \leqslant CL_n$, where the constant C depends only on p and q, and

$$d_{n,p} = \inf_{0 \leqslant \varepsilon \leqslant 1} \left[\varepsilon^{3-p} \sum_j \int_{\{|X_{nj}| \leqslant \varepsilon\}} |X_{nj}|^p \, dP + \sum_j \int_{\{|X_{nj}| > \varepsilon\}} |X_{nj}|^p dP \right].$$

In case (b) when the X_j are i.i.d. this yields an error bound of $o(n^{-(p-2)/2})$, slightly improving on the $O(n^{-(p-2)/2})$ bound due to von Bahr. However, there is no such improvement in case (a).

Our results are most significant in the case $3 < p < 4$, where there is no published improvement of von Bahr's. We state them for the larger range $2 < p \leqslant 4$. They contain those of von Bahr in the i.i.d. case, except when $0 < q < p-2$, when his rates are slightly better than ours. The reason for this is that we have used a simpler method for estimating the integrals of the tails of characteristic functions. (See his remark on p. 817.) In the case of a general triangular array this meant a considerable simplification of the proofs and the results.

THEOREM. *Let X_{nj}, $1 \leqslant j \leqslant k_n$, be a sequence of independent random variables with zero means and finite variances satisfying $\sum_j E(X_{nj}^2) = 1$, and let $S_n = \sum_j X_{nj}$. Suppose that $L_n = \sum_j E|X_{nj}|^p < \infty$, where $2 < p < 4$. Let μ_p denote the absolute pth moment of the standard normal distribution. There exists an absolute constant C_0 such that for all q in $0 < q \leqslant p$, and whenever $L_n \leqslant 1$ (and $|ES_n^3| \leqslant 1$ if $p > 3$),*

(1) $$\left| E|S_n|^q - \mu_q \right| \leqslant C_0^{1/(p-2)} |\sin(\tfrac{1}{2}q\pi)| M_n/q(4-p),$$

where

$$M_n = \begin{cases} L_n + L_n^{q/(p-2)}, & \text{if } 2 < p \leqslant 3, \\ L_n + L_n^{q/(p-2)} + |ES_n^3|^2, & \text{if } 3 < p < 4. \end{cases}$$

In the case $p = 4$, $\left| E|S_n|^4 - \mu_4 \right| \leqslant C_0^{\frac{1}{2}}(\sum_j EX_{nj}^4 + |ES_n^3|^2)$ if $\sum_j EX_{nj}^4 \leqslant 1$ and $|ES_n^3| \leqslant 1$.

REMARK. It follows from von Bahr's Theorem 2 that in the i.i.d. case, $|E|S_n|^3 - \mu_3| = O(n^{-1})$ when X_1 has finite fourth moment, whereas $|ES_n^3| = O(n^{-\frac{1}{2}})$ if $E(X_1^3) \neq 0$. Hence the rate of convergence of $E|S_n|^p$ to μ_p may be considerably faster than the rate of convergence of $|ES_n^3|$ to 0, although in general it will not be faster than the rate of convergence of $|ES_n^3|^2$ to 0.

3. The proofs

We will begin by establishing two lemmas. In our proofs the symbol C denotes a generic constant depending only on p, not necessarily the same at each appearance, while C_1, C_2, C_3 and C_4 denote particular versions of C. The symbols θ, φ and ψ denote generic complex- or real-valued functions of the real variable t, dominated by 1 for all t. Rl z denotes the real part of z.

Let f_{nj} and f_n be the characteristic functions of X_{nj} and S_n, respectively, and let $r_{nj} = f_{nj} - 1$. If $E|S_n|^p < \infty$ we will write

$$f_{nj}(t) = 1 + \sum_{r=2}^m (it)^r EX_{nj}^r/r! + \alpha_{nj}(t),$$

$$f_n(t) = 1 + \sum_{r=2}^m (it)^r ES_n^r/r! + \alpha_n(t)$$

and

$$e^{-\frac{1}{2}t^2} = 1 - \tfrac{1}{2}t^2 + \alpha(t),$$

where $m = 2$ if $2 < p \leqslant 3$, $m = 3$ if $3 < p \leqslant 4$, and $\alpha_{nj}(t)$ is of the form

$$\alpha_{nj}(t) = C\theta_{nj}(t)|t|^p E|X_{nj}|^p.$$

(See Loève (1960), p. 199.)

