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THE CLOSED GRAPH THEOREM WITHOUT CATEGORY

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We show that a diagonal theorem of P. Antosik can be used to give a proof of the Closed Graph Theorem for normed spaces which does not depend upon the Baire Category Theorem.

The Closed Graph Theorem is one of the fundamental and most important results in functional analysis. The usual proofs of the Closed Graph Theorem for Banach or Frechet spaces utilizes the completeness in the domain space by means of the Baire Category Theorem ([3]); the same is the case for another of the basic results of functional analysis, the Uniform Boundedness Principle ([2] II. 1.11). However "gliding hump" proofs of the Uniform Bounded Principle do appear, for example , section 7.2 . In [1] it was shown that the proof of the Uniform Boundedness Principle could be based on a very simple result concerning infinite matrices instead of the Baire Category Theorem. In this note we would like to point out that a proof of the Closed Graph Theorem can also be obtained by employing the matrix methods of [1] in the domain space and using some basic results on domains of adjoint operators. This method of proof offers an interesting contrast to the usual Baire category proofs of the Closed Graph Theorem.

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Throughout the rest of this note let X and Y be normed linear spaces. We follow the notation and terminology of [2]. A sequence $\{x_i\}$ in X is called a <u>H-convergent sequence</u> if every subsequence of $\{x_i\}$ has a subsequence $\{x_{n_i}\}$ such that the series $\sum_{i=1}^{\infty} x_{n_i}$ is (norm) convergent to an element $x \in X$. Note that any H-convergent sequence converges to 0, and if X is complete any sequence $\{x_i\}$ which converges to 0 is also H-convergent (if $||x_i|| \to 0$, then any subsequence has a subsequence $\{x_{n_i}\}$ with $||x_{n_i}|| < 1/2^i$ so the series $\sum_{i=1}^{\infty} x_{n_i}$ converges by completeness). In general, a sequence which converges to 0 may not be H-convergent; for example, if c_{00} is the space of real sequences which are eventually 0 and if e_i is the sequence with a 1 in the i^{th} coordinate and 0 elsewhere, then $\{e_i/i\}$ converges to 0 but is not H-convergent. A normed space is called a <u>H-space</u> if every sequence which converges to 0 is H-convergent. The requirement that any sequence which converges to 0 is H-convergent is a kind of "completeness-type condition", but it is not equivalent to a space being complete; Klis has given an example of a normed H-space which is not complete ([4]).

Suppose now that $T : X \quad Y$ is linear. Then the <u>adjoint</u>, T', of T is defined as follows: the domain of T' is

$$D(T') = \{y' \in Y' : y'T \text{ is continuous}\}$$

and $T' : D(T') \quad X'$ is defined by $T'y' = y'T$. In order to establish the Closed Graph Theorem for closed operators $T : X \to Y$ it suffices to establish that the adjoint operator T' has domain $D(T') = Y'$ and is (norm) continuous from Y' into X'. We now give conditions under which this holds.

First concerning the continuity of the adjoint operator, we have the following result which is due to E. Pap for inner product spaces ([6]). For this we use the following matrix theorem of P. Antosik ([1] 2.2):

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THEOREM 1. *Let* $x_{ij} \in X$, $i, j = 1, 2, \dots$. *Suppose that*

(I) $\lim_i x_{ij} = x_j$ *exists for each* j *and*

(II) *for each increasing sequence of positive integers* $\{m_j\}$ *there is a subsequence* $\{n_j\}$ *of* $\{m_j\}$ *such that* $\sum_{j=1}^{\infty} x_{in_j} = z_i$ *converges and* $\{z_i\}$ *converges.*

Then $\lim_i x_{ii} = 0$.

THEOREM 2. *Let* X *be a* H*-space. Then*

(i) T' *carries weak* bounded sets into norm bounded sets.*

(ii) T' *carries norm bounded sets into norm bounded sets and is, therefore, continuous with respect to the norm topologies.*

Proof. It suffices to establish (i) since (ii) follows immediately. Let $\{y_i'\}$ be weak* bounded in $D(T')$. For each i , pick $x_i \in X$, $\|x_i\| = 1$, such that $\langle T'y_i', x_i\rangle \geq \|T'y_i'\| - 1/i$. It suffices to show that $\{\langle T'y_i', x_i\rangle\}$ is bounded or that $\{t_i\langle T'y_i', x_i\rangle\}$ converges to 0 for any positive sequence of scalars $\{t_i\}$ which converges to 0 . We use Theorem 1 to establish this.