LEMMA 1. *There exist constants A_p and B_p depending only on p such that whenever $2 < p < 4$, $0 \leqslant q \leqslant p$, $L_n \leqslant 1$ (and $|ES_n^3| \leqslant 1$ if $p > 3$),*

$$\left| \int_0^{B_p} (\mathrm{Rl}\, f_n(t) - e^{-\frac{1}{2}t^2}) t^{-(q+1)} dt \right| \leqslant A_p M_n/(4-p).$$

A_p *is bounded away from ∞ and B_p is bounded away from zero.*

PROOF. We will give a proof in the case $3 < p < 4$. Now,

$$|r_{nj}| \leqslant \tfrac{1}{2}t^2 E(X_{nj}^2) + \tfrac{1}{6}|t|^3 E|X_{nj}|^3 + C|t|^p E|X_{nj}|^p$$

$$\leqslant C_1\{t^2(E|X_{nj}|^p)^{2/p} + |t|^3(E|X_{nj}|^p)^{3/p} + |t|^p E|X_{nj}|^p\}$$

$$\leqslant C_1\{t^2 L_n^{2/p} + |t|^3 L_n^{3/p} + |t|^p L_n\} \leqslant \tfrac{1}{2}$$

if $|t| \leqslant B_p^{(1)} L_n^{-1/p}$, where $B_p^{(1)} = \min\{1, (6C_1)^{-\frac{1}{2}}\}$. Furthermore,

$$|r_{nj}|^2 \leqslant 4\{(\tfrac{1}{2}t^2 EX_{nj}^2)^2 + (\tfrac{1}{6}|t|^3 E|X_{nj}|^3)^2 + (C|t|^p E|X_{nj}|^p)^2\}$$

$$\leqslant C|t|^4\{(E|X_{nj}|^p)^{4/p} + (E|X_{nj}|^p)^{6/p} + (E|X_{nj}|^p)^2\} \quad \text{if } t \leqslant 1$$

$$\leqslant C|t|^4 E|X_{nj}|^p$$

if $p \leqslant 4$ and $L_n \leqslant 1$. It follows that if $|t| \leqslant B_p^{(1)}$ and $L_n \leqslant 1$ then the principal-valued logarithm of f_{nj} is given by

$$\log f_{nj}(t) = \tfrac{1}{2}(it)^2 EX_{nj}^2 + \tfrac{1}{6}(it)^3 EX_{nj}^3 + \alpha_{nj}(t) + C\theta_{nj}(t)|t|^4 E|X_{nj}|^p$$

and that of f_n is given by

$$\log f_n(t) = -\tfrac{1}{2}t^2 - \tfrac{1}{6}it^3 ES_n^3 + \sum_j \alpha_{nj}(t) + C\theta_n(t)|t|^4 L_n.$$

Hence if $|t| \leqslant B_p^{(1)}$ and $L_n \leqslant 1$,

$$f_n(t) = e^{-\frac{1}{2}t^2} \exp\{-\tfrac{1}{6}it^3 ES_n^3 + \sum_j \alpha_{nj}(t) + C\theta_n(t)|t|^4 L_n\}$$

$$= e^{-\frac{1}{2}t^2} \exp(s_n),$$

say. Now,

$$|s_n| \leqslant C_2\{|t|^3 |ES_n^3| + |t|^p L_n + |t|^4 L_n\} \leqslant \tfrac{1}{2}$$

if $L_n \leqslant 1$, $|ES_n^3| \leqslant 1$ and $|t| \leqslant B_p = \min\{B_p^{(1)}, (6C_2)^{-\frac{1}{2}}\}$. Also,

$$|s_n|^2 \leqslant C|t|^4(L_n + |ES_n^3|^2)$$

if $L_n \leqslant 1$ and $|t| \leqslant 1$, and so

$$f_n(t) = e^{-\frac{1}{2}t^2}\{1 - \tfrac{1}{6}it^3 ES_n^3 + \sum_j \alpha_{nj}(t) + C\varphi_n(t)|t|^4(L_n + |ES_n^3|^2)\}.$$