Consider the infinite matrix z_{ij} , where $z_{ij} = \langle\sqrt{t_i}T'y_i', \sqrt{t_j}\, x_j\rangle$. Now $\lim_i z_{ij} = \lim_i \langle\sqrt{t_i}\, y_i', T(\sqrt{t_j}\, x_j)\rangle = 0$ since $\{y_i'\}$ is weak* bounded so that (I) of Theorem 1 is satisfied. For (II) if $\{m_j\}$ is any increasing sequence of positive integers, then since $\sqrt{t_j}\, x_j \to 0$ and X is a H-space there is a subsequence $\{n_j\}$ such that $x = \sum_{j=1}^{\infty} \sqrt{t_{n_j}} x_{n_j}$. By continuity, $\sum_{j=1}^{\infty} z_{in_j} = \langle\sqrt{t_i}\, T'y_i', x\rangle$ and $\langle\sqrt{t_i}\, T'y_i', x\rangle = \langle\sqrt{t_i}\, y_i', T_x\rangle \to 0$ since $\{y_i'\}$ is weak* bounded. Thus, (II) holds and by Theorem 1,

$$\lim_i z_{ii} = \lim_i t_i\langle T'y_i', x_i\rangle = 0 .$$

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Condition (ii) was established in 3.11 of [1] by directly applying Theorem 1.

Concerning the domain of T', we first recall the following well-known result which guarantees that the adjoint of a closed operator has a non-trivial domain ([8] IV. 8.1).

THEOREM 3. *If* $T : X \to Y$ *is closed, then* $D(T')$ *separates the points of* Y *and, therefore,* $D(T')$ *is weak* dense in* Y'.

From Theorem 3, in order to establish that the domain, $D(T')$, of the adjoint of a closed operator T is Y', it suffices to show that $D(T')$ is weak* closed in Y'. We now establish this for the case when Y is complete and T' is continuous.

THEOREM 4. *If* Y *is a Banach space and* $T' : D(T') \to X'$ *is (norm) continuous, then* $D(T')$ *is weak* closed in* Y'.

Proof. By the Krelin-Smulian Theorem, it suffices to show that $D(T') \cap S'$ is weak* closed, where $S' = \{y' \in Y' : \|y'\| \le 1\}$ is the unit ball of Y' ([2] V. 5.7). Suppose that $\{y'_\nu\}$ is a net in $D(T') \cap S'$ which is weak* convergent to $y' \in Y'$. We must show that $y' \in D(T')$ or $y'T$ is continuous. If $\|x\| \le 1$, then $|\langle y'_\nu, Tx\rangle| = |\langle T'y'_\nu, x\rangle| \le \|T'y'_\nu\| \le \|T'\|$ so $\lim|\langle y'_\nu, Tx\rangle| = |\langle y', Tx\rangle| \le \|T'\|$ and $y'T$ is continuous.

Combining Theorems 2, 3 and 4, we obtain the following version of the Closed Graph Theorem.

THEOREM 5. *Let* X *be a* H*-space,* Y *a Banach space and* $T : X \quad Y$ *closed. Then* T *is continuous.*

Proof. From Theorems 2, 3, and 4, $T' : Y' \to X'$ is continuous so if $x \in X$,

$$\|Tx\| = \sup\{|\langle y', Tx\rangle| : \|y'\| \le 1\}$$
$$= \sup\{|\langle T'y', x\rangle| : \|y'\| \le 1\} \le \|T'\|\|x\|,$$

and T is continuous.

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Since the Open Mapping Theorem is an easy consequence of the Closed Graph Theorem, this proof also gives a proof of the Open Mapping Theorem which does not rely on the Baire Category Theorem as is usually the case ([2] II. 2.1).

Of course, there are much more general versions of the Closed Graph Theorem for locally convex spaces, but their proofs employ much more sophisticated methods that the versions for either Banach or Frechet spaces ([6]).

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