Consequently,

$$\mathrm{Rl}\, f_n(t) - e^{-\frac{1}{2}t^2} = e^{-\frac{1}{2}t^2} \sum_j \mathrm{Rl}\,\alpha_{nj}(t) + C\psi_n(t)|t|^4(L_n + |ES_n^3|^2).$$

The functions $\mathrm{Rl}\,\alpha_{nj}(t) = E\{\cos(tX_{nj}) - 1 + \frac{1}{2}t^2 X_{nj}^2\}$ are non-negative, and since $B_p \leqslant 1$,

$$\left| \int_0^{B_p} (\mathrm{Rl}\, f_n(t) - e^{-\frac{1}{2}t^2}) t^{-(q+1)} dt \right|$$

$$\leqslant \sum_j \int_0^{B_p} \mathrm{Rl}\,\alpha_{nj}(t) t^{-(p+1)} dt + C(L_n + |ES_n^3|^2) \int_0^{B_p} t^{3-p} dt.$$

For each $\lambda > 0$,

$$\int_0^\lambda \mathrm{Rl}\,\alpha_{nj}(t) t^{-(p+1)} dt = E(|X_{nj}|^p A(\lambda|X_{nj}|)],$$

where $A(x) = \int_0^x (\cos t - 1 + \frac{1}{2}t^2) t^{-(p+1)} dt$ is a continuous bounded increasing function with $A(0) = 0$ and $A(\infty) = \pi/2\Gamma(p+1)|\sin \frac{1}{2}p\pi|$. See Pitman (1968), p. 428.) Hence

$$\left| \int_0^{B_p} (\mathrm{Rl}\, f_n(t) - e^{-\frac{1}{2}t^2}) t^{-(q+1)} dt \right|$$

$$\leqslant C\{(L_n/|\sin \tfrac{1}{2}p\pi|) + B_p^{4-q}(L_n + |ES_n^3|^2)/(4-p)\}$$

$$\leqslant A_p(L_n + |ES_n^3|^2)/(4-p),$$

where A_p is bounded as $p \to 4$. B_p is bounded away from zero. This completes the proof of Lemma 1.

LEMMA 2. *There exist constants A_p and B_p depending only on p such that whenever $2 < p \leqslant 4$ and $|t| \leqslant B_p L_n^{-1/(p-2)}$,*

$$|\mathrm{Rl}\, f_n(t) - e^{-\frac{1}{2}t^2}| \leqslant A_p(t^6 + |t|^p) e^{-t^2/3} M_n.$$

A_p is bounded away from ∞ and B_p^{p-2} is bounded away from zero.

PROOF. We will give a proof in the case $3 < p \leqslant 4$. As in the proof of Lemma 1, we can show that if $|t| \leqslant C_1 L_n^{-1/p}$ then $|r_{nj}| \leqslant \frac{1}{2}$. Furthermore,

$$|r_{nj}|^2 \leqslant 4\{(|t|^p E|X_{nj}|^p)^{4/p} + (|t|^p E|X_{nj}|^p)^{6/p} + (|t|^p E|X_{nj}|^p)^2\},$$

and so if $|t| \leqslant C_1 L_p^{-1/p} \leqslant C_1(E|X_{nj}|^p)^{-1/p}$ then

$$|r_{nj}|^2 \leqslant 4\{C_1^{4-p} + C_1^{6-p} + C_1^p\}|t|^p E|X_{nj}|^p.$$

Consequently the principal-valued logarithm of f_{nj} is given by

$$\log f_{nj}(t) = \tfrac{1}{2}(it)^2 EX_{nj}^2 + \tfrac{1}{6}(it)^3 EX_{nj}^3 + C\theta_{nj}(t)|t|^p E|X_{nj}|^p$$

and that of f_n, by

$$\log f_n(t) = -\tfrac{1}{2}t^2 - \tfrac{1}{6}it^3 ES_n^3 + C_2\,\theta_n(t)|t|^p L_n.$$

Hence if $|t| \leqslant C_1 L_n^{-1/p}$ then

$$|\mathrm{Rl} f_n(t) - e^{-\frac{1}{2}t^2}| \leqslant |f_n(t) - \exp(-\tfrac{1}{2}t^2 - \tfrac{1}{6}it^3 ES_n^3)|$$

$$+ |\mathrm{Rl}\exp(-\tfrac{1}{2}t^2 - \tfrac{1}{6}it^3 ES_n^3) - e^{-\frac{1}{2}t^2}|$$

$$\leqslant C_2 |t|^p L_n \exp(-\tfrac{1}{2}t^2 + C_2 |t|^p L_n) + e^{-\frac{1}{2}t^2}\{1 - \cos(\tfrac{1}{6}t^3 ES_n^3)\}$$

$$\leqslant C_2 |t|^p e^{-t^2/3} L_n + t^6 e^{-\frac{1}{2}t^2} |ES_n^3|^2$$

if $|t| \leqslant (6C_2 L_n)^{-1/(p-2)}$. Let $A_p^{(1)} = C_2 + 1$ and $B_p^{(1)} = (6C_2)^{-1/(p-2)}$. If

$$|t| \leqslant \min\{C_1 L_n^{-1/p}, B_p^{(1)} L_n^{-1/(p-2)}\}$$

then

$$|\mathrm{Rl} f_n(t) - e^{-\frac{1}{2}t^2}| \leqslant A_p^{(1)} (t^6 + |t|^p) e^{-t^2/3} M_n.$$

Now suppose that $|t| > C_1 L_n^{-1/p}$. Let $\tilde{X}_{nj} = X_{nj} - Y_{nj}$, where Y_{nj} is independent of X_{nj} but with the same distribution. Then

$$E\tilde{X}_{nj}^2 = 2EX_{nj}^2, \quad E\tilde{X}_{nj}^3 = 0 \quad \text{and} \quad E|\tilde{X}_{nj}|^p \leqslant 2^p E|X_{nj}|^p,$$

so that

$$|f_{nj}(t)|^2 = 1 - \tfrac{1}{2}t^2 E\tilde{X}_{nj}^2 + C\theta_{nj}|t|^p E|\tilde{X}_{nj}|^p$$

$$\leqslant 1 - t^2 EX_{nj}^2 + C_3 |t|^p E|X_{nj}|^p$$

$$\leqslant \exp(-t^2 EX_{nj}^2 + C_3 |t|^p E|X_{nj}|^p).$$

Consequently

$$|f_n(t)|^2 = \prod_j |f_{nj}(t)|^2 \leqslant \exp(-t^2 + C_3 |t|^p L_n) \leqslant e^{-2t^2/3}$$

if $|t| \leqslant (3C_3 L_n)^{-1/(p-2)}$. Let $B_p^{(2)} = (3C_3)^{-1/(p-2)}$. If

$$C_1 L_n^{-1/p} < |t| \leqslant B_p^{(2)} L_n^{-1/(p-2)}$$

then

$$|f_n(t) - e^{-\frac{1}{2}t^2}| \leqslant |f_n(t)| + e^{-\frac{1}{2}t^2}$$

$$\leqslant 2e^{-t^2/3}$$

$$\leqslant 2C_1^{-p} |t|^p e^{-t^2/3} L_n.$$

Let $A_p^{(2)} = 2C_1^{-p}$, $A_p = \max\{A_p^{(1)}, A_p^{(2)}\}$ and $B_p = \min\{B_p^{(1)}, B_p^{(2)}\}$ to obtain the desired result.

PROOF OF THEOREM. Suppose that $L_n \leqslant 1$, $|ES_n^3| \leqslant 1$, $0 < q \leqslant p < 4$ and $q \neq 2$. Let $k_q = \pi/[\Gamma(q+1)|\sin\tfrac{1}{2}q\pi|]$. By Lemma 1 of Brown (1970),

$$k_q |E|S_n|^q - \mu_q| = \left| \int_0^\infty (\mathrm{Rl}\,\alpha_n(t) - \alpha(t)) t^{-(q+1)} dt \right|$$

$$= \left| \int_0^\infty (\mathrm{Rl} f_n(t) - e^{-\frac{1}{2}t^2}) t^{-(q+1)} dt \right|$$

$$\leqslant \left| \int_0^{C_1} (\mathrm{Rl} f_n(t) - e^{-\frac{1}{2}t^2}) t^{-(q+1)} dt \right|$$

$$+ \int_{C_1}^{C_2} |\mathrm{Rl} f_n(t) - e^{-\frac{1}{2}t^2}| t^{-(q+1)} dt + 2 \int_{C_2}^\infty t^{-(q+1)} dt$$

for any constants $C_1 \leqslant C_2$. Our Lemmas 1 and 2 now imply that

$$k_q |E|S_n|^q - \mu_q|$$

$$\leqslant C_3\{(M_n/(4-p)) + M_n \int_{C_1}^{\infty} (t^6 + |t|^p) e^{-t^2/3} dt + \int_{C_4 L_n^{-1/(p-2)}}^{\infty} t^{-(q+1)} dt\},$$

where C_3 is bounded away from ∞, and C_1 and C_4^{p-2} are bounded away from zero,

$$\leqslant C^{1/(p-2)} M_n / q(4-p),$$

where C is bounded away from ∞. Hence

$$|E|S_n|^q - \mu_q| \leqslant C_0^{1/(p-2)} |\sin \tfrac{1}{2} q\pi| M_n / q(4-p),$$

where C_0 is an absolute constant. By setting $q = p$ and taking the limit as $p \uparrow 4$, and noting that $E|X|^p \to E|X|^4$ as $p \uparrow 4$, we see that

$$|E|S_n|^4 - \mu_4| \leqslant C_0^{\frac{1}{2}}(\tfrac{1}{2}\pi)(2L_n + |ES_n^3|^2)/4$$

$$\leqslant C_0^{\frac{1}{2}}(L_n + |ES_n^3|^2),$$

provided that $L_n = \sum_j EX_{nj}^4 \leqslant 1$ and $|ES_n^3| \leqslant 1$.

REFERENCES

B. von Bahr (1965), "On the convergence of moments in the central limit theorem", *Ann. Math. Statist.* **36**, 808–818.

S. Bernstein (1939), "Quelques remarques sur le théorème limite Liapounoff', *Dokl. Akad. Nauk SSR (Comptes rendus)* **24**, 3–8.

R. N. Bhattacharya and R. R. Rao (1976), *Normal Approximation and Asymptotic Expansions* (Wiley, New York).

B. M. Brown (1969), "Moments of a stopping rule related to the central limit theorem", *Ann. Math. Statist.* **40**, 1236–1249.

B. M. Brown (1970), "Characteristic functions, moments and the central limit theorem", *Ann. Math. Statist.* **41**, 658–664.

M. Loève (1960), *Probability Theory* (Van Nostrand, Princeton).

R. Michel (1976), "Nonuniform central limit bounds with applications to probabilities of deviations", *Ann. Probability* **4**, 102–106.

E. J. G. Pitman (1968), "On the behaviour of the characteristic function of a probability distribution in the neighbourhood of the origin", *J. Austral. Math. Soc.* **8**, 423–443.

